Research Article

Stability of Runge-Kutta Methods for Neutral Delay Differential Equations

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This paper is concerned with the numerical stability of a class of nonlinear neutral delay differential equations. The numerical stability results are obtained for (k, l) -algebraically stable Runge-Kutta methods when they are applied to this type of problem. Numerical examples are given to confirm our theoretical results.

1. Introduction

Let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathbb{C}^N$ and let $\| \cdot \|$ be the corresponding norm. Consider the initial value problems (IVPs) of nonlinear neutral delay differential equations (NDDEs) as follows (cf. [1]):

$$\begin{aligned} y'(t) &= f(t, y(t), y(t - \tau_1), y'(t - \tau_2)), \quad t \geq t_0, \\ y(t) &= \varphi(t), \quad t_0 - \tau \leq t \leq t_0, \end{aligned} \quad (1)$$

where τ_1, τ_2 are real positive constants, $\tau = \max\{\tau_1, \tau_2\}$, $\varphi : [t_0 - \tau, t_0] \rightarrow \mathbb{C}^N$ is a continuously differentiable function, and $f : [t_0, +\infty) \times \mathbb{C}^N \times \mathbb{C}^N \times \mathbb{C}^N \rightarrow \mathbb{C}^N$ is a continuous function satisfying

$$\begin{aligned} \operatorname{Re} \langle u_1 - u_2, f(t, u_1, v, w) - f(t, u_2, v, w) \rangle \\ \leq \alpha \|u_1 - u_2\|^2, \quad \forall t \geq t_0, \quad u_1, u_2, v, w \in \mathbb{C}^N, \end{aligned} \quad (2)$$

$$\begin{aligned} \|f(t, u, v_1, w) - f(t, u, v_2, w)\| &\leq \beta_1 \|v_1 - v_2\|, \\ \forall t \geq t_0, \quad u, v_1, v_2, w &\in \mathbb{C}^N, \end{aligned} \quad (3)$$

$$\begin{aligned} \|f(t, u, v, w_1) - f(t, u, v, w_2)\| &\leq \beta_2 \|w_1 - w_2\|, \\ \forall t \geq t_0, \quad u, v, w_1, w_2 &\in \mathbb{C}^N, \end{aligned} \quad (4)$$

$$\begin{aligned} \|F(t, u, v_1, x_1, y, z) - F(t, u, v_2, x_2, y, z)\| \\ \leq \gamma_1 \|v_1 - v_2\| + \gamma_2 \|x_1 - x_2\|, \end{aligned} \quad (5)$$

$$\forall t \geq t_0, \quad u, v_1, v_2, x_1, x_2, y, z \in \mathbb{C}^N,$$

where

$$F(t, u, v, x, y, z) := f(t, u, v, f(t - \tau_2, x, y, z)) \quad (6)$$

and $\alpha, \beta_1, \beta_2, \gamma_1$, and γ_2 are real constants and $\beta_1 \geq 0, 0 \leq \beta_2 < 1, \gamma_1 > 0$, and $\gamma_2 > 0$.

The existence and uniqueness of solution to (1) are discussed in [2, 3]. Throughout this paper we assume that problem (1) has a unique true solution $y(t)$, and we will still use the symbol $\mathbb{D}(\alpha, \beta_1, \beta_2, \gamma_1, \gamma_2)$ presented in literature [1] to denote the problem class consisting of all problem (1) satisfying conditions (2)–(5).

In order to investigate the stability of NDDEs, we have to introduce the perturbed problem of (1), which is defined by the same function $f(t, u, v, w)$, but with another initial condition:

$$\begin{aligned} z'(t) &= f(t, z(t), z(t - \tau_1), z'(t - \tau_2)), \quad t \geq t_0, \\ z(t) &= \psi(t), \quad t_0 - \tau \leq t \leq t_0, \end{aligned} \quad (7)$$

where $\psi : [t_0 - \tau, t_0] \rightarrow \mathbb{C}^N$ is a continuous and differentiable function. The unique exact solution of problem (7) is denoted by $z(t)$.

Remark 1. In 2004, Wang and Li [4] considered nonlinear NDDEs of form as

$$\begin{aligned} y'(t) &= f(t, y(t), y(t-\tau), y'(t-\tau)), \quad t \geq 0, \\ y(t) &= \varphi(t), \quad -\tau \leq t \leq 0, \end{aligned} \quad (8)$$

and they obtained a series of stability results of theoretical solution and numerical solution which was given by backward Euler methods. Recently, they further investigated the stability of one-leg methods [5], Runge-Kutta methods [6], and continuous Runge-Kutta-type methods [7] for the solution to problem (8), respectively. More research in this field can be found in literatures [2, 3, 8–15]. The research of numerical methods for more extensive problem class can be seen in [16–19].

