Research Article

Bounded Fatou Components of Composite Transcendental Entire Functions with Gaps

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We prove that composite transcendental entire functions with certain gaps have no unbounded Fatou component.

1. Introduction

Let f be a transcendental entire function. We write $f^1 = f$, and $f^n = f \circ f^{n-1}$ for $n \geq 2$ for the n th iterate of f . The Fatou set or set of normality $F(f)$ of f consists of all z in the complex plane $\mathbb{C}$ which has a neighborhood U such that the family $\{f^n | U : n \geq 1\}$ is a normal family. The Julia set $J(f)$ of f is $J(f) = \mathbb{C} \setminus F(f)$. For the fundamental results in the iteration theory of rational and entire functions, we refer to the original papers of Fatou [1–3], and Julia [4] and the books of Beardon [5], Carleson and Gamelin [6], Milnor [7], and Ren [8].

Let U be a connected component of $F(f)$. Then $f^n(U) \subseteq U_n$, where U_n is a component of $F(f)$. If there is a smallest positive integer p such that $U_p = U$, then U is periodic of period p . In particular, if $p = 1$, then U is called invariant. If for some integer $n \geq 1$, U_n is periodic, while U is not periodic then U is called preperiodic. If U is periodic and $f^n | U \rightarrow \infty$ then U is called a Baker domain. If all the U_n are disjoint, that is, for $n \neq m$, $U_n \neq U_m$ then U is called a wandering domain. Let f be a transcendental entire function. In 1981, Baker [9] proposed whether every component of $F(f)$ is bounded if the growth of f is sufficiently small. The appropriate growth condition would appear to be of order $1/2$, minimal type at most. In [9], Baker observed that this condition is best possible in the following sense: for any sufficiently large positive a , the function

$$f(z) = z^{-1/2} \sin \sqrt{z} + z + a \quad (1)$$

is of order $1/2$, mean type, and has an unbounded component U of $F(f)$ containing a segment $[x_0, \infty)$ of the positive real

axis, such that $f^n(z) \rightarrow \infty$ as $n \rightarrow \infty$, locally uniformly in U .

It is conjectured that if the order of f is less than $1/2$, minimal type, then every component of $F(f)$ is bounded. It is still open for wandering domains, although there are several remarkable results for wandering domains under the assumptions that the growth satisfies, in addition, some regular conditions; see [10–17].

Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an entire function with gaps; that is, some of the a_n are zero, in a certain sense. Then the function has the form $f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}$. We say that $f(z)$ has Fabry gaps if $n_k/k \rightarrow \infty$ as $k \rightarrow \infty$, and $f(z)$ has Fejér gaps if $\sum_{k=1}^{\infty} (1/n_k) < \infty$.

Wang [18] proved that every component of the Fatou set of an entire function with certain gaps is bounded, by using the properties of the entire functions with such gaps. Wang [18] obtained the following result.

Theorem 1. Let $f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}$ be an entire function with $0 < \mu \leq \rho < \infty$. If $f(z)$ has Fabry gaps, then every component of $F(f)$ is bounded.

For Fejér gap, Wang [18] proposed the following problem. Let $f(z)$ be an entire function with Fejér gaps, that is,

$$\sum_{k=1}^{\infty} \frac{1}{n_k} < \infty. \quad (2)$$

Is every component of $F(f)$ bounded?

For composite of entire function, Qiao [19] proved the following result.

Theorem 2. Let $h(z) = f_N \circ f_{N-1} \circ \cdots \circ f_1$ be a transcendental entire function, where f_j ($j = 1, 2, \dots, N$) are entire functions with order $\rho_j < 1/2$. Then every nonwandering component is bounded.

Cao and Wang [20] proved the following result.

Theorem 3. Let $h(z) = f_N \circ f_{N-1} \circ \cdots \circ f_1$, where f_j ($j = 1, 2, \dots, N$) are nonconstant holomorphic maps, each having order less than $1/2$. If there is a number $i \in \{1, 2, \dots, N\}$ such that the lower order of f_i is greater than 0, then every component of $F(h)$ is bounded.

Singh [21] proved the following result.

