Research Article

Common Fixed Points of Generalized Cocyclic Mappings in Complex Valued Metric Spaces

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We present fixed point results of mappings satisfying generalized contractive conditions in complex valued metric spaces. As an application, we obtain a common fixed point of a pair of weakly compatible mappings. Some common fixed point results of generalized contractive-type mappings involved in cocyclic representation of a nonempty subset of a complex valued metric space are also obtained. Some examples are also presented to support the results proved herein. These results extend and generalize many results in the existing literature.

1. Introduction and Preliminaries

Fixed point theory is one of the famous and traditional theories in mathematics that provide important and traditional tools for proving the existence of solutions of many problems in both applied and pure mathematics. In metric fixed point theory, the interplay between contractive condition and the existence and uniqueness of a fixed point has been very strong and fruitful. There are many applications to the study of fixed points of mappings which satisfy contractive conditions related to the solutions of differential and integral equations [1–3]. Fixed point results for mappings inspired and motivated the development of many other important kinds of points like equilibrium points, coincidence points, periodic points, sectional points, intersection points, and so forth.

Banach's fixed point theorem is one of the basic and most widely applied fixed point theorems in mathematical analysis. Kirk et al. [4] established fixed point theorems for mappings satisfying cyclical contractive conditions. The cyclic contraction mappings need not be continuous, while the Banach contraction mappings are continuous mappings. Păcurar and Rus [5] proved some fixed point results for cyclic weakly contractive mappings. Karapinar [6] obtained fixed point

results for cyclic weakly contractive mappings. Derafshpour and Rezapour [7] proved some results on the existence of best proximity points of cyclic contractions. Recently, Abbas et al. [8] established some common fixed point results of rational-type cocyclic mappings in the setup of multiplicative metric spaces. For further interesting results in this direction, we refer readers to [9–17]. There are many generalizations of metric spaces such as partial metric spaces, generalized metric spaces, cone metric spaces, and quasimetric spaces. Recently, Azam et al. [18] obtained the generalization of Banach's contraction principle by introducing the concept of complex valued metric space and established some common fixed point theorems for mappings involving rational expressions which are not meaningful in cone metric spaces. Later, some other useful results appeared in [19–24]. Abbas et al. [25] obtained the well-posedness and periodic point property of mappings satisfying a rational inequality in an ordered complex valued metric space. Recently, Nashine et al. [26] proved common fixed point theorems under rational contractions in complex valued metric spaces.

In this paper, we establish fixed point result of a generalized contractive mapping in the setup of complex valued metric spaces. Further, by employing this result, a common fixed

point of a pair of weakly compatible mappings is obtained. We study the sufficient conditions for the existence of common fixed points of pair of rational contractive-type mappings involved in cocyclic representation of a nonempty subset of a complex valued metric space. Our results generalize and extend comparable results.

By $\mathbb{R}, \mathbb{R}^+, \mathbb{C}, \mathbb{C}^+$, and $\mathbb{N}$ we denote the set of all real numbers, the set of all nonnegative real numbers, the set of all complex numbers, the set of all nonnegative complex numbers, and the set of all natural numbers, respectively.

Consistent with Azam et al. [18], the following definitions and results will be needed in the sequel.

Let $z_1, z_2 \in \mathbb{C}$. Define a partial order $\leq$ on $\mathbb{C}$ as follows:

$z_1 \leq z_2$ if and only if $\operatorname{Re}(z_1) \leq \operatorname{Re}(z_2)$, $\operatorname{Im}(z_1) \leq \operatorname{Im}(z_2)$.

It follows that $z_1 \leq z_2$ if one of the following conditions is satisfied:

- (1) $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$, $\operatorname{Im}(z_1) < \operatorname{Im}(z_2)$.
- (2) $\operatorname{Re}(z_1) < \operatorname{Re}(z_2)$, $\operatorname{Im}(z_1) = \operatorname{Im}(z_2)$.
- (3) $\operatorname{Re}(z_1) < \operatorname{Re}(z_2)$, $\operatorname{Im}(z_1) < \operatorname{Im}(z_2)$.
- (4) $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$, $\operatorname{Im}(z_1) = \operatorname{Im}(z_2)$.

In particular, we will write $z_1 \leq z_2$ if one of (1), (2), and (3) is satisfied and we will write $z_1 < z_2$ if only (3) is satisfied.

The partial order $\leq$ on $\mathbb{C}$ satisfies the following properties:

- (i) If $\theta \leq z_1 \leq z_2$, then $|z_1| \leq |z_2|$, where θ is the zero of complex number.
- (ii) $z_1 \leq z_2$ is equivalent to $z_1 - z_2 \leq \theta$.
- (iii) If $z_1 \leq z_2$ and $r \geq 0$ is a real number, then $rz_1 \leq rz_2$.
- (iv) If $\theta \leq z_1$ and $\theta \leq z_2$ with $z_1 + z_2 \neq \theta$, then $z_1^2/(z_1 + z_2) \leq z_1$.
- (v) $\theta \leq z_1$ and $\theta \leq z_2$ do not imply $\theta \leq z_1 z_2$.
- (vi) $\theta \leq z_1$ does not imply $\theta \leq 1/z_1$. Moreover, if $\theta \leq z_1$ and $\theta \leq 1/z_1$, then $\operatorname{Im}(z_1) = 0$.

Definition 1. Let X be a nonempty set. Suppose that the mapping $d : X \times X \rightarrow \mathbb{C}$ satisfies the conditions

- (1) $\theta \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
- (2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a *complex valued metric* on X and (X, d) is called a complex valued metric space.

A point $x \in X$ is called an *interior point* of a set $A \subseteq X$ whenever there exists $\theta < r \in \mathbb{C}$ such that $B(x, r) = \{y \in X : d(x, y) < r\} \subseteq A$. A subset A in X is called *open* whenever each point of A is an interior point of A . The family $F = \{B(x, r) : x \in X, \theta < r\}$ is a subbasis for a Hausdorff topology τ on X .

A point $x \in X$ is called a *limit point* of A whenever, for every $\theta < r \in \mathbb{C}$, $B(x, r) \cap (A \setminus x) \neq \emptyset$. A subset $B \subseteq X$ is called *closed* whenever each limit point of B belongs to B .

Let $\{x_n\}$ be a sequence in X and $x \in X$. If, for every $c \in \mathbb{C}$, with $\theta < c$ there is $n_0 \in \mathbb{N}$ such that, for all $n > n_0$, $d(x_n, x) < c$, then x is called the *limit* of $\{x_n\}$ and we write $\lim_{n \rightarrow \infty} x_n = x$ or $x_n \rightarrow x$ as $n \rightarrow \infty$. If for every $c \in \mathbb{C}$, with $\theta < c$, there is $n_0 \in \mathbb{N}$ such that, for all $n > n_0$, $d(x_n, x_{n+m}) < c$, then $\{x_n\}$ is called a *Cauchy sequence* in (X, d) . If every Cauchy sequence is convergent in (X, d) , then (X, d) is called a *complete complex valued metric space*.

Lemma 2 (see [18]). *Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \rightarrow 0$ as $n \rightarrow \infty$.*

Lemma 3. *Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ is a Cauchy sequence if and only if $\sup_{m \geq 1} |d(x_n, x_{n+m})| \rightarrow 0$ as $n \rightarrow \infty$.*

Recall that if f and g are two self-maps on a set X and $u = fx = gx$ for some x in X , then x is called a *coincidence point* of f and g and u is called a *point of coincidence* of f and g .

Definition 4 (see [27]). Two self-maps, f and g , on a nonempty set X are called weakly compatible if they commute at their coincidence point.

We will also need the following proposition from [27].

Proposition 5. *Let f and g be weakly compatible self-maps on a set X . If f and g have a unique point of coincidence— $w = fx = gx$ —then w is the unique common fixed point of f and g .*

Definition 6 (see [12]). Let $\{X_i : i = 1, 2, \dots, p\}$ be a finite collection of nonempty subsets of a set X , where p is some positive integer and $f, g : X \rightarrow X$. The set X is said to have a cocyclic representation with respect to the collection $\{X_i : i = 1, 2, \dots, p\}$ and a pair (f, g) if

- (1) $X = \cup_{i=1}^p X_i$,
- (2) $f(X_1) \subseteq g(X_2), \dots, f(X_{p-1}) \subseteq g(X_p), f(X_p) \subseteq g(X_1)$.