The difference between (1) and (8) is that the quantities τ_1, τ_2 in (1) can be unequal. However, many real-world phenomena can be described by (1) but not with (8). For the application of this type in the real world, one is referred to [1], because, in these problems, the change of the state variable depends on the state of the past some time and, in addition, a state change of another some time (cf. [20, 21]). It is easy to see that the existent analytic and numerical stability results for (8) in the abovementioned literatures can not be applied to problem (1).

Remark 2. In 2012, the authors of the present paper considered problem (1), and a sufficient condition for the stability of the problem itself is given [1]. This result is described below.

Theorem 3 (cf. [1]). *Assume that (1) belongs to $\mathbb{D}(\alpha, \beta_1, \beta_2, \gamma_1, \gamma_2)$ with $\alpha < 0$ and*

$$\frac{\gamma_1 + \gamma_2 - \beta_2\alpha}{-\alpha} \leq 1. \quad (9)$$

Then one has

$$\|y(t) - z(t)\| \leq \phi_1 + \frac{\beta_1\phi_1 + \beta_2\phi_2}{-\alpha} \sum_{i=0}^{\lfloor \tau_2/\tau_1 \rfloor} \left(1 + \frac{\beta_1}{-\alpha}\right)^i, \quad (10)$$

where

$$\begin{aligned} \phi_1 &= \sup_{t_0 - \tau \leq s \leq t_0} \|\varphi(s) - \psi(s)\|, \\ \phi_2 &= \sup_{t_0 - \tau \leq s \leq t_0} \|\varphi'(s) - \psi'(s)\|, \end{aligned} \quad (11)$$

and the notation $\lfloor x \rfloor$ denotes the largest integer smaller than or equal to x .

Remark 4. In [1], the numerical stability results are obtained for A-stable one-leg methods when they are applied to problem (1). This paper pursues this and further investigates the stability of Runge-Kutta methods for problem (1). It is well known that the Runge-Kutta-type methods are a class of important and common numerical methods for solving differential equations. Therefore, it is important to analyse whether or not Runge-Kutta methods inherit the stability of the underlying problem when they are applied to (1). This also is the motivation of this paper.

The organization of this paper is as follows. In Section 2, the numerical stability results of (k, l) -algebraically stable Runge-Kutta methods for (1) are given. Finally, in Section 3, some numerical experiments are given which confirm the theoretical results obtained in this paper.

2. The Stability of Runge-Kutta Methods

The adaptation of the s stage Runge-Kutta method

$$(A, b, c) = \begin{array}{c|cccc} c_1 & a_{11} & a_{12} & \cdots & a_{1s} \\ c_2 & a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \cdots & \cdots & \cdots & \cdots \\ c_s & a_{s1} & a_{s2} & \cdots & a_{ss} \\ \hline & b_1 & b_2 & \cdots & b_s \end{array} \quad (12)$$

for ODEs can generally lead to an s stage Runge-Kutta method

$$\begin{aligned} Y_i^{(n)} &= y_n + h \sum_{j=1}^s a_{ij} f(t_n + c_j h, Y_j^{(n)}, \bar{Y}_j^{(n)}, \tilde{Y}_j^{(n)}), \\ i &= 1, 2, \dots, s, \end{aligned} \quad (13)$$

$$y_{n+1} = y_n + h \sum_{j=1}^s b_j f(t_n + c_j h, Y_j^{(n)}, \bar{Y}_j^{(n)}, \tilde{Y}_j^{(n)}),$$

$$n = 0, 1, \dots$$

for solving problem (1) in NDDEs; here, $h > 0$ is the step size and $t_n = t_0 + nh$. The values $y_n, Y_i^{(n)}, \bar{Y}_i^{(n)}$, and $\tilde{Y}_i^{(n)}$ are approximations to $y(t_n), y(t_n + c_i h), y(t_n + c_i h - \tau_1)$, and $y'(t_n + c_i h - \tau_2)$, respectively. $y_0 = \varphi(t_0)$. The symbols A and b are linear mappings corresponding to the matrices $A = [a_{ij}] \in R^{s \times s}$ and $b^T = [b_1, b_2, \dots, b_s] \in R^s$, respectively. We always assume that $\sum_{i=1}^s b_i = 1$; each $c_i = \sum_{j=1}^s a_{ij} \in [0, 1]$.