Theorem 4. Let $\mathcal{L}$ be the set of all entire functions f such that, for given $\varepsilon > 0$,

$$\log m(r, f) \geq (1 - \varepsilon) \log M(r, f) \quad (3)$$

holds for all r outside a set of logarithmic density 0. Let $\mathcal{F} = \bigcup_{K \geq 1} \mathcal{F}_K$ where $\mathcal{F}_K$ is the set of all transcendental entire functions f such that

$$\log \log M(r, f) \geq (\log r)^{1/K}. \quad (4)$$

If $h(z) = f_N \circ f_{N-1} \circ \cdots \circ f_1(z)$, where $f_j \in \mathcal{F} \cap \mathcal{L}$ ($j = 1, 2, \dots, N$), then every component of $F(h)$ is bounded.

2. Preliminaries

We use the standard notations for the maximum modulus $M(r, f)$, minimum modulus $m(r, f)$, order of growth ρ , and lower order of growth μ of a function f ; namely,

$$\begin{aligned} M(r, f) &= \max \{|f(z)| : |z| = r\}, \\ m(r, f) &= \min \{|f(z)| : |z| = r\}, \\ \rho &= \limsup_{r \rightarrow \infty} \frac{\log \log M(r, f)}{\log r}, \\ \mu &= \liminf_{r \rightarrow \infty} \frac{\log \log M(r, f)}{\log r}. \end{aligned} \quad (5)$$

Briefly, we also denote maximum modulus $M(r, f)$ and minimum modulus $m(r, f)$ by $M(r)$ and $m(r)$.

Let E be a set in $\mathbb{C}$. The logarithmic measure of a set E is defined by $\int_E (dt/t)$. If $E \subset [1, \infty)$, $E(a, b)$ denote the part of E in the interval (a, b) , that is, $E(a, b) = E \cap (a, b)$, then the upper logarithmic density of the set E is defined by

$$\overline{\text{logdens}} E = \limsup_{r \rightarrow \infty} \frac{1}{\log r} \int_{E(1, r)} \frac{dt}{t}; \quad (6)$$

the lower logarithmic density of the set E is defined by

$$\underline{\text{logdens}} E = \liminf_{r \rightarrow \infty} \frac{1}{\log r} \int_{E(1, r)} \frac{dt}{t}. \quad (7)$$

If the upper and lower logarithmic density are equal, their common value is called the logarithmic density of E .

Lemma 5 (see [22]). Let f be a transcendental entire function. Then there exists $R > 0$ such that, for all $r \geq R$ and all $c > 1$,

$$\log M(r^c, f) \geq c \log M(r, f). \quad (8)$$

Lemma 6 (see [23]). Let f be an entire function of finite order with Fabry gaps. Then for given $\varepsilon > 0$,

$$\log m(r, f) \geq (1 - \varepsilon) \log M(r, f) \quad (9)$$

holds for all r outside a set of logarithmic density 0.

Lemma 7 (see [24]). Let $f = \sum_{k=0}^{\infty} a_k z^{n_k}$ be an entire function with

$$n_k > k \log k (\log \log k)^\alpha \quad \text{as } k \rightarrow \infty \quad (10)$$

for some $\alpha > 2$. Then for given $\varepsilon > 0$,

$$\log m(r, f) \geq (1 - \varepsilon) \log M(r, f) \quad (11)$$

holds for all r outside a set of logarithmic density 0.

Lemma 8 (see [25]). If $f = \sum_{k=0}^{\infty} a_k z^{n_k}$ satisfies the gap-condition $n_k > k(\log k)^{2+\eta}$, then for given $\varepsilon > 0$,

$$\log m(r, f) \geq (1 - \varepsilon) \log M(r, f) \quad (12)$$

holds for all r outside a set of finite logarithmic measure.

Lemma 9 (see [26]). For an entire function $f(z)$ with Fejér gaps and $\varepsilon > 0$,

$$\log m(r, f) \geq (1 - \varepsilon) \log M(r, f). \quad (13)$$

Lemma 10 (see [9]). Let D be a domain and K a compact subset of D . Let G be the family of all holomorphic functions g on D which omit the values 0, 1 and satisfy the condition that $|g| \geq 1$ on K . Then there exist constants A and B such that $|g(z')| < A|g(z)|^B$, for every $g \in G$ and every $z, z' \in K$.

Lemma 11. Let f be a transcendental entire function of finite order with Fabry gaps. Then there exist $L > 1$ and $R > 0$ such that, for all $r \geq R$, there exists σ satisfying $r \leq \sigma \leq r^L$ and $m(\sigma, f) = M(r, f)$.