Recall that a function $f : \mathbb{C} \rightarrow \mathbb{C}$ is said to be lower semicontinuous at z_0 in $\mathbb{C}$ if for every $\varepsilon > \theta$ there exists a neighborhood U of z_0 such that $f(z) \geq f(z_0) - \varepsilon$ for all z in U . This can be expressed as $\liminf_{z \rightarrow z_0} f(z) \geq f(z_0)$. Also f is upper semicontinuous at z_0 in $\mathbb{C}$ if for every $\varepsilon > \theta$ there exists a neighborhood U of z_0 such that $f(z) \leq f(z_0) + \varepsilon$ for all z in U . This can be expressed as $\limsup_{z \rightarrow z_0} f(z) \leq f(z_0)$.

Definition 7. The control functions ψ and ϕ are defined as follows:

- (i) $\psi : \mathbb{C}^+ \rightarrow \mathbb{C}^+$ is a continuous nondecreasing function with $\psi(z) = \theta$ if and only if $z = \theta$.
- (ii) $\phi : \mathbb{C}^+ \rightarrow \mathbb{C}^+$ is a lower semicontinuous function with $\phi(z) = \theta$ if and only if $z = \theta$.

2. Main Results

Now, we prove fixed and common fixed point theorems in the setup of complex valued metric space. Our first result is the following.

Theorem 8. Let (X, d) be a complete complex valued metric space and $f : X \rightarrow X$. Suppose that there exist control functions ψ and φ such that, for $x, y \in X$,

$$\psi(d(fx, fy)) \leq \psi(U_f(x, y)) - \varphi(U_f(x, y)) \quad (1)$$

holds, where

$$U_f(x, y) \in \left\{ d(x, y), \frac{d(fx, x)d(fy, y)}{1 + d(x, y)}, \frac{d(fx, y)d(fy, y)}{1 + d(x, y)}, \frac{d(fx, y)d(fy, x)}{1 + d(x, y)} \right\}. \quad (2)$$

Then f has almost one fixed point.

Proof. Let x_0 be an arbitrary element in X . Construct a sequence $\{x_n\}$ in X as $x_{n+1} = fx_n$ or equivalently as $x_n = f^n x_0$ for $n \in \mathbb{N}$. If, for some $n \in \mathbb{N}$, $x_n = x_{n+1}$ holds, then we have $fx_n = x_{n+1} = x_n$ and x_n is the fixed point of f . So we assume that $x_{n+1} \neq x_n$, for all $n \in \mathbb{N}$; that is, $d(x_{n+1}, x_n) > \theta$ for all $n \in \mathbb{N}$. From (1), we have

$$\begin{aligned} \psi(d(x_{n+1}, x_{n+2})) &= \psi(d(fx_n, fx_{n+1})) \\ &\leq \psi(U_f(x_n, x_{n+1})) \\ &\quad - \varphi(U_f(x_n, x_{n+1})), \end{aligned} \quad (3)$$

where

$$\begin{aligned} U_f(x_n, x_{n+1}) &\in \left\{ d(x_n, x_{n+1}), \right. \\ &\quad \frac{d(fx_n, x_n)d(fx_{n+1}, x_{n+1})}{1 + d(x_n, x_{n+1})}, \\ &\quad \frac{d(fx_n, x_{n+1})d(fx_{n+1}, x_{n+1})}{1 + d(x_n, x_{n+1})}, \\ &\quad \left. \frac{d(fx_n, x_{n+1})d(fx_{n+1}, x_n)}{1 + d(x_n, x_{n+1})} \right\} = \left\{ d(x_n, x_{n+1}), \right. \\ &\quad \frac{d(x_{n+1}, x_n)d(x_{n+2}, x_{n+1})}{1 + d(x_n, x_{n+1})}, \\ &\quad \frac{d(x_{n+1}, x_{n+1})d(x_{n+2}, x_{n+1})}{1 + d(x_n, x_{n+1})}, \\ &\quad \left. \frac{d(x_{n+1}, x_{n+1})d(x_{n+2}, x_n)}{1 + d(x_{n+1}, x_n)} \right\} = \left\{ d(x_n, x_{n+1}), \right. \\ &\quad \left. \frac{d(x_{n+1}, x_n)d(x_{n+2}, x_{n+1})}{1 + d(x_n, x_{n+1})}, \theta \right\}. \end{aligned} \quad (4)$$

If $U_f(x_n, x_{n+1}) = d(x_n, x_{n+1})$, then

$$\begin{aligned} \psi(d(x_{n+1}, x_{n+2})) &\leq \psi(d(x_n, x_{n+1})) - \varphi(d(x_n, x_{n+1})) \\ &\leq \psi(d(x_n, x_{n+1})), \end{aligned} \quad (5)$$

and, by the property of ψ , we obtain $d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1})$.

In case $U_f(x_n, x_{n+1}) = d(x_{n+1}, x_n)d(x_{n+2}, x_{n+1})/(1 + d(x_n, x_{n+1}))$, then

$$\begin{aligned} &\psi(d(x_{n+1}, x_{n+2})) \\ &\leq \psi\left(\frac{d(x_{n+1}, x_n)d(x_{n+2}, x_{n+1})}{1 + d(x_n, x_{n+1})}\right) \\ &\quad - \varphi\left(\frac{d(x_{n+1}, x_n)d(x_{n+2}, x_{n+1})}{1 + d(x_n, x_{n+1})}\right) \\ &\leq \psi(d(x_{n+2}, x_{n+1})) \\ &\quad - \varphi\left(\frac{d(x_{n+1}, x_n)d(x_{n+2}, x_{n+1})}{1 + d(x_n, x_{n+1})}\right) \end{aligned} \quad (6)$$

implies $\varphi(d(x_{n+1}, x_n)d(x_{n+2}, x_{n+1})/(1 + d(x_n, x_{n+1}))) \leq \theta$, a contradiction.

Finally if $U_f(x_n, x_{n+1}) = \theta$, then $\psi(d(x_{n+1}, x_{n+2})) \leq \theta$ gives a contradiction.

Thus

$$d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1}) \quad (7)$$

for all $n \in \mathbb{N}$ and $\{d(x_n, x_{n+1})\}$ is the decreasing of positive real numbers. Consequently, there exists $\theta \leq c \in \mathbb{C}^+$ such that $\{d(x_n, x_{n+1})\}$ converges to c . Suppose that $c > \theta$. Now

$$\begin{aligned} \psi(d(x_{n+1}, x_{n+2})) &\leq \psi(U_f(x_n, x_{n+1})) \\ &\quad - \varphi(U_f(x_n, x_{n+1})), \end{aligned} \quad (8)$$

where $U_f(x_n, x_{n+1}) \in \{d(x_n, x_{n+1}), d(x_{n+1}, x_n)d(x_{n+2}, x_{n+1})/(1 + d(x_n, x_{n+1})), \theta\}$.

If $U_f(x_n, x_{n+1}) = d(x_n, x_{n+1})$, then lower semicontinuity of φ gives that

$$\begin{aligned} &\limsup_{n \rightarrow \infty} \psi(d(x_{n+1}, x_{n+2})) \\ &\leq \limsup_{n \rightarrow \infty} \psi(d(x_n, x_{n+1})) \\ &\quad - \liminf_{n \rightarrow \infty} \varphi(d(x_n, x_{n+1})), \end{aligned} \quad (9)$$

which implies $\psi(c) \leq \psi(c) - \varphi(c) < \psi(c)$, a contradiction as $\varphi(c) > \theta$.