In this paper, we always let $h = \tau_2/m_2$ for some positive integer m_2 and $\tau_1 = (m_1 - \delta)h$ with integer $m_1 \geq 1$ and $\delta \in [0, 1]$. Let $Y_j^{(i)} = \varphi(t_i + c_j h)$ for $i < 0, 1 \leq j \leq s$. Define (cf. [22])

$$\bar{Y}_j^{(n)} = \begin{cases} \sum_{i=-\mu}^{\nu} L_i(\delta) Y_j^{(n-m_1+i)}, & t_n + c_j h - \tau_1 > t_0, \\ \varphi(t_n + c_j h - \tau_1), & t_n + c_j h - \tau_1 \leq t_0, \end{cases} \quad (14)$$

$$j = 1, 2, \dots, s,$$

where

$$L_i(x) = \prod_{k=-\mu, k \neq i}^{\nu} \frac{x-k}{i-k}, \quad x \in [0, 1), \quad (15)$$

and we assume integers $\mu, \nu \geq 0$ and $m_1 \geq \nu + 1$ so as to guarantee that, in the interpolation procedure for $\bar{Y}_j^{(n)}$, no unknown values $Y_j^{(k)}$ with $k \geq n$ are used.

The arguments $\tilde{Y}_i^{(n)}$ are given by

$$\tilde{Y}_i^{(n)} = \begin{cases} f(t_n + c_i h - \tau_2, Y_i^{(n-m_2)}, \bar{Y}_i^{(n-m_2)}, \tilde{Y}_i^{(n-m_2)}), & t_n + c_i h - \tau_2 > t_0, \\ \varphi'(t_n + c_i h - \tau_2), & t_n + c_i h - \tau_2 \leq t_0, \end{cases} \quad i = 1, 2, \dots, s. \quad (16)$$

Similarly, the adaptation of the Runge-Kutta method (A, b, c) with the same interpolation procedures to problem (7) leads to the following process:

$$\begin{aligned} Z_i^{(n)} &= z_n + h \sum_{j=1}^s a_{ij} f(t_n + c_j h, Z_j^{(n)}, \bar{Z}_j^{(n)}, \tilde{Z}_j^{(n)}), \\ &\quad i = 1, 2, \dots, s, \\ z_{n+1} &= z_n + h \sum_{j=1}^s b_j f(t_n + c_j h, Z_j^{(n)}, \bar{Z}_j^{(n)}, \tilde{Z}_j^{(n)}), \\ &\quad n = 0, 1, \dots, \end{aligned} \quad (17)$$

where $z_0 = \psi(t_0)$.

Definition 5 (cf. [23]). Let k, l be real constants. A method (A, b, c) is said to be (k, l) -algebraically stable if there exists a diagonal nonnegative matrix $D = \text{diag}(d_1, d_2, \dots, d_s)$ such that $M = [M_{ij}]$ is nonnegative definite, where

$$\begin{aligned} M &= \begin{pmatrix} k - 1 - 2le^T De & e^T D - b^T - 2le^T DA \\ De - b - 2lA^T De & DA + A^T D - bb^T - 2lA^T DA \end{pmatrix}. \end{aligned} \quad (18)$$

As an important special case, a $(1, 0)$ -algebraically stable method is called algebraically stable for short.

In the following, we give the main result of this paper and its proof.

Theorem 6. Assume that method (12) is (k, l) -algebraically stable with $0 < k \leq 1$ and that problems (1) and (7) belong to $\mathbb{D}(\alpha, \beta_1, \beta_2, \gamma_1, \gamma_2)$ with $\alpha < 0$, $\beta_1 \leq \gamma_1$. Then when

$$\left(2\alpha + \frac{\gamma_1 + L_0^2 (\mu + \nu + 1)^2 \gamma_1 + 2\gamma_2}{1 - \beta_2} \right) h \leq 2l, \quad (19)$$

method (13) with (14) and (16) is stable; that is,

$$\begin{aligned} \|y_n - z_n\| &\leq C \max \left\{ \max_{t_0 - \tau \leq t \leq t_0} \|\varphi(t) - \psi(t)\|, \right. \\ &\quad \left. \max_{t_0 - \tau \leq t \leq t_0} \|\varphi'(t) - \psi'(t)\| \right\}, \end{aligned} \quad (20)$$

where the constant $C > 0$ depends only on $\alpha, \beta_1, \beta_2, \gamma_1, \gamma_2, \tau_1$, and τ_2 and the sequences $\{y_n\}$ and $\{z_n\}$ are two approximation solutions produced by (13) and (17), respectively.