Proof. By Lemma 6, given any $\varepsilon > 0$, we have $m(r, f) > M(r, f)^{1-\varepsilon}$, for all r outside a set E of logarithmic density 0. Let $c > 1$. Then there exists R_0 such that, for all $R > R_0$, there exists $s \in [R, R^c]$ such that $m(s, f) > M(s, f)^{1-\varepsilon}$. If not,

then there exists a sequence $R_j \rightarrow \infty$ such that $m(s, f) \leq M(s, f)^{1-\varepsilon}$, for every $s \in [R_j, R_j^c]$. Thus $[R_j, R_j^c] \subset E$. So

$$\begin{aligned} \underline{\log \text{dens}} E &= \liminf_{r \rightarrow \infty} \frac{1}{\log r} \int_{E(1, r)} \frac{dt}{t} \\ &\geq \liminf_{j \rightarrow \infty} \frac{1}{\log R_j^c} \int_{E(1, R_j^c)} \frac{dt}{t} \\ &\geq \liminf_{j \rightarrow \infty} \frac{1}{c \log R_j} \int_{R_j}^{R_j^c} \frac{dt}{t} \\ &= 1 - \frac{1}{c}, \end{aligned} \quad (14)$$

contradicting $\log \text{dens}(E) = 0$.

Set $R^{1-2\varepsilon} = r$. There exists $s \in [r^{1/(1-2\varepsilon)}, r^{c/(1-2\varepsilon)}]$ such that $m(s, f) > M(s, f)^{1-\varepsilon}$. By Lemma 6,

$$\begin{aligned} m(s, f) &> M(s, f)^{1-\varepsilon} \geq M(r^{1/(1-2\varepsilon)}, f)^{1-\varepsilon} \\ &\geq M(r, f)^{(1-\varepsilon)/(1-2\varepsilon)} \geq M(r, f). \end{aligned} \quad (15)$$

Since $m(r, f) \leq M(r, f)$, there exists $\sigma \in [r, s] \subset [r, r^L]$, where $L = c/(1 - 2\varepsilon)$ such that $m(\sigma, f) = M(r, f)$. This completes the proof of Lemma 11. $\square$

By Lemma 9 and the same method in the proof of Lemma 11, we can prove the following result.

Lemma 12. *Let f be a transcendental entire function of finite order with Fejér gaps. Then there exist $L > 1$ and $R > 0$ such that, for all $r \geq R$, there exists σ satisfying $r \leq \sigma \leq r^L$ and $m(\sigma, f) = M(r, f)$.*

By Lemma 7 and the same method in the proof of Lemma 11, we can prove the following result.

Lemma 13. *Let f be a transcendental entire function of finite order with*

$$n_k > k \log k (\log \log k)^\alpha \quad \text{as } k \rightarrow \infty \quad (16)$$

for some $\alpha > 2$. Then there exist $L > 1$ and $R > 0$ such that, for all $r \geq R$, there exists σ satisfying $r \leq \sigma \leq r^L$ and $m(\sigma, f) = M(r, f)$.

3. Main Results

In 2012, the authors proved some results on the bounded Fatou components of transcendental entire functions with gaps; see [27]. In this paper, we investigate the iteration of the composite entire functions with gaps and obtain the following results.

Theorem 14. *Let $h(z) = f_N \circ f_{N-1} \circ \dots \circ f_1(z)$ be a transcendental entire function, where $f_j = \sum_{k=0}^{\infty} a_{jk} z^{n_{jk}}$ ($j = 1, 2, \dots, N$) have Fabry gaps of finite order. If*

$$M(r, f_j) \geq \exp(\exp(\log r)^p), \quad 0 < p < 1, \quad (17)$$

then $F(h)$ has no unbounded component.