In case $U_f(x_n, x_{n+1}) = d(x_{n+1}, x_n)d(x_{n+2}, x_{n+1})/(1 + d(x_n, x_{n+1}))$, then we obtain

$$\begin{aligned} &\limsup_{n \rightarrow \infty} \psi(d(x_{n+1}, x_{n+2})) \\ &\leq \limsup_{n \rightarrow \infty} \psi\left(\frac{d(x_{n+1}, x_n)d(x_{n+2}, x_{n+1})}{1 + d(x_n, x_{n+1})}\right) \\ &\quad - \liminf_{n \rightarrow \infty} \varphi\left(\frac{d(x_{n+1}, x_n)d(x_{n+2}, x_{n+1})}{1 + d(x_n, x_{n+1})}\right), \end{aligned} \quad (10)$$

which implies that $\psi(c) \leq \psi(c^2/(1+c)) - \varphi(c^2/(1+c)) < \psi(c^2/(1+c))$, and so $c \leq (c^2/(1+c))$ which implies $1+c \leq c$, a contradiction.

If $U_f(x_n, x_{n+1}) = \theta$, then we have $\limsup_{n \rightarrow \infty} \psi(d(x_{n+1}, x_{n+2})) \leq \theta$, a contradiction.

Therefore $c = \theta$; that is, $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = \theta$.

Now, we claim that $\lim_{n,m \rightarrow \infty} d(x_n, x_m) = \theta$. If not, then there exist $\theta < \varepsilon \in \mathbb{C}^+$ and sequences $\{n_k\}, \{m_k\}$ in $\mathbb{N}$, with $n_k > m_k \geq k$, such that $d(x_{n_k}, x_{m_k}) \geq \varepsilon$ for all $k \in \mathbb{N}$. Without any loss of generality, we can assume that $d(x_{n_k}, x_{m_{k-1}}) < \varepsilon$. Since $\{d(x_{m_{k-1}}, x_{m_k})\}$ is a subsequence of convergent sequence $\{d(x_{n-1}, x_n)\} \rightarrow \theta$ as $n \rightarrow \infty$, therefore, $\{d(x_{m_{k-1}}, x_{m_k})\} \rightarrow \theta$ as $k \rightarrow \infty$. Now

$$\varepsilon \leq d(x_{m_k}, x_{n_k}) \leq d(x_{n_k}, x_{m_{k-1}}) + d(x_{m_{k-1}}, x_{m_k}) \quad (11)$$

implies that

$$\lim_{k \rightarrow \infty} d(x_{m_k}, x_{n_k}) = \varepsilon. \quad (12)$$

From (12) and inequality $d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{m_{k-1}}) + d(x_{m_{k-1}}, x_{n_k})$, it follows that $\varepsilon \leq \lim_{k \rightarrow \infty} d(x_{m_{k-1}}, x_{n_k})$. Also, the inequality $d(x_{m_{k-1}}, x_{n_k}) \leq d(x_{m_{k-1}}, x_{m_k}) + d(x_{m_k}, x_{n_k})$ and (12) give that $\lim_{k \rightarrow \infty} d(x_{m_{k-1}}, x_{n_k}) \leq \varepsilon$, and hence we have

$$\lim_{k \rightarrow \infty} d(x_{m_{k-1}}, x_{n_k}) = \varepsilon. \quad (13)$$

From (13) and inequality $d(x_{m_{k-1}}, x_{n_k}) \leq d(x_{m_{k-1}}, x_{n_{k+1}}) + d(x_{n_k}, x_{n_{k+1}})$, it is implied that $\varepsilon \leq \lim_{k \rightarrow \infty} d(x_{m_{k-1}}, x_{n_{k+1}})$, while inequality $d(x_{m_{k-1}}, x_{n_{k+1}}) \leq d(x_{m_{k-1}}, x_{n_k}) + d(x_{n_k}, x_{n_{k+1}})$ and (13) imply that $\lim_{k \rightarrow \infty} d(x_{m_{k-1}}, x_{n_{k+1}}) \leq \varepsilon$, and hence we have

$$\lim_{k \rightarrow \infty} d(x_{m_{k-1}}, x_{n_{k+1}}) = \varepsilon. \quad (14)$$

From (12) and $d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{n_{k+1}}) + d(x_{n_k}, x_{n_{k+1}})$, we have $\varepsilon \leq \lim_{k \rightarrow \infty} d(x_{m_k}, x_{n_{k+1}})$, and the inequality $d(x_{m_k}, x_{n_{k+1}}) \leq d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_{k+1}})$ and (12) give that $\lim_{k \rightarrow \infty} d(x_{m_k}, x_{n_{k+1}}) \leq \varepsilon$. So

$$\lim_{k \rightarrow \infty} d(x_{m_k}, x_{n_{k+1}}) = \varepsilon. \quad (15)$$

As

$$\begin{aligned} M_f(x_{n_k}, x_{m_{k-1}}) &\in \left\{ d(x_{n_k}, x_{m_{k-1}}), \right. \\ &\frac{d(fx_{n_k}, x_{n_k})d(fx_{m_{k-1}}, x_{m_{k-1}})}{1+d(x_{n_k}, x_{m_{k-1}})}, \\ &\frac{d(fx_{n_k}, x_{m_{k-1}})d(fx_{m_{k-1}}, x_{m_{k-1}})}{1+d(x_{n_k}, x_{m_{k-1}})}, \end{aligned}$$

$$\begin{aligned} &\frac{d(fx_{n_k}, x_{m_{k-1}})d(fx_{m_{k-1}}, x_{n_k})}{1+d(x_{n_k}, x_{m_{k-1}})} \Big\} \\ &= \left\{ d(x_{n_k}, x_{m_{k-1}}), \frac{d(x_{n_{k+1}}, x_{n_k})d(x_{m_k}, x_{m_{k-1}})}{1+d(x_{n_k}, x_{m_{k-1}})}, \right. \\ &\frac{d(x_{n_{k+1}}, x_{m_{k-1}})d(x_{m_k}, x_{m_{k-1}})}{1+d(x_{n_k}, x_{m_{k-1}})}, \\ &\left. \frac{d(x_{n_{k+1}}, x_{m_{k-1}})d(x_{m_k}, x_{n_k})}{1+d(x_{n_k}, x_{m_{k-1}})} \right\}, \end{aligned} \quad (16)$$

so we have

$$\lim_{k \rightarrow \infty} U_f(x_{n_k}, x_{m_{k-1}}) \in \left\{ \varepsilon, \theta, \theta, \frac{\varepsilon^2}{1+\varepsilon} \right\}. \quad (17)$$

Now, if $\lim_{k \rightarrow \infty} U_f(x_{n_k}, x_{m_{k-1}}) = \varepsilon$, then, from (1), it follows that

$$\begin{aligned} \psi(d(x_{n_{k+1}}, x_{m_k})) &= \psi(d(fx_{n_k}, fx_{m_{k-1}})) \\ &\leq \psi(U_f(x_{n_k}, x_{m_{k-1}})) \\ &\quad - \varphi(U_f(x_{n_k}, x_{m_{k-1}})) \end{aligned} \quad (18)$$

and taking upper limit as $k \rightarrow \infty$ implies that $\psi(\varepsilon) \leq \psi(\varepsilon) - \varphi(\varepsilon) < \psi(\varepsilon)$, a contradiction as $\varepsilon > \theta$.

In case $\lim_{k \rightarrow \infty} U_f(x_{n_k}, x_{m_{k-1}}) = \theta$, then, from (1), it follows that $\limsup_{k \rightarrow \infty} \psi(d(x_{n_{k+1}}, x_{m_k})) \leq \theta$, a contradiction.

When $\lim_{k \rightarrow \infty} U_f(x_{n_k}, x_{m_{k-1}}) = \varepsilon^2/(1+\varepsilon)$, then, from (1), it follows that

$$\begin{aligned} \psi(\varepsilon) &= \limsup_{k \rightarrow \infty} \psi(d(x_{n_{k+1}}, x_{m_k})) \\ &= \limsup_{k \rightarrow \infty} \psi(d(fx_{n_k}, fx_{m_{k-1}})) \\ &\leq \limsup_{k \rightarrow \infty} \psi(U_f(x_{n_k}, x_{m_{k-1}})) \\ &\quad - \liminf_{k \rightarrow \infty} \varphi(U_f(x_{n_k}, x_{m_{k-1}})) \\ &= \psi\left(\frac{\varepsilon^2}{1+\varepsilon}\right) - \varphi\left(\frac{\varepsilon^2}{1+\varepsilon}\right) \leq \psi\left(\frac{\varepsilon^2}{1+\varepsilon}\right), \end{aligned} \quad (19)$$

which implies $\varepsilon \leq \varepsilon^2/(1+\varepsilon)$; that is, $1+\varepsilon \leq \varepsilon$, a contradiction.