Proof. For any $n > 0$, let $q_i^{(n)} = \lfloor (n + c_i)/m_2 \rfloor$, $i = 1, 2, \dots, s$; then we have

$$t_0 + q_i^{(n)} \tau_2 \leq t_n + c_i h \leq t_0 + (q_i^{(n)} + 1) \tau_2. \quad (21)$$

Let

$$\begin{aligned} w_n &= y_n - z_n, \\ W_i^{(n)} &= Y_i^{(n)} - Z_i^{(n)}, \\ \bar{W}_i^{(n)} &= \bar{Y}_i^{(n)} - \bar{Z}_i^{(n)}, \\ Q_i^{(n)} &= f(t_n + c_i h, Y_i^{(n)}, \bar{Y}_i^{(n)}, \tilde{Y}_i^{(n)}) \\ &\quad - f(t_n + c_i h, Z_i^{(n)}, \bar{Z}_i^{(n)}, \tilde{Z}_i^{(n)}). \end{aligned} \quad (22)$$

Then from (13) and (17) we obtain that

$$\begin{aligned} W_i^{(n)} &= w_n + h \sum_{j=1}^s a_{ij} Q_j^{(n)}, \quad i = 1, 2, \dots, s, \\ w_{n+1} &= w_n + h \sum_{j=1}^s b_j Q_j^{(n)}. \end{aligned} \quad (23)$$

As in [23], we can easily obtain that

$$\begin{aligned} \|w_{n+1}\|^2 - k \|w_n\|^2 &- 2 \sum_{j=1}^s d_j \text{Re} \langle W_j^{(n)}, h Q_j^{(n)} - l W_j^{(n)} \rangle \\ &= - \sum_{i=1}^{s+1} \sum_{j=1}^{s+1} M_{ij} \langle u_i, u_j \rangle, \end{aligned} \quad (24)$$

where

$$\begin{aligned} u_1 &= w_n, \\ u_i &= h Q_{i-1}^{(n)}, \quad i = 2, 3, \dots, s+1. \end{aligned} \quad (25)$$

Noting that the method is (k, l) -algebraically stable; then (24) reads

$$\begin{aligned} \|w_{n+1}\|^2 &\leq k \|w_n\|^2 \\ &\quad + 2 \sum_{j=1}^s d_j \text{Re} \langle W_j^{(n)}, h Q_j^{(n)} - l W_j^{(n)} \rangle. \end{aligned} \quad (26)$$

It follows from (2) that

$$\begin{aligned} 2\operatorname{Re} \langle W_i^{(n)}, hQ_i^{(n)} \rangle &= 2h\operatorname{Re} \left\langle W_i^{(n)}, f \left(t_n + c_i h, Y_i^{(n)}, \bar{Y}_i^{(n)}, \tilde{Y}_i^{(n)} \right) \right. \\ &\quad \left. - f \left(t_n + c_i h, Z_i^{(n)}, \bar{Z}_i^{(n)}, \tilde{Z}_i^{(n)} \right) \right\rangle \leq 2h\alpha \|W_i^{(n)}\|^2 \\ &\quad + 2h \|W_i^{(n)}\| \Gamma_i^{(n)}, \end{aligned} \quad (27)$$

where

$$\begin{aligned} \Gamma_i^{(n)} &= \left\| f \left(t_n + c_i h, Z_i^{(n)}, \bar{Y}_i^{(n)}, \tilde{Y}_i^{(n)} \right) \right. \\ &\quad \left. - f \left(t_n + c_i h, Z_i^{(n)}, \bar{Z}_i^{(n)}, \tilde{Z}_i^{(n)} \right) \right\|. \end{aligned} \quad (28)$$

When $t_n + c_i h \leq t_0 + \tau_2$, from (3), (4), and (16), we have

$$\Gamma_i^{(n)} \leq \beta_1 \|\bar{W}_i^{(n)}\| + \beta_2 \phi_2. \quad (29)$$

Otherwise, when $t_n + c_i h > t_0 + \tau_2$, from (4), (5), and (16), we have

$$\begin{aligned} \Gamma_i^{(n)} &= \left\| f \left(t_n + c_i h, Z_i^{(n)}, \bar{Y}_i^{(n)}, \tilde{Y}_i^{(n)} \right) \right. \\ &\quad \left. f \left(t_n + c_i h - \tau_2, Y_i^{(n-m_2)}, \bar{Y}_i^{(n-m_2)}, \tilde{Y}_i^{(n-m_2)} \right) \right) - f \left(t_n \right. \\ &\quad \left. + c_i h, Z_i^{(n)}, \bar{Z}_i^{(n)}, \tilde{Z}_i^{(n)} \right) \\ &\quad \left. f \left(t_n + c_i h - \tau_2, Z_i^{(n-m_2)}, \bar{Y}_i^{(n-m_2)}, \tilde{Y}_i^{(n-m_2)} \right) \right) + f \left(t_n \right. \\ &\quad \left. + c_i h, Z_i^{(n)}, \bar{Z}_i^{(n)}, \tilde{Z}_i^{(n)} \right) \\ &\quad \left. f \left(t_n + c_i h - \tau_2, Z_i^{(n-m_2)}, \bar{Y}_i^{(n-m_2)}, \tilde{Y}_i^{(n-m_2)} \right) \right) - f \left(t_n \right. \\ &\quad \left. + c_i h, Z_i^{(n)}, \bar{Z}_i^{(n)}, \tilde{Z}_i^{(n)} \right) \\ &\quad \left. f \left(t_n + c_i h - \tau_2, Z_i^{(n-m_2)}, \bar{Z}_i^{(n-m_2)}, \tilde{Z}_i^{(n-m_2)} \right) \right) \Big\| \\ &\leq \gamma_1 \|\bar{W}_i^{(n)}\| + \gamma_2 \|W_i^{(n-m_2)}\| + \beta_2 \Gamma_i^{(n-m_2)}. \end{aligned} \quad (30)$$