Proof. The proof follows the idea of Theorems 3 and 4. Since f_j is a transcendental entire function of finite order with Fabry gaps, by Lemma 11, there exist $L_j > 1$ and r_j sufficiently large such that, for $r \geq r_j$, there exists $\sigma^{(j)}$ such that $r \leq \sigma^{(j)} \leq r^{L_j}$ and

$$m(\sigma^{(j)}, f_j) = M(r, f_j), \quad j = 1, 2, \dots, N. \quad (18)$$

Since f_j is a transcendental entire function of finite order, h must be a transcendental entire function. It follows that there exists a number $r_0 > 1$ such that $M(r, h) > r^2$ for all $r > r_0$, and there exists a number $t_j > 1$ such that

$$\exp(r^{t_j}) \geq M(r, f_j)^{(\log M(r, f_j))^{1/p}} \quad (19)$$

for all r sufficiently large. In fact, if there is a sequence $\{r_n\}$ which tend to ∞ such that

$$\exp(r_n^{t_j}) < M(r_n, f_j)^{(\log M(r_n, f_j))^{1/p}}, \quad (20)$$

then

$$t \log r_n < \frac{1+p}{p} \cdot \log \log M(r_n, f_j). \quad (21)$$

So

$$t \leq \frac{1+p}{p} \lim_{n \rightarrow \infty} \frac{\log \log M(r_n, f_j)}{\log r_n} \leq \frac{1+p}{p} \rho. \quad (22)$$

Which gives a contradiction if we take any $t > ((1+p)/p)\rho$.

Given $R_1^{(1)} > r_0$, define inductively

$$\begin{aligned} R_n^{(j+1)} &= M(R_n^{(j)}, f_j), \quad j = 1, 2, \dots, N-1; \\ R_{n+1}^{(1)} &= M(R_n^{(N)}, f_N). \end{aligned} \quad (23)$$

It is easy to see that, for all $n \in \mathbb{N}$,

$$(R_n^{(1)})^2 < M(R_n^{(1)}, h) \leq R_{n+1}^{(1)}. \quad (24)$$

Thus

$$R_{n+1}^{(1)} > (R_n^{(1)})^{2^n} \rightarrow \infty \quad \text{as } n \rightarrow \infty \quad (25)$$

and $R_n^{(j)} \rightarrow \infty$ as $n \rightarrow \infty$ for all $j = 1, 2, \dots, N$.

Take number n_0 sufficiently large, for $n \geq n_0$, by Lemma 11, there exists $\sigma_n^{(j)}$ such that

$$\begin{aligned} \exp((R_n^{(j)})^{t_j}) &\leq \sigma_n^{(j)} \leq \exp(L_j (R_n^{(j)})^{t_j}), \\ m(\sigma_n^{(j)}, f_j) &= M(\exp((R_n^{(j)})^{t_j}), f_j) \end{aligned} \quad (26)$$

for $n \geq n_0$. By the hypotheses of Theorem 14, (19), and Lemma 11, we have

$$\begin{aligned}
& m(\sigma_n^{(j)}, f_j) \\
&= M\left(\exp\left((R_n^{(j)})^{t_j}\right), f_j\right) \\
&\geq M\left(M(R_n^{(j)}, f_j)^{(\log M(R_n^{(j)}, f_j))^{1/p}}, f_j\right) \\
&\geq \exp\left(\exp\left(\log M(R_n^{(j)}, f_j)^{(\log M(R_n^{(j)}, f_j))^{1/p}}\right)^p\right) \\
&= \exp\left(\exp\left((\log M(R_n^{(j)}, f_j))(\log M(R_n^{(j)}, f_j))^p\right)\right) \\
&= \exp\left(M(R_n^{(j)}, f_j)^{(\log M(R_n^{(j)}, f_j))^p}\right) \\
&\geq \exp\left(L_{j+1}M(R_n^{(j)}, f_j)^{t_{j+1}}\right) \\
&= \exp\left(L_{j+1}(R_n^{(j+1)})^{t_{j+1}}\right)
\end{aligned} \tag{27}$$

for $j = 1, 2, \dots, N-1$; and

$$\begin{aligned}
& m(\sigma_n^{(N)}, f_N) \\
&= M\left(\exp\left((R_n^{(N)})^{t_N}\right), f_N\right) \\
&\geq M\left(M(R_n^{(N)}, f_N)^{(\log M(R_n^{(N)}, f_N))^{1/p}}, f_N\right) \\
&\geq \exp\left(\exp\left(\log M(R_n^{(N)}, f_N)^{(\log M(R_n^{(N)}, f_N))^{1/p}}\right)^p\right) \\
&= \exp\left(\exp\left((\log M(R_n^{(N)}, f_N))(\log M(R_n^{(N)}, f_N))^p\right)\right) \\
&= \exp\left(M(R_n^{(N)}, f_N)^{(\log M(R_n^{(N)}, f_N))^p}\right) \\
&\geq \exp\left(L_1M(R_n^{(N)}, f_N)^{t_1}\right) \\
&= \exp\left(L_1(R_{n+1}^{(1)})^{t_1}\right).
\end{aligned} \tag{28}$$