Thus, we obtain that $\lim_{n,m \rightarrow \infty} d(x_n, x_m) = \theta$, and hence $\{x_n\}$ is a Cauchy sequence in the complex valued metric space (X, d) .

Next, we assume there exists a point $u \in X$ such that $\lim_{n \rightarrow \infty} d(u, x_n) = \theta$.

Thus,

$$\begin{aligned} U_f(u, x_n) &\in \left\{ d(u, x_n), \frac{d(fu, u)d(fx_n, x_n)}{1 + d(u, x_n)}, \right. \\ &\quad \left. \frac{d(fu, x_n)d(fx_n, x_n)}{1 + d(u, x_n)}, \frac{d(fu, x_n)d(fx_n, u)}{1 + d(u, x_n)} \right\} \\ &= \left\{ d(u, x_n), \frac{d(fu, u)d(x_{n+1}, x_n)}{1 + d(u, x_n)}, \right. \\ &\quad \left. \frac{d(fu, x_n)d(x_{n+1}, x_n)}{1 + d(u, x_n)}, \frac{d(fu, x_n)d(x_{n+1}, u)}{1 + d(u, x_n)} \right\}. \end{aligned} \quad (20)$$

Taking limit as $n \rightarrow \infty$, we conclude that $\lim_{n \rightarrow \infty} M_f(u, x_n) = \theta$. Hence

$$\psi(d(fu, fx_{n+1})) \leq \psi(U_f(u, x_n)) - \varphi(U_f(u, x_n)) \quad (21)$$

and taking upper limit as $n \rightarrow \infty$ implies that

$$\psi(d(fu, u)) \leq \theta, \quad (22)$$

which further implies that $fu = u$.

To prove the uniqueness of fixed point of f , assume on the contrary that $fv = v$ and $fw = w$ with $v \neq w$. Note that

$$\begin{aligned} \psi(d(v, w)) &= \psi(d(fv, fw)) \\ &\leq \psi(U_f(v, w)) - \varphi(U_f(v, w)), \end{aligned} \quad (23)$$

where

$$\begin{aligned} U_f(v, w) &\in \left\{ d(v, w), \frac{d(fv, v)d(fw, w)}{1 + d(v, w)}, \right. \\ &\quad \left. \frac{d(fv, w)d(fw, w)}{1 + d(v, w)}, \frac{d(fv, w)d(fw, v)}{1 + d(v, w)} \right\} \\ &= \left\{ d(v, w), \frac{d(v, v)d(w, w)}{1 + d(v, w)}, \frac{d(v, w)d(w, w)}{1 + d(v, w)}, \right. \\ &\quad \left. \frac{d(v, w)d(w, v)}{1 + d(v, w)} \right\} = \left\{ d(v, w), \theta, \theta, \frac{d(v, w)^2}{1 + d(v, w)} \right\}. \end{aligned} \quad (24)$$

Now, if $U_f(v, w) = d(v, w)$, then from (23) it follows that

$$\begin{aligned} \psi(d(v, w)) &= \psi(d(fv, fw)) \\ &\leq \psi(d(v, w)) - \varphi(d(v, w)) \\ &< \psi(d(v, w)), \end{aligned} \quad (25)$$

a contradiction as $d(v, w) > \theta$. Hence $v = w$.

In case $U_f(v, w) = \theta$, then (23) gives a contradiction.

Finally, if $U_f(v, w) = d(v, w)^2/(1 + d(v, w))$, then, from (23),

$$\begin{aligned} \psi(d(v, w)) &\leq \psi\left(\frac{d(v, w)^2}{1 + d(v, w)}\right) - \varphi\left(\frac{d(v, w)^2}{1 + d(v, w)}\right) \\ &< \psi\left(\frac{d(v, w)^2}{1 + d(v, w)}\right), \end{aligned} \quad (26)$$

which gives $1 + d(v, w) \leq d(v, w)$, a contradiction. Hence for all cases we have $v = w$. $\square$

Corollary 9. Let (X, d) be a complete complex valued metric space and $f : X \rightarrow X$. Suppose that there exist control functions ψ and φ such that for $x, y \in X$

$$\psi(d(f^n x, f^n y)) \leq \frac{\psi(U_{f^n}(x, y))}{\varphi(U_{f^n}(x, y))} \quad (27)$$

holds for $n \in \mathbb{N}$, where

$$\begin{aligned} U_{f^n}(x, y) &\in \left\{ d(x, y), \frac{d(f^n x, x)d(f^n y, y)}{1 + d(x, y)}, \right. \\ &\quad \left. \frac{d(f^n x, y)d(f^n y, y)}{1 + d(x, y)}, \frac{d(f^n x, y)d(f^n y, x)}{1 + d(x, y)} \right\}. \end{aligned} \quad (28)$$

Then f has almost one fixed point u in X and $f^n u = fu$.

Proof. Taking $T = f^n$. It follows from Theorem 8 that T has a unique fixed point u . Also $f(u) = f(Tu) = f^{n+1}(u) = f^n(fu) = T(fu)$ implies that fu is also the fixed point of T . From the uniqueness of fixed point of T , it is implied that $fu = u$. $\square$

Recently, Haghi et al. [28] proved the following lemma.

Lemma 10. Let X be a nonempty set and let f be a self-map on X . Then there exists a subset $E \subseteq X$ such that $f : E \rightarrow X$ is one-to-one and $f(E) = f(X)$.

Theorem 11. Let (X, d) be a complex valued metric space and $f, g : X \rightarrow X$. Suppose that there exist control functions ψ and φ such that for any $x, y \in X$

$$\psi(d(fx, fy)) \leq \psi(U(gx, gy)) - \varphi(U(gx, gy)) \quad (29)$$

holds, where

$$\begin{aligned} U(x, y) &\in \left\{ d(gx, gy), \frac{d(fx, gx)d(fy, gy)}{1 + d(gx, gy)}, \right. \\ &\quad \left. \frac{d(fx, gy)d(fy, gy)}{1 + d(gx, gy)}, \frac{d(fx, gy)d(fy, gx)}{1 + d(gx, gy)} \right\}. \end{aligned} \quad (30)$$

If $g(X)$ is a complete subspace of X , then f and g have at most one coincidence point in X . Furthermore, if f and g are weakly compatible, then f and g have a unique common fixed point in X .

Proof. It follows from Lemma 10 that there exists $E \subseteq X$ such that $g : E \rightarrow X$ is one-to-one and $g(E) = g(X)$. Define a map $h : g(E) \rightarrow g(E)$ by $h(gx) = fx$. Since g is one-to-one on E , h is well-defined. Note that

$$\begin{aligned} \psi(d(h(gx), h(gy))) \\ \leq \psi(U(gx, gy)) - \varphi(U(gx, gy)), \end{aligned} \quad (31)$$

for all $gx, gy \in g(E)$, where

$$U(gx, gy) \in \left\{ d(gx, gy), \frac{d(h(gx), gx)d(h(gy), gy)}{1 + d(gx, gy)}, \frac{d(h(gx), gy)d(h(gy), gy)}{1 + d(gx, gy)}, \frac{d(h(gx), gy)d(h(gy), gx)}{1 + d(gx, gy)} \right\}. \quad (32)$$

Since $g(E) = g(X)$ is complete, by Theorem 8, there exists $x^* \in X$ such that $h(gx^*) = gx^* = fx^*$. Hence, f and g have at most one point of coincidence in X . By Proposition 5, f and g have at most one common fixed point. $\square$

3. Cocyclic Contraction Mappings

In this section, we prove common fixed point results for self-maps satisfying cocyclic contractions defined on a complex valued metric space. Our first result is the following.