From this and by induction we have

$$\begin{aligned} \Gamma_i^{(n)} &\leq \gamma_1 \sum_{j=1}^{q_i^{(n)}} \beta_2^{j-1} \|\bar{W}_i^{(n-(j-1)m_2)}\| + \gamma_2 \sum_{j=1}^{q_i^{(n)}} \beta_2^{j-1} \|W_i^{(n-jm_2)}\| \\ &\quad + \beta_2^{q_i^{(n)}} \Gamma_i^{(n-q_i^{(n)}m_2)} \\ &\leq \gamma_1 \sum_{j=1}^{q_i^{(n)}+1} \beta_2^{j-1} \|\bar{W}_i^{(n-(j-1)m_2)}\| \\ &\quad + \gamma_2 \sum_{j=1}^{q_i^{(n)}} \beta_2^{j-1} \|W_i^{(n-jm_2)}\| + \beta_2^{q_i^{(n)}} \beta_2 \phi_2, \end{aligned} \quad (31)$$

where we have used (29) and $\beta_1 \leq \gamma_1$. Apparently, inequality (29) can be integrated to (31) by $q_i^{(n)} = 0$. Note that, here and later, when $q < p$, the value of sum $\sum_p^q = 0$.

Inequality (27) together with (31) yields

$$\begin{aligned} 2\operatorname{Re} \langle W_i^{(n)}, hQ_i^{(n)} \rangle &\leq h \left(2\alpha + \frac{\gamma_1 + \gamma_2}{1 - \beta_2} \right) \|W_i^{(n)}\|^2 \\ &\quad + h\gamma_1 \sum_{j=1}^{q_i^{(n)}+1} \beta_2^{j-1} \|\bar{W}_i^{(n-(j-1)m_2)}\|^2 \\ &\quad + h\gamma_2 \sum_{j=1}^{q_i^{(n)}} \beta_2^{j-1} \|W_i^{(n-jm_2)}\|^2 \\ &\quad + h \frac{\beta_2^{q_i^{(n)}} \beta_2^2}{\gamma_2} \phi_2^2. \end{aligned} \quad (32)$$

Denote

$$\mu_1 = h \left(2\alpha + \frac{\gamma_1 + \gamma_2}{1 - \beta_2} \right) - 2l. \quad (33)$$

Then, from (26) and noting that $0 < k \leq 1$, we can easily obtain

$$\begin{aligned} \|w_{n+1}\|^2 &\leq k \|w_n\|^2 + \mu_1 \sum_{i=1}^s d_i \|W_i^{(n)}\|^2 \\ &\quad + h \sum_{i=1}^s d_i \left(\gamma_1 \sum_{j=1}^{q_i^{(n)}+1} \beta_2^{j-1} \|\bar{W}_i^{(n-(j-1)m_2)}\|^2 \right. \\ &\quad \left. + \gamma_2 \sum_{j=1}^{q_i^{(n)}} \beta_2^{j-1} \|W_i^{(n-jm_2)}\|^2 + \frac{\beta_2^{q_i^{(n)}} \beta_2^2}{\gamma_2} \phi_2^2 \right) \leq \|w_0\|^2 \\ &\quad + \mu_1 \sum_{j=0}^n \sum_{i=1}^s d_i \|W_i^{(j)}\|^2 \\ &\quad + h\gamma_1 \sum_{\mu=0}^n \sum_{i=1}^s d_i \sum_{j=1}^{q_i^{(\mu)}+1} \beta_2^{j-1} \|\bar{W}_i^{(\mu-(j-1)m_2)}\|^2 \\ &\quad + h\gamma_2 \sum_{\mu=0}^n \sum_{i=1}^s d_i \sum_{j=1}^{q_i^{(\mu)}} \beta_2^{j-1} \|W_i^{(\mu-jm_2)}\|^2 \\ &\quad + h \sum_{\mu=0}^n \sum_{i=1}^s d_i \beta_2^{q_i^{(\mu)}} \frac{\beta_2^2}{\gamma_2} \phi_2^2 \leq \|w_0\|^2 \\ &\quad + \mu_1 \sum_{i=1}^s d_i \sum_{j=0}^n \|W_i^{(j)}\|^2 + \frac{h\gamma_2}{1 - \beta_2} \sum_{i=1}^s d_i \sum_{j=0}^n \|W_i^{(j)}\|^2 \\ &\quad + \frac{h\gamma_1}{1 - \beta_2} \sum_{i=1}^s d_i \sum_{j=0}^n \|\bar{W}_i^{(j)}\|^2 + \frac{hdm_2}{1 - \beta_2} \frac{\beta_2^2}{\gamma_2} \phi_2^2 \leq \|w_0\|^2 \end{aligned}$$