Suppose on contrary that $F(h)$ has an unbounded component D . Without loss of generality we may assume 0, 1 belong to $J(h)$. Hence each map h^n omits the values 0, 1 in D . It follows from the unboundedness and connectivity of D that there exists $n_1 \geq n_0 \in \mathbb{N}$ such that D meets the circles

$$\begin{aligned}
\alpha_n^{(j)} &= \{z : |z| = R_n^{(j)}\}, \\
\beta_n^{(j)} &= \{z : |z| = \exp\left(L_j(R_n^{(j)})^{t_j}\right)\}, \\
\gamma_n^{(j)} &= \{z : |z| = \sigma_n^{(j)}\}
\end{aligned} \tag{29}$$

for all $n \geq n_1$, $j = 1, 2, \dots, N$.

We choose a value $k \in \mathbb{N}$ such that $k \geq n_1$ and note that D must contain a path Γ joining a point $\omega_k^{(1)} \in \alpha_k^{(1)}$ to a point $\tau_{k+1}^{(1)} \in \beta_{k+1}^{(1)}$. It is clear that Γ contains two subsets Γ' , Γ'' such that Γ' joins $\omega_k^{(1)} \in \alpha_k^{(1)}$ to $\eta_k^{(1)} \in \beta_k^{(1)}$ and contains $\xi_k^{(1)} \in \gamma_k^{(1)}$ and Γ'' joins $\delta_{k+1}^{(1)} \in \alpha_{k+1}^{(1)}$ to $\tau_{k+1}^{(1)} \in \beta_{k+1}^{(1)}$. We know that $R_k^{(2)} = M(R_k^{(1)}, f_1)$ and so $R_k^{(2)} \geq |f_1(\omega_k^{(1)})|$. Also $m(\sigma_k^{(1)}, f_1) \geq \exp(L_2(R_k^{(2)})^{t_2})$ and so $|f_1(\xi_k^{(1)})| \geq \exp(L_2(R_k^{(2)})^{t_2})$. Hence $f_1(\Gamma')$ contains an arc joining a point $\omega_k^{(2)} \in \alpha_k^{(2)}$ to a point $\eta_k^{(2)} \in \beta_k^{(2)}$. Similarly $f_1(\Gamma'')$ contains an arc joining a point $\delta_{k+1}^{(2)} \in \alpha_{k+1}^{(2)}$ to a point $\tau_{k+2}^{(2)} \in \beta_{k+2}^{(2)}$.

Repeating the process inductively we obtain that $h(\Gamma') = f_N \circ f_{N-1} \circ \dots \circ f_1(\Gamma')$ contains an arc joining $\omega_{k+1}^{(1)} \in \alpha_{k+1}^{(1)}$ to a point $\eta_{k+1}^{(1)} \in \beta_{k+1}^{(1)}$ and $h(\Gamma'')$ contains an arc joining a point $\delta_{k+2}^{(1)} \in \alpha_{k+2}^{(1)}$ to a point $\tau_{k+2}^{(1)} \in \beta_{k+2}^{(1)}$.

Since Γ' and Γ'' are two subsets of Γ , it follows that $h(\Gamma)$ must contain an arc joining $\omega_{k+1}^{(1)} \in \alpha_{k+1}^{(1)}$ to the point $\tau_{k+2}^{(1)} \in \beta_{k+2}^{(1)}$. By induction it now follows that $h^n(\Gamma)$ contains an arc joining a point $\omega_{k+n}^{(1)} \in \alpha_{k+n}^{(1)}$ to the point $\tau_{k+n+1}^{(1)} \in \beta_{k+n+1}^{(1)}$.