Theorem 12. Let (X, d) be a complex valued metric space, let $A_1, A_2, \dots, A_p$ be nonempty closed subsets of X , and let $Y = \cup_{i=1}^p A_i$. Suppose that $f, g : Y \rightarrow Y$ are such that

- (a) Y has a cocyclic representation with respect to pair (f, g) and to the collection $\{A_i : i = 1, 2, \dots, p\}$;
- (b) there exist control functions ψ and φ such that, for any $(x, y) \in A_i \times A_{i+1}$, $i = 1, 2, \dots, p$,

$$\psi(d(fx, fy)) \leq \psi(U_{f,g}(x, y)) - \varphi(U_{f,g}(x, y)) \quad (33)$$

holds where

$$U_{f,g}(x, y) \in \left\{ d(gx, gy), \frac{d(fx, gx)d(fy, gy)}{1 + d(gx, gy)}, \frac{d(fx, gy)d(fy, gy)}{1 + d(gx, gy)}, \frac{d(fx, gy)d(fy, gx)}{1 + d(gx, gy)} \right\}, \quad (34)$$

with $A_{p+1} = A_1$.

If $g(A_i)$ is complete subspace of X for each $i \in \{1, 2, \dots, p\}$, then f and g have a unique coincidence point $z \in \cap_{i=1}^p g(A_i) = K$ provided that $f(K) \subseteq g(K) \subseteq K$ and $g(K)$ is closed. Furthermore, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Let $x_0 \in \cup_{i=1}^p A_i$ be a given element. By the given condition, there exists $i_0 \in \{1, 2, \dots, p\}$ such that $x_0 \in A_{i_0}$. Since $f(A_{i_0}) \subseteq g(A_{i_0+1})$, we can choose a point $x_1 \in A_{i_0+1}$ such that $f(x_0) = g(x_1)$. Continuing this process, for $n > 0$, there exists $i_n \in \{1, 2, \dots, p\}$ such that, having chosen $x_n \in A_{i_n}$, we obtain $x_{n+1} \in A_{i_{n+1}}$ such that $f(x_n) = g(x_{n+1})$.

If for some $n_0 \geq 0$, $f(x_{n_0}) = f(x_{n_0+1})$; that is, $f(x_{n_0}) = g(x_{n_0})$ and x_{n_0} is the coincidence point of f and g . Taking $d(f(x_n), f(x_{n+1})) > \theta$ for all n , we have

$$\begin{aligned} \psi(d(gx_{n+1}, gx_{n+2})) &= \psi(d(fx_n, fx_{n+1})) \\ &\leq \psi(U_{f,g}(x_n, x_{n+1})) \\ &\quad - \varphi(U_{f,g}(x_n, x_{n+1})) \\ &\leq \psi(U_{f,g}(x_n, x_{n+1})), \end{aligned} \quad (35)$$

where

$$\begin{aligned} U_{f,g}(x_n, x_{n+1}) &\in \left\{ d(gx_n, gx_{n+1}), \frac{d(fx_n, gx_n)d(fx_{n+1}, gx_{n+1})}{1 + d(gx_n, gx_{n+1})}, \frac{d(fx_n, gx_{n+1})d(fx_{n+1}, gx_{n+1})}{1 + d(gx_n, gx_{n+1})}, \frac{d(fx_n, gx_{n+1})d(fx_{n+1}, gx_n)}{1 + d(gx_n, gx_{n+1})} \right\} = \left\{ d(gx_n, gx_{n+1}), \frac{d(gx_{n+1}, gx_n)d(gx_{n+2}, gx_{n+1})}{1 + d(gx_n, gx_{n+1})}, \frac{d(gx_{n+1}, gx_{n+1})d(gx_{n+2}, gx_{n+1})}{1 + d(gx_n, gx_{n+1})^2}, \frac{d(gx_{n+1}, gx_{n+1})d(gx_{n+2}, gx_n)}{1 + d(gx_n, gx_{n+1})} \right\} \\ &= \left\{ d(gx_n, gx_{n+1}), \frac{d(gx_{n+1}, gx_n)d(gx_{n+2}, gx_{n+1})}{1 + d(gx_n, gx_{n+1})}, \theta, \theta \right\}. \end{aligned} \quad (36)$$

If, for some n , $U_{f,g}(x_n, x_{n+1}) = d(gx_n, gx_{n+1})$, then

$$\begin{aligned} \psi(d(gx_{n+1}, gx_{n+2})) &\leq \psi(d(gx_n, gx_{n+1})) \\ &\quad - \varphi(d(gx_n, gx_{n+1})) \end{aligned} \quad (37)$$

implies that $d(gx_{n+2}, gx_{n+1}) \leq d(gx_n, gx_{n+1})$.

In case $U_{f,g}(x_n, x_{n+1}) = d(gx_{n+1}, gx_n)d(gx_{n+2}, gx_{n+1})/(1 + d(gx_n, gx_{n+1}))$, then

$$\begin{aligned} & \psi(d(gx_{n+1}, gx_{n+2})) \\ & \leq \psi\left(\frac{d(gx_{n+1}, gx_n)d(gx_{n+2}, gx_{n+1})}{1 + d(gx_n, gx_{n+1})}\right) \\ & \quad - \varphi\left(\frac{d(gx_{n+1}, gx_n)d(gx_{n+2}, gx_{n+1})}{1 + d(gx_n, gx_{n+1})}\right) \quad (38) \\ & \leq \psi(d(gx_{n+2}, gx_{n+1})) \\ & \quad - \varphi\left(\frac{d(gx_{n+1}, gx_n)d(gx_{n+2}, gx_{n+1})}{1 + d(gx_n, gx_{n+1})}\right) \end{aligned}$$

implies $\varphi(d(gx_{n+1}, gx_n)d(gx_{n+2}, gx_{n+1})/(1 + d(gx_n, gx_{n+1}))) \leq \theta$, a contradiction.

When $U_{f,g}(x_n, x_{n+1}) = \theta$, then $\psi(d(gx_{n+1}, gx_{n+2})) \leq \theta$ gives a contradiction.

Thus

$$d(gx_{n+1}, gx_{n+2}) \leq d(gx_n, gx_{n+1}) \quad (39)$$

for all $n \in \mathbb{N}$ and $\{d(gx_n, gx_{n+1})\}$ is the decreasing of positive real numbers. Consequently, there exists $\theta \leq \delta \in \mathbb{C}^+$ such that $\{d(x_n, x_{n+1})\}$ converges to δ . Suppose that $\delta > \theta$. Now

$$\begin{aligned} \psi(d(gx_{n+1}, gx_{n+2})) & \leq \psi(U_{f,g}(x_n, x_{n+1})) \\ & \quad - \varphi(U_{f,g}(x_n, x_{n+1})), \end{aligned} \quad (40)$$

where

$$\begin{aligned} U_{f,g}(x_n, x_{n+1}) & \in \left\{ d(gx_n, gx_{n+1}), \right. \\ & \quad \left. \frac{d(gx_{n+1}, gx_n)d(gx_{n+2}, gx_{n+1})}{1 + d(gx_n, gx_{n+1})}, \theta \right\}. \end{aligned} \quad (41)$$

If $U_{f,g}(x_n, x_{n+1}) = d(gx_n, gx_{n+1})$, then lower semicontinuity of φ gives that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \psi(d(gx_{n+1}, gx_{n+2})) \\ & \leq \limsup_{n \rightarrow \infty} \psi(d(gx_n, gx_{n+1})) \quad (42) \\ & \quad - \liminf_{n \rightarrow \infty} \varphi(d(gx_n, gx_{n+1})), \end{aligned}$$

which implies that $\psi(\delta) \leq \psi(\delta) - \varphi(\delta) < \psi(\delta)$, a contradiction as $\varphi(\delta) > \theta$.

In case $U_{f,g}(x_n, x_{n+1}) = d(gx_{n+1}, gx_n)d(gx_{n+2}, gx_{n+1})/(1 + d(gx_n, gx_{n+1}))$, then we obtain

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \psi(d(gx_{n+1}, gx_{n+2})) \\ & \leq \limsup_{n \rightarrow \infty} \psi\left(\frac{d(gx_{n+1}, gx_n)d(gx_{n+2}, gx_{n+1})}{1 + d(gx_n, gx_{n+1})}\right) \quad (43) \\ & \quad - \liminf_{n \rightarrow \infty} \varphi\left(\frac{d(gx_{n+1}, gx_n)d(gx_{n+2}, gx_{n+1})}{1 + d(gx_n, gx_{n+1})}\right), \end{aligned}$$

which implies that $\psi(\delta) \leq \psi(\delta^2/(1 + \delta)) - \varphi(\delta^2/(1 + \delta)) < \psi(\delta^2/(1 + \delta))$, and so $\delta \leq \delta^2/(1 + \delta)$ which implies $1 + \delta \leq \delta$, a contradiction.