$$\begin{aligned}
& + \left(\mu_1 + \frac{h\gamma_2}{1 - \beta_2} \right) \sum_{i=1}^s d_i \sum_{j=0}^n \|W_i^{(j)}\|^2 + \frac{h\gamma_1}{1 - \beta_2} \\
& \cdot \sum_{i=1}^s d_i \sum_{j=0}^n \|\overline{W}_i^{(j)}\|^2 + \frac{d\tau_2}{1 - \beta_2} \frac{\beta_2^2}{\gamma_2} \phi_2^2,
\end{aligned} \quad (34)$$

where $d = \sum_{i=1}^s d_i$.

In order to estimate the term $\|\overline{W}_i^{(j)}\|$ in (34), we denote $J_0 = \{1, 2, \dots, s\}$, $J_1 = \{j \in J_0 \mid c_j + \delta \leq 1\}$. Then when $n \geq m_1$ or $n = m_1 - 1$ but $j \in J_0 - J_1$, from (14), we have

$$\begin{aligned}
\|\overline{W}_j^{(n)}\| &= \|\overline{Y}_j^{(n)} - \overline{Z}_j^{(n)}\| \\
&= \left\| \sum_{i=-\mu}^{\nu} L_i(\delta) (Y_j^{(n-m_1+i)} - Z_j^{(n-m_1+i)}) \right\| \\
&\leq L_0 \sum_{i=-\mu}^{\nu} \|W_j^{(n-m_1+i)}\|,
\end{aligned} \quad (35)$$

where

$$L_0 = \max_{-\mu \leq i \leq \nu} \sup_{x \in [0,1]} |L_i(x)|. \quad (36)$$

Therefore, we obtain that

$$\|\overline{W}_j^{(n)}\|^2 \leq L_0^2 (\mu + \nu + 1) \sum_{i=-\mu}^{\nu} \|W_j^{(n-m_1+i)}\|^2. \quad (37)$$

Otherwise, when $n \leq m_1 - 2$ or $n = m_1 - 1$ but $j \in J_1$, we have

$$\|\overline{W}_j^{(n)}\| \leq \phi_1. \quad (38)$$

Therefore, when $n \leq m_1 - 2$, (34) with (38) gives

$$\begin{aligned}
\|w_{n+1}\|^2 &\leq \|w_0\|^2 + \left(\mu_1 + \frac{h\gamma_2}{1 - \beta_2} \right) \sum_{i=1}^s d_i \sum_{j=0}^n \|W_i^{(j)}\|^2 \\
&+ \frac{h\gamma_1}{1 - \beta_2} \sum_{i=1}^s d_i \sum_{j=0}^n \phi_1^2 + \frac{d\tau_2}{1 - \beta_2} \frac{\beta_2^2}{\gamma_2} \phi_2^2 \\
&\leq \|w_0\|^2 + \left(\mu_1 + \frac{h\gamma_2}{1 - \beta_2} \right) \sum_{i=1}^s d_i \sum_{j=0}^n \|W_i^{(j)}\|^2 \\
&+ \frac{d\tau_1\gamma_1}{1 - \beta_2} \phi_1^2 + \frac{d\tau_2}{1 - \beta_2} \frac{\beta_2^2}{\gamma_2} \phi_2^2.
\end{aligned} \quad (39)$$