Thus $h^n(D)$ is a component of $F(h)$ containing $h^n(\Gamma)$, and, on Γ , h^n takes a value of modulus at least $R_{k+n}^{(1)}$ and $R_{k+n}^{(1)} \rightarrow \infty$ as $n \rightarrow \infty$. Thus we conclude that $R_{k+n}^{(1)} \rightarrow \infty$ locally uniformly in D . Hence there exists $N \in \mathbb{N}$ such that, for all $z \in \Gamma$, $|h^n(z)| > 1$ for all $n > N$. Thus the family $\{h^n\}_{n>N}$ satisfies the conditions of Lemma 10 on Γ , and so there exist constants A, B such that $|h^n(z')| < A|h^n(z)|^B$ for all $n > N$ and for all $z, z' \in \Gamma$. Choose $z_n, z'_n \in \Gamma$ with $n > N$ such that $h^n(z_n) = \omega_{k+n}^{(1)} \in \alpha_{k+n}^{(1)}$ and $h^n(z'_n) = \tau_{k+n+1}^{(1)} \in \beta_{k+n+1}^{(1)}$; we have

$$\begin{aligned}
M(R_{k+n}^{(1)}, h) &= M(R_{k+n}^{(1)}, f_N \circ f_{N-1} \circ \dots \circ f_1) \\
&\leq M(R_{k+n}^{(N)}, f_N) = R_{k+n+1}^{(1)} \\
&< \exp\left(L_1(R_{k+n+1}^{(1)})^{t_1}\right) = |h^n(z'_n)| \\
&< A|h^n(z_n)|^B = A|R_{k+n}^{(1)}|^B
\end{aligned} \tag{30}$$

for all $n > N$ which contradicts the fact that h is a transcendental entire function and $R_{k+n}^{(1)} \rightarrow \infty$ as $n \rightarrow \infty$. This completes the proof of Theorem 14. $\square$

Corollary 15 (see [27]). *Let $f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}$ be a transcendental entire function of finite order with Fabry gaps. If*

$$M(r, f) \geq \exp(\exp(\log r)^p), \quad 0 < p < 1, \tag{31}$$

then $F(f)$ has no unbounded component.

Remark 16. If f satisfies $M(r, f) \geq \exp(\exp(\log r)^p)$, $0 < p < 1$, then $\rho \geq 0$, so Corollary 15 is an extension of Theorem 1.

By Lemma 12 and the same method used in the proof of Theorem 14, we can show the following result.

Theorem 17. Let $h(z) = f_N \circ f_{N-1} \circ \cdots \circ f_1(z)$ be a transcendental entire function, where $f_j = \sum_{k=0}^{\infty} a_{jk} z^{n_{jk}}$ ($j = 1, 2, \dots, N$) have Fejér gaps. If

$$M(r, f_j) \geq \exp(\exp(\log r)^p), \quad 0 < p < 1, \quad (32)$$

then $F(h)$ has no unbounded component.

By Theorem 17, we have the following.

Corollary 18 (see [27]). Let $f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}$ be a transcendental entire function with Fejér gaps. If

$$M(r, f) \geq \exp(\exp(\log r)^p), \quad 0 < p < 1, \quad (33)$$

then $F(f)$ has no unbounded component.

Remark 19. Corollary 18 partly answers the problem of Wang on the Fejér gaps.

By Lemma 13 and the same method of the proof of Theorem 14, we can show Theorem 20.

Theorem 20. Let $h(z) = f_N \circ f_{N-1} \circ \cdots \circ f_1(z)$ be a transcendental entire function, where $f_j = \sum_{k=0}^{\infty} a_{jk} z^{n_{jk}}$ ($j = 1, 2, \dots, N$) are entire functions with the gap-conditions

$$n_{jk} > k \log k (\log \log k)^\alpha \quad \text{as } k \rightarrow \infty \text{ for some } \alpha > 2. \quad (34)$$

If

$$M(r, f_j) \geq \exp(\exp(\log r)^p), \quad 0 < p < 1, \quad (35)$$

then $F(h)$ has no unbounded component.

By Theorem 20, we have the following.

Corollary 21. Let $h(z) = f_N \circ f_{N-1} \circ \cdots \circ f_1(z)$ be a transcendental entire function, where $f_j = \sum_{k=0}^{\infty} a_{jk} z^{n_{jk}}$ ($j = 1, 2, \dots, N$) are entire functions with the gap-conditions $n_{jk} > k(\log k)^{2+\eta}$ for some $\eta > 0$. If

$$M(r, f_j) \geq \exp(\exp(\log r)^p), \quad 0 < p < 1, \quad (36)$$

then $F(h)$ has no unbounded component.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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