If $U_{f,g}(x_n, x_{n+1}) = \theta$, then we have $\psi(\delta) = \limsup_{n \rightarrow \infty} \psi(d(gx_{n+1}, gx_{n+2})) \leq \theta$, a contradiction. Therefore $\delta = \theta$; that is,

$$\lim_{n \rightarrow \infty} d(gx_{n+1}, gx_n) = \theta. \quad (44)$$

Now we claim that $\{gx_n\}$ is a complex valued Cauchy sequence. If not, there are $\varepsilon > \theta$ and even integers n_k and m_k with $m_k > n_k > k$ such that

$$d(gx_{m_k}, gx_{n_k}) \geq \varepsilon \quad (45)$$

and $d(gx_{m_k-1}, gx_{n_k}) < \varepsilon$. Note that

$$\begin{aligned} \varepsilon & \leq d(gx_{m_k}, gx_{n_k}) \\ & \leq d(gx_{n_k}, gx_{m_k-1}) + d(gx_{m_k-1}, gx_{m_k}). \end{aligned} \quad (46)$$

From (44) and (45), it follows that

$$\lim_{k \rightarrow \infty} d(gx_{m_k}, gx_{n_k}) = \varepsilon. \quad (47)$$

By (47) and

$$\begin{aligned} d(gx_{n_k+1}, gx_{m_k}) & \leq d(gx_{n_k+1}, gx_{n_k}) \\ & \quad + d(gx_{n_k}, gx_{m_k}), \end{aligned} \quad (48)$$

we have $\lim_{k \rightarrow \infty} d(gx_{n_k+1}, gx_{m_k}) \leq \varepsilon$. Also, by (47) and

$$\begin{aligned} d(gx_{m_k}, gx_{n_k}) & \leq d(gx_{n_k+1}, gx_{n_k}) \\ & \quad + d(gx_{n_k+1}, gx_{m_k}), \end{aligned} \quad (49)$$

we obtain that $\varepsilon \leq \lim_{k \rightarrow \infty} d(gx_{n_k+1}, gx_{m_k})$. Hence

$$\lim_{k \rightarrow \infty} d(gx_{n_k+1}, gx_{m_k}) = \varepsilon. \quad (50)$$

From (44) and

$$\begin{aligned} d(gx_{n_k+1}, gx_{m_k-1}) & \leq d(gx_{n_k+1}, gx_{n_k}) \\ & \quad + d(gx_{n_k}, gx_{m_k-1}), \end{aligned} \quad (51)$$

we have $\lim_{k \rightarrow \infty} d(gx_{n_k+1}, gx_{m_k-1}) \leq \varepsilon$. By (50) and the inequality

$$\begin{aligned} d(gx_{n_k+1}, gx_{m_k}) & \leq d(gx_{n_k+1}, gx_{m_k-1}) \\ & \quad + d(gx_{m_k-1}, gx_{m_k}), \end{aligned} \quad (52)$$

we have $\varepsilon \leq \lim_{k \rightarrow \infty} d(gx_{n_k+1}, gx_{m_k-1})$. Thus

$$\lim_{k \rightarrow \infty} d(gx_{n_k+1}, gx_{m_k-1}) = \varepsilon. \quad (53)$$

As

$$\begin{aligned}
U_{f,g}(x_{n_k}, x_{m_k-1}) &\in \left\{ d(gx_{n_k}, gx_{m_k-1}), \right. \\
&\frac{d(fx_{n_k}, gx_{n_k})d(fx_{m_k-1}, gx_{m_k-1})}{1 + d(gx_{n_k}, gx_{m_k-1})}, \\
&\frac{d(fx_{n_k}, gx_{m_k-1})d(fx_{m_k-1}, gx_{m_k-1})}{1 + d(gx_{n_k}, gx_{m_k-1})}, \\
&\left. \frac{d(fx_{n_k}, gx_{m_k-1})d(fx_{m_k-1}, gx_{n_k})}{1 + d(gx_{n_k}, gx_{m_k-1})} \right\} \\
&= \left\{ d(gx_{n_k}, gx_{m_k-1}), \right. \\
&\frac{d(gx_{n_k+1}, gx_{n_k})d(gx_{m_k}, gx_{m_k-1})}{1 + d(gx_{n_k}, gx_{m_k-1})}, \\
&\frac{d(gx_{n_k+1}, gx_{m_k-1})d(gx_{m_k}, gx_{m_k-1})}{1 + d(gx_{n_k}, gx_{m_k-1})}, \\
&\left. \frac{d(gx_{n_k+1}, gx_{m_k-1})d(gx_{m_k}, gx_{n_k})}{1 + d(gx_{n_k}, gx_{m_k-1})} \right\},
\end{aligned} \tag{54}$$

so we have $\lim_{k \rightarrow \infty} U_{f,g}(x_{n_k}, x_{m_k-1}) \in \{\varepsilon, \theta, \varepsilon^2/(1+\varepsilon)\}$.

Now, if $\lim_{k \rightarrow \infty} U_{f,g}(x_{n_k}, x_{m_k-1}) = \varepsilon$, then, from (b), it follows that

$$\begin{aligned}
\psi(d(gx_{n_k+1}, gx_{m_k})) &= \psi(d(fx_{n_k}, fx_{m_k-1})) \\
&\leq \psi(M_{f,g}(x_{n_k}, x_{m_k-1})) \\
&\quad - \varphi(M_{f,g}(x_{n_k}, x_{m_k-1}))
\end{aligned} \tag{55}$$

and taking upper limit as $k \rightarrow \infty$ implies that $\psi(\varepsilon) \leq \psi(\varepsilon) - \varphi(\varepsilon) < \psi(\varepsilon)$, a contradiction as $\varepsilon > \theta$.

In case $\lim_{k \rightarrow \infty} M_{f,g}(x_{n_k}, x_{m_k-1}) = \theta$, then, from (b), it follows that $\limsup_{k \rightarrow \infty} \psi(d(gx_{n_k+1}, gx_{m_k})) \leq \theta$, a contradiction.

When $\lim_{k \rightarrow \infty} M_{f,g}(x_{n_k}, x_{m_k-1}) = \varepsilon^2/(1+\varepsilon)$, then, from (b), it follows that

$$\begin{aligned}
\psi(\varepsilon) &= \limsup_{k \rightarrow \infty} \psi(d(gx_{n_k+1}, gx_{m_k})) \\
&= \limsup_{k \rightarrow \infty} \psi(d(fx_{n_k}, fx_{m_k-1})) \\
&\leq \limsup_{k \rightarrow \infty} \psi(U_{f,g}(x_{n_k}, x_{m_k-1})) \\
&\quad - \liminf_{k \rightarrow \infty} \varphi(U_{f,g}(x_{n_k}, x_{m_k-1})) \\
&= \psi\left(\frac{\varepsilon^2}{1+\varepsilon}\right) - \varphi\left(\frac{\varepsilon^2}{1+\varepsilon}\right) \leq \psi\left(\frac{\varepsilon^2}{1+\varepsilon}\right),
\end{aligned} \tag{56}$$

which implies $\varepsilon \leq \varepsilon^2/(1+\varepsilon)$; that is, $1+\varepsilon \leq \varepsilon$, a contradiction.

Hence $\{gx_n\}$ is a Cauchy sequence in $\cup_{i=1}^p g(A_i)$. Since $\cup_{i=1}^p g(A_i)$ is complete, there exists $y \in \cup_{i=1}^p g(A_i)$ such that

$$\lim_{n \rightarrow \infty} d(gx_n, y) = \theta. \tag{57}$$

Consequently, we can find a point z in Y such that $g(z) = y$.