When $n = m_1 - 1$, (34) gives

$$\begin{aligned}
&\|w_{m_1}\|^2 \\
&\leq \|w_0\|^2 + \left(\mu_1 + \frac{h\gamma_2}{1 - \beta_2} \right) \sum_{i=1}^s d_i \sum_{j=0}^{m_1-1} \|W_i^{(j)}\|^2 \\
&+ \frac{h\gamma_1}{1 - \beta_2} \sum_{i=1}^s d_i \sum_{j=0}^{m_1-1} \|\overline{W}_i^{(j)}\|^2 + \frac{d\tau_2}{1 - \beta_2} \frac{\beta_2^2}{\gamma_2} \phi_2^2 \\
&\leq \|w_0\|^2 + \left(\mu_1 + \frac{h\gamma_2}{1 - \beta_2} \right) \sum_{i=1}^s d_i \sum_{j=0}^{m_1-1} \|W_i^{(j)}\|^2 \\
&+ \frac{2d\tau_1\gamma_1}{1 - \beta_2} \phi_1^2 + \frac{d\tau_2}{1 - \beta_2} \frac{\beta_2^2}{\gamma_2} \phi_2^2 \\
&+ \frac{h\gamma_1}{1 - \beta_2} \sum_{i \in J_0 - J_1} d_i \|\overline{W}_i^{(m_1-1)}\|^2 \\
&\leq \|w_0\|^2 + \left(\mu_1 + \frac{h\gamma_2}{1 - \beta_2} \right) \sum_{i=1}^s d_i \sum_{j=0}^{m_1-1} \|W_i^{(j)}\|^2 \\
&+ \frac{2d\tau_1\gamma_1}{1 - \beta_2} \phi_1^2 + \frac{d\tau_2}{1 - \beta_2} \frac{\beta_2^2}{\gamma_2} \phi_2^2 \\
&+ L_0^2 (\mu + \nu + 1) \frac{h\gamma_1}{1 - \beta_2} \sum_{i \in J_0 - J_1} d_i \sum_{j=-\mu}^{\nu} \|W_i^{(-1+j)}\|^2 \\
&\leq \|w_0\|^2 + \left(\mu_1 + \frac{h\gamma_2}{1 - \beta_2} \right) \sum_{i=1}^s d_i \sum_{j=0}^{m_1-1} \|W_i^{(j)}\|^2 \\
&+ \frac{2d\tau_1\gamma_1}{1 - \beta_2} \phi_1^2 + \frac{d\tau_2}{1 - \beta_2} \frac{\beta_2^2}{\gamma_2} \phi_2^2 \\
&+ L_0^2 (\mu + \nu + 1) \frac{h\gamma_1}{1 - \beta_2} \sum_{i=1}^s d_i \sum_{j=-\mu}^{\nu} \|W_i^{(-1+j)}\|^2.
\end{aligned} \quad (40)$$

When $n \geq m_1$, from (34) and (37) we have

$$\begin{aligned}
\|w_{n+1}\|^2 &\leq \|w_0\|^2 + \left(\mu_1 + \frac{h\gamma_2}{1 - \beta_2} \right) \sum_{i=1}^s d_i \sum_{j=0}^n \|W_i^{(j)}\|^2 \\
&+ \frac{2d\tau_1\gamma_1}{1 - \beta_2} \phi_1^2 + \frac{d\tau_2}{1 - \beta_2} \frac{\beta_2^2}{\gamma_2} \phi_2^2 + L_0^2 (\mu + \nu + 1) \frac{h\gamma_1}{1 - \beta_2} \\
&\cdot \sum_{k=m_1-1}^n \sum_{i=1}^s d_i \sum_{j=-\mu}^{\nu} \|W_i^{(k-m_1+j)}\|^2 \leq \|w_0\|^2 \\
&+ \left(h \left(2\alpha + \frac{\gamma_1 + L_0^2 (\mu + \nu + 1)^2 \gamma_1 + 2\gamma_2}{1 - \beta_2} \right) - 2l \right) \\
&\cdot \sum_{i=1}^s d_i \sum_{j=0}^n \|W_i^{(j)}\|^2
\end{aligned}$$

$$\begin{aligned}
& + \left(2 + L_0^2 (\mu + \nu + 1) \frac{(\mu + 1)(\mu + 2)}{2} \right) \frac{d\tau_1 \gamma_1}{1 - \beta_2} \phi_1^2 \\
& + \frac{d\tau_2}{1 - \beta_2} \frac{\beta_2^2}{\gamma_2} \phi_2^2.
\end{aligned} \quad (41)$$

By condition (19), a combination of (39), (40), and (41) shows the method is stable, which completes the proof of Theorem 6. $\square$

For the case where the Runge-Kutta method is algebraically stable, from Theorem 6 we can obtain following result.

Corollary 7. Assume that method (12) is algebraically stable and that problems (1) and (7) belong to $\mathbb{D}(\alpha, \beta_1, \beta_2, \gamma_1, \gamma_2)$ with $\alpha < 0$, $\beta_1 \leq \gamma_1$. Then when

$$2\alpha + \frac{\gamma_1 (1 + L_0^2 (\mu + \nu + 1)^2) + 2\gamma_2}{1 - \beta_2} \leq 0, \quad (42)$$

method (13) with (14) and (16) is unconditionally stable for step size h .

Remark 8. It is well known that the Gauss, Radau IA, Radau IIA, and Lobatto IIIC Runge-Kutta methods are all algebraically stable; therefore, from Corollary 7, these methods with (14) and (16) are stable when applied to (1) $\in \mathbb{D}(\alpha, \beta_1, \beta_2, \gamma_1, \gamma_2)$ with $\alpha < 0$, $\beta_1 \leq \gamma_1$ provided (42) satisfied. Paper [24] also shows that the Gauss and Radau methods have some good stability properties for delay equations. This is in harmony with the results in this paper.

Remark 9. When $\tau_1 = \tau_2$ in right-hand side function of problem (1), from Theorem 6 we can obtain the corresponding stability result which is similar to that obtained by Wang in [6], but the result of Theorem 6 is more extensive than that in [6], because, in [6], for processing the delay items, only piecewise linear interpolation is considered, which will lead to higher order method appear order reduction phenomenon for the problem which possesses sufficiently smooth solution.

Remark 10. For the method which has order at most 2, we can choose $\nu = 1$, $\mu = 0$ in (15) to process $\bar{Y}_j^{(n)}$. Here (14) can be reduced to

$$\begin{aligned}
& \bar{Y}_j^{(n)} \\
& = \begin{cases} (1 - \delta) Y_j^{(n-m_1)} + \delta Y_j^{(n-m_1+1)}, & t_n + c_j h - \tau_1 > t_0, \\ \varphi(t_n + c_j h - \tau_1), & t_n + c_j h - \tau_1 \leq t_0, \end{cases} \quad (43) \\
& j = 1, 2, \dots, s,
\end{aligned}$$

which is widely used by many authors for numerically solving differential equations with delay term, such as [6, 8, 25]. Theorem 6 and Corollary 7 are suitable for this situation certainly. Notable is, in this case, the conclusion of the theorem can be more perfect, if we deduce it directly based on (43).

3. Numerical Experiment

We consider the following nonlinear NDDE:

$$\begin{aligned}
y_1'(t) &= -6y_1(t) + 0.1 \sin y_2(t-1) \sin y_1'(t-0.33) \\
&+ \phi_1(t), \quad t \geq 0, \\
y_2'(t) &= -4y_2(t) - 0.2 \cos y_1(t-1) \cos y_2'(t-0.33) \\
&+ \phi_2(t), \quad t \geq 0, \\
y_1(t) &= \sin 5t, \\
y_2(t) &= \cos 5t,
\end{aligned} \quad (44)$$

$$-1 \leq t \leq 0,$$

where

$$\begin{aligned}
\phi_1(t) &= 5 \cos 5t + 6 \sin 5t \\
&- 0.1 \sin(\cos 5(t-1)) \sin(5 \cos 5(t-0.33)), \\
\phi_2(t) &= -5 \sin 5t + 4 \cos 5t \\
&+ 0.2 \cos(\sin 5(t-1)) \cos(5 \sin 5(t-0.33)).
\end{aligned} \quad (45)$$

Let $\alpha = -4$, $\beta_1 = 0.2$, $\beta_2 = 0.2$, $\gamma_1 = 0.2$, and $\gamma_2 = 1.2$; then problem (44) belongs to $\mathbb{D}(\alpha, \beta_1, \beta_2, \gamma_1, \gamma_2)$ and condition (9) is satisfied, where the inner product is standard inner product. Theorem 3 implies that system (44) is stable.

In order to solve problem (44), we consider the 2-stage Radau IIA Runge-Kutta method:

$$\begin{aligned}
Y_1^{(n)} &= y_n + \frac{5}{12} h f\left(t_n + \frac{1}{3} h, Y_1^{(n)}\right) \\
&- \frac{1}{12} h f\left(t_n + h, Y_2^{(n)}\right), \\
Y_2^{(n)} &= y_n + \frac{3}{4} h f\left(t_n + \frac{1}{3} h, Y_1^{(n)}\right) \\
&+ \frac{1}{4} h f\left(t_n + h, Y_2^{(n)}\right), \\
y_{n+1} &= Y_2^{(n)}.
\end{aligned} \quad (46)$$

Because method (46) is A-stable, then from Theorem 6 the numerical solutions will preserve the stability of problem (44) itself. Let $h = 0.033$ and $h = 0.0033$; the numerical solutions y_{1n} , y_{2n} and z_{1n} , z_{2n} are computed, and the errors $\|y_n - z_n\|$ are shown in Figures 1 and 2, respectively, where $y_n = [y_{1n}, y_{2n}]^T$ and $z_n = [z_{1n}, z_{2n}]^T$ are given by applying (46) to (44) and its perturbed problem, which takes initial functions

$$\begin{aligned}
y_1(t) &= \sin(6t) + 0.1, \\
y_2(t) &= \cos(4t) - 0.2,
\end{aligned} \quad (47)$$

$$-1 \leq t \leq 0.$$

These results further confirm our conclusions.

FIGURE 1: The norm of difference $\|y_n - z_n\|$ with step size $h = 0.033$.

FIGURE 2: The norm of difference $\|y_n - z_n\|$ with step size $h = 0.0033$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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