Now we show that $y \in \cap_{i=1}^p g(A_i) = K$. From condition (a) and $x_0 \in A_{i_0}$ for some $i_0 \in \{1, 2, \dots, p\}$, we can choose a subsequence $\{x_{n_k}\}$ in A_{i_0} out of the sequence $\{x_n\}$. Obviously, $\{gx_{n_k}\} \subseteq g(A_{i_0})$. As $g(A_{i_0})$ is closed, so $y \in g(A_{i_0})$. Similarly, we can choose a subsequence $\{x_{n_{k+1}}\}$ in A_{i_0+1} out of the sequence $\{x_n\}$. Obviously, $\{gx_{n_{k+1}}\} \subseteq g(A_{i_0+1})$. As $g(A_{i_0+1})$ is closed, so $y \in g(A_{i_0+1})$. Continuing this way, we obtain that $y \in \cap_{i=1}^p g(A_i)$ and hence $\cap_{i=1}^p g(A_i) \neq \emptyset$.

Now we show that $f(z) = g(z)$. Since $z \in Y$, there exists some i in $\{1, 2, \dots, p\}$ such that $z \in A_i$. Choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $x_{n_k} \in A_{i-1}$. From (b), we have

$$\begin{aligned}
\psi(d(gx_{n_k+1}, fz)) &= \psi(d(fx_{n_k}, fz)) \\
&\leq \psi(U_{f,g}(x_{n_k}, z)) \\
&\quad - \varphi(U_{f,g}(x_{n_k}, z)),
\end{aligned} \tag{58}$$

where

$$\begin{aligned}
U_{f,g}(x_{n_k}, z) &\in \left\{ d(gx_{n_k}, gz), \right. \\
&\frac{d(fx_{n_k}, gx_{n_k})d(fz, gz)}{1 + d(gx_{n_k}, gz)}, \frac{d(fx_{n_k}, gz)d(fz, gz)}{1 + d(gx_{n_k}, gz)}, \\
&\frac{d(fx_{n_k}, gz)d(fz, gx_{n_k})}{1 + d(gx_{n_k}, gz)} \left. \right\} = \left\{ d(gx_{n_k}, gz), \right. \\
&\frac{d(gx_{n_k+1}, gx_{n_k})d(fz, gz)}{1 + d(gx_{n_k}, gz)}, \\
&\frac{d(gx_{n_k+1}, gz)d(fz, gz)}{1 + d(gx_{n_k}, gz)}, \\
&\left. \frac{d(gx_{n_k+1}, gz)d(fz, gx_{n_k})}{1 + d(gx_{n_k}, gz)} \right\},
\end{aligned} \tag{59}$$

which gives $\lim_{k \rightarrow \infty} U_{f,g}(x_{n_k}, z) = \theta$. Thus $\psi(d(gz, fz)) \leq \theta$ and hence $f(z) = g(z)$. Thus z is the coincidence point of f and g .

Thus K being a finite intersection of closed sets is closed and hence complete. Consider the restrictions mappings $f|_K, g|_K : K \rightarrow K$. Clearly, $f|_K(K) \subseteq g|_K(K) \subseteq K$. Also $g|_Z(K)$ is closed and hence complete. From Theorem 11, it follows that $f|_K$ and $g|_K$ have at most one coincidence point in K . As f and g are weakly compatible, from Proposition 5, it follows that $f|_K$ and $g|_K$ have at most one common fixed point. $\square$

Corollary 13. Let (X, d) be a complex valued metric space, let $A_1, A_2, \dots, A_p$ be nonempty closed subsets of X , and let $Y = \cup_{i=1}^p A_i$. Suppose that $f, g : Y \rightarrow Y$ are such that

- (a) Y has a cocyclic representation with respect to pair (f^m, g^n) and to the collection $\{A_i : i = 1, 2, \dots, p\}$ for some $m, n \in \mathbb{N}$;
- (b) there exist two control functions, ψ and φ , such that, for any $(x, y) \in A_i \times A_{i+1}$, $i = 1, 2, \dots, p$,

$$\psi(d(f^m x, f^m y)) \leq \psi(U_{f,g}(x, y)) - \varphi(U_{f,g}(x, y)) \quad (60)$$

holds for some $m, n \in \mathbb{N}$, where

$$U_f(x, y) \in \left\{ d(g^n x, g^n y), \frac{d(f^m x, g^n x) d(f^m y, g^n y)}{1 + d(g^n x, g^n y)}, \frac{d(f^m x, g^n y) d(f^m y, g^n y)}{1 + d(g^n x, g^n y)^2}, \frac{d(f^m x, g^n y) d(f^m y, g^n x)}{1 + d(g^n x, g^n y)} \right\}, \quad (61)$$

with $A_{p+1} = A_1$.

If $g^n(A_i)$ is a complete subspace of X for each $i \in \{1, 2, \dots, p\}$, then f^m and g^n have a unique coincidence point $z \in \cap_{i=1}^p g^n(A_i) = K$ provided that $f^m(K) \subseteq g^n(K) \subseteq K$ and $g^n(K)$ is closed. Moreover, if f^m and g^n are weakly compatible, then f^m and g^n have a unique common fixed point. Furthermore, f and g have at most one common fixed point provided that f and g commute.

Proof. Set $F = f^m$ and $G = g^n$. From Theorem 12, F and G have a unique common fixed point u . Now $F(fu) = f(Fu) = fu$ and $G(fu) = f(Gu) = fu$ imply that fu is the common fixed point of F and G . Also, $G(gu) = g(Gu) = gu$ and $F(gu) = g(Fu) = gu$ imply that gu is also the common fixed point of F and G . By the uniqueness of common fixed point of F and G , we have $fu = gu = u$. $\square$

Example 14. Let $X = \mathbb{R}$ and let $d : X \times X \rightarrow \mathbb{C}^+$ be the complex valued metric defined by $d(x, y) = |x - y|e^{i\theta}$, where $\theta \in [0, \pi/2]$. Suppose $A_1 = [-c, 0]$, $A_2 = [-kc, kc]$ and $A_3 = [-kc/2, c]$ and $A_4 = A_1$, where $c > 0$ and $k \in [0, 1)$. Define $f, g : \cup_{i=1}^3 A_i \rightarrow \cup_{i=1}^3 A_i$ by

$$fx = \begin{cases} -\frac{\alpha x}{c}, & \text{if } x > 0, \\ 0, & \text{otherwise,} \end{cases} \quad (62)$$

$$gx = \frac{\beta x}{c},$$

where $\alpha, \beta \geq 1$ with $\alpha < 2\beta \leq c$. Note that $f(A_1) = \{0\} \subseteq [-k\beta, k\beta] = g(A_2)$, $f(A_2) = [-k\alpha, 0] \subseteq [-k\beta/2, \beta] = g(A_3)$,

and $f(A_3) = [-\alpha, 0] \subseteq [-\beta, 0] = g(A_1)$. Hence $Y = A_1 \cup A_2 \cup A_3$ is a cocyclic representation with respect to pair (f, g) and the collection $\{A_1, A_2, A_3\}$. Define $\psi, \varphi : \mathbb{C}^+ \rightarrow \mathbb{C}^+$ by

$$\psi(z) = \beta z, \quad \varphi(z) = \begin{cases} \frac{\beta z}{4}, & \text{if } z \in [\theta, 2\beta e^{i\theta}] \\ \beta z, & \text{if } t > 2\beta e^{i\theta}. \end{cases} \quad (63)$$

Clearly ψ is continuous and nondecreasing and φ is lower semicontinuous with $\psi(z) = \varphi(z) = \theta$ if and only if $z = \theta$.

We show that condition (b) is satisfied:

- (I) Let $x \in A_1$ and $y \in A_2$. If $y \in [-kc, 0]$, then $d(fx, fy) = \theta$ and hence (b) holds. If $y \in (0, kc]$, then we have

$$\begin{aligned} \psi(d(fx, fy)) &= \psi\left(d\left(0, -\frac{\alpha y}{c}\right)\right) = \frac{\alpha\beta}{c} ye^{i\theta} \\ &\leq \frac{3\beta^2}{4c} (y - x) e^{i\theta} = \frac{3\beta}{4} d\left(\frac{\beta x}{c}, \frac{\beta y}{c}\right) \\ &= \frac{3\beta}{4} d(gx, gy) \leq \frac{3\beta}{4} U_{f,g}(x, y) \\ &= \beta U_{f,g}(x, y) - \frac{\beta}{4} U_{f,g}(x, y) \\ &= \psi(U_{f,g}(x, y)) - \varphi(U_{f,g}(x, y)). \end{aligned} \quad (64)$$

- (II) Let $x \in A_2$, $y \in A_3$. If $x \in [-kc, 0]$ and $y \in [-kc/2, 0]$, then $d(fx, fy) = \theta$ and hence (b) is satisfied. If $x \in [-kc, 0]$ and $y \in (0, c]$, then we obtain that

$$\begin{aligned} \psi(d(fx, fy)) &= \psi\left(d\left(0, -\frac{\beta y}{c}\right)\right) = \frac{\beta^2}{c} ye^{i\theta} \\ &< \frac{3\beta(\alpha + \beta)}{4c} ye^{i\theta} = \frac{3\beta}{4} d\left(-\frac{\alpha y}{c}, \frac{\beta y}{c}\right) \\ &= \frac{3\beta}{4} d(fy, gy) \leq \frac{3\beta}{4} U_{f,g}(x, y) \\ &= \beta U_{f,g}(x, y) - \frac{\beta}{4} U_{f,g}(x, y) \\ &= \psi(U_{f,g}(x, y)) - \varphi(U_{f,g}(x, y)). \end{aligned} \quad (65)$$

When $x \in (0, kc]$ and $y \in [-kc/2, 0]$, we have

$$\begin{aligned} \psi(d(fx, fy)) &= \psi\left(d\left(-\frac{\alpha x}{c}, 0\right)\right) = \frac{\alpha\beta}{c} xe^{i\theta} \\ &< \frac{3\beta(\alpha + \beta)}{4c} xe^{i\theta} = \frac{3\beta}{4} d\left(-\frac{\alpha x}{c}, \frac{\beta x}{c}\right) \\ &= \frac{3\beta}{4} d(fx, gx) \leq \frac{3\beta}{4} U_{f,g}(x, y) \\ &= \beta U_{f,g}(x, y) - \frac{\beta}{4} U_{f,g}(x, y) \\ &= \psi(U_{f,g}(x, y)) - \varphi(U_{f,g}(x, y)). \end{aligned} \quad (66)$$

For $x \in (0, kc]$ and $y \in (0, c]$, we obtain that

$$\begin{aligned}
 \psi(d(fx, fy)) &= \psi\left(d\left(-\frac{\alpha x}{c}, -\frac{\alpha y}{c}\right)\right) \\
 &= \frac{\alpha\beta}{c} |x - y| e^{i\theta} < \frac{3\beta^2}{4c} |x - y| e^{i\theta} \\
 &= \frac{3\beta}{4} d\left(\frac{\beta x}{c}, \frac{\beta y}{c}\right) = \frac{3\beta}{4} d(gx, gy) \\
 &\leq \frac{3\beta}{4} U_{f,g}(x, y) \\
 &= \beta U_{f,g}(x, y) - \frac{\beta}{4} U_{f,g}(x, y) \\
 &= \psi(U_{f,g}(x, y)) - \varphi(U_{f,g}(x, y)).
 \end{aligned} \tag{67}$$

(III) Let $x \in A_3, y \in A_1$. If $x \in [-kc/2, 0]$, then $d(fx, fy) = \theta$ and (b) is satisfied. If $x \in (0, c]$, then we have

$$\begin{aligned}
 \psi(d(fx, fy)) &= \psi\left(d\left(-\frac{\alpha x}{c}, 0\right)\right) = \frac{\alpha\beta}{c} x e^{i\theta} \\
 &< \frac{3\beta(\alpha + \beta)}{4c} x e^{i\theta} = \frac{3\beta}{4} d\left(-\frac{\alpha x}{c}, \frac{\beta x}{c}\right) \\
 &= \frac{3\beta}{4} d(fx, gx) \leq \frac{3\beta}{4} U_{f,g}(x, y) \\
 &= \beta U_{f,g}(x, y) - \frac{\beta}{4} U_{f,g}(x, y) \\
 &= \psi(U_{f,g}(x, y)) - \varphi(U_{f,g}(x, y)).
 \end{aligned} \tag{68}$$

Hence f and g satisfy all the conditions of Theorem 12. Moreover, 0 is a unique common fixed point of f and g in $\cap_{i=1}^3 A_i$.

We denote $Y = \{\phi : \mathbb{C}^+ \rightarrow \mathbb{C}^+ : \phi \text{ is Lebesgue integrable with finite integral on each compact subset of } \mathbb{C}^+ \text{ and, for each } \theta < \varepsilon \in \mathbb{C}^+, \int_{\theta}^{\varepsilon} \phi(z) dz > \theta\}$. As a consequence of Theorem 12, we obtain the following common fixed point result for mappings satisfying cocyclic contractive condition of integral type in a complex valued metric space.

Corollary 15. Let (X, d) be a complex valued metric space, let $A_1, A_2, \dots, A_p$, be nonempty closed subsets of X , and let $Y = \cup_{i=1}^p A_i$. Suppose that $f, g : Y \rightarrow Y$ are mappings such that

- Y has a cocyclic representation with respect to pair (f, g) and to the collection $\{A_i : i = 1, 2, \dots, p\}$;
- there exist control functions ψ and φ such that, for any $(x, y) \in A_i \times A_{i+1}$, $i = 1, 2, \dots, p$,

$$\begin{aligned}
 \int_{\theta}^{\psi(d(fx, fy))} \phi(w) dw &\leq \int_{\theta}^{\psi(U_{f,g}(x, y))} \phi(w) dw \\
 &\quad - \int_{\theta}^{\varphi(U_{f,g}(x, y))} \phi(w) dw
 \end{aligned} \tag{69}$$

is satisfied, where $\phi \in Y$ and

$$U_{f,g}(x, y) \in \left\{ d(gx, gy), \frac{d(fx, gx)d(fy, gy)}{1 + d(gx, gy)}, \frac{d(fx, gy)d(fy, gy)}{1 + d(gx, gy)^2}, \frac{d(fx, gy)d(fy, gx)}{1 + d(gx, gy)} \right\} \tag{70}$$

with $A_{p+1} = A_1$.

If $g(A_i)$ is a complete subspace of X for each $i \in \{1, 2, \dots, p\}$, then f and g have a unique coincidence point $z \in \cap_{i=1}^p g(A_i) = K$ provided that $f(K) \subseteq g(K) \subseteq K$ and $g(K)$ is closed. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Define $\Psi : \mathbb{C}^+ \rightarrow \mathbb{C}^+$ by $\Psi(z) = \int_{\theta}^z \phi(w) dw$. From (69), it follows that

$$\begin{aligned}
 \Psi(\psi(d(fx, fy))) &\leq \Psi(\psi(U_{f,g}(x, y))) \\
 &\quad - \Psi(\varphi(U_{f,g}(x, y))),
 \end{aligned} \tag{71}$$

which can be written as

$$\psi^*(d(fx, fy)) \leq \psi^*(U_{f,g}(x, y)) - \varphi^*(U_{f,g}(x, y)), \tag{72}$$

where $\psi^* = \Psi \circ \psi$ and $\varphi^* = \Psi \circ \varphi$. Clearly, $\psi^*, \varphi^* : \mathbb{C}^+ \rightarrow \mathbb{C}^+$, ψ^* is continuous and nondecreasing, φ^* is lower semicontinuous, and $\psi^*(z) = \varphi^*(z) = \theta$ if and only if $z = \theta$. Hence, by Theorem 12, f and g have a unique common fixed point. $\square$

Remark 16. It is worth mentioning that the results obtained in this paper are distinct from the results in [8] in the following sense:

- The results in this paper are proved in the setup of complex valued metric spaces in which the distance between any two elements is a complex number but the results proved in [8] are in the setup of multiplicative metric spaces, where the distance between any two elements is a real number greater than or equal to 1.
- The contractive conditions imposed on the mappings established in this paper are substantially different from contractions used in [8].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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