Research Article

Multiplicity of Solutions for a Modified Schrödinger-Kirchhoff-Type Equation in $\mathbb{R}^N$

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We study the existence of infinitely many solutions for a class of modified Schrödinger-Kirchhoff-type equations by the dual method and the nonsmooth critical point theory.

1. Introduction and Main Result

In this paper, we study the following modified Schrödinger-Kirchhoff-type equations of the form

$$\begin{aligned} & - \left(a + b \int_{\mathbb{R}^N} |\nabla u|^2 dx \right) \Delta u - a u \Delta u^2 + V(x) u \\ & = g(x, u), \quad x \in \mathbb{R}^N, \end{aligned} \quad (1)$$

where $a > 0$, $b \geq 0$, $N \geq 3$, $g \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, and $V \in C(\mathbb{R}^N, \mathbb{R})$.

When $a u \Delta u^2 = 0$, (1) is reduced to the following Kirchhoff-type problem:

$$\begin{aligned} & - \left(a + b \int_{\mathbb{R}^N} |\nabla u|^2 dx \right) \Delta u + V(x) u = g(x, u), \\ & x \in \mathbb{R}^N. \end{aligned} \quad (2)$$

If $V(x) = 0$, problem (2) is related to the stationary analogue of the Kirchhoff equation

$$u_{tt} - \left(a + b \int_{\mathbb{R}^N} |\nabla u|^2 dx \right) \Delta u = g(x, u) \quad (3)$$

proposed by Kirchhoff in [1] as an existence of the classical D'Alembert's wave equation for free vibrations of elastic string. Kirchhoff's model takes into account the changes for free vibrations of elastic strings. Recently, there have been

many papers concerned with the Kirchhoff-type problems by variational methods; see [2–8] and the references therein. Many studies of them are concentrated on a bounded smooth domain Ω of $\mathbb{R}^N$ ($N = 1, 2, 3$); it is well known that the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^t(\mathbb{R}^N)$ ($2 \leq t < 2^*$) is not compact. Hence, if we look for solution by variational methods, it is very difficult to prove (PS) condition. Moreover, in order to check the (PS) condition or some of its variants, one has to impose certain conditions.

When $b = 0$ and $a = 1$, (1) is reduced to the following modified nonlinear Schrödinger equation:

$$-\Delta u - u \Delta u^2 + V(x) u = g(x, u), \quad x \in \mathbb{R}^N. \quad (4)$$

Solutions of (4) are standing waves of the following quasilinear Schrödinger equations:

$$\begin{aligned} & i\psi_t + \Delta \psi - V(x) \psi + k \Delta \left(\alpha(|\psi|^2) \right) \alpha'(|\psi|^2) \psi \\ & + g(x, \psi) = 0, \quad x \in \mathbb{R}^N, \end{aligned} \quad (5)$$

where k is a real constant and α and g are real functions. Equations (5) are derived as model of several physical phenomena, such as [9, 10]. Many achievements had been obtained on the existence of ground states, infinitely many solutions, and soliton solutions for (4), by a dual approach, Nehari method, and the minimax methods in critical point theory, applying the perturbation approach and the Lusternik-Schnirelmann category theory; see [11–18].

Recently, K. Wu and X. Wu [19] obtained infinitely many small energy solutions of (1) by applying Clark's theorem to a perturbation functional. In (1), set $g(x, u) = \alpha k(x)|u|^{p-2}u + \beta u^{2^*-1}$, in which α and β are real parameters, $2^* = 2N/(N-2)$ ($N \geq 3$); Liang and Shi [20] obtained soliton solutions of (1) by using the concentration-compactness principle and minimax methods.

In this paper, we transform (1) to another equation with a continuous energy functional in some Banach space. We obtain the existence of multiple solution for problem (1) via using nonsmooth critical point theory and using some new techniques. Throughout this paper, the main ideas used here come from Colin and Jeanjean [12] and Liu et al. [14].

We need the following several notations. Let

$$H^1(\mathbb{R}^N) := \{u \in L^2(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\} \quad (6)$$

with the inner product

$$\langle u, v \rangle_{H^1} = \int_{\mathbb{R}^N} [\nabla u \cdot \nabla v + uv] dx \quad (7)$$

and the norm

$$\|u\|_{H^1}^2 = \langle u, u \rangle_{H^1} = \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) dx. \quad (8)$$

Let

$$H_V^1(\mathbb{R}^N) := \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x) u^2 dx < \infty \right\} \quad (9)$$

and the norm

$$\|u\|_{H_V^1}^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x) u^2) dx. \quad (10)$$

Let the following assumption (V) hold:

$$(V) \quad V \in C(\mathbb{R}^N, \mathbb{R}), \quad V_0 := \inf_{x \in \mathbb{R}^N} V(x) > 0, \text{ and } \lim_{|x| \rightarrow \infty} V(x) = \infty.$$

Moreover, we need the following assumptions:

- (g_1) Let $g \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$; $g(x, t)$ is odd in t .
 (g_2) There exist constants $C > 0$ and $4 < p < 22^*$ such that

$$|g(x, t)| \leq C(1 + |t|^{p-1}) \quad (11)$$

for all $(x, t) \in \mathbb{R}^N \times \mathbb{R}$, where $2^* = 2N/(N-2)$ is the Sobolev critical exponent.

- (g_3) $g(x, t) = o(|t|)$ uniformly in x as $|t| \rightarrow 0$.
 (g_4) There exists $\mu > 4$ such that $\mu G(x, t) \leq g(x, t)t$, $\forall (x, t) \in \mathbb{R}^N \times \mathbb{R}$, and $\inf_{x \in \mathbb{R}^N, |t|=1} G(x, t) > 0$, where $G(x, t) := \int_0^t g(x, s) ds$.

The main result of this paper is as follows.

Theorem 1. Assume that (V) and (g_1)–(g_4) are satisfied. Then problem (1) has a sequence $\{u_n\}$ of solutions such that $J(u_n) \rightarrow \infty$ as $n \rightarrow \infty$.

Throughout the paper, $\rightarrow$ and $\rightharpoonup$ denote the strong and weak convergence, respectively. C , c , C_i , and c_i express distinct constants. For $1 \leq s < \infty$, the usual Lebesgue space is endowed with the norm

$$\|u\|_s := \left(\int_{\mathbb{R}^N} |u|^s dx \right)^{1/s}. \quad (12)$$

The paper is organized as follows. In Section 2, we reformulate our problem into a new one which has an associated functional well defined in a suitable space and present some preliminary results. In Section 3, we introduce some notions and results of nonsmooth critical point theory and we show that the functional I satisfies $(PS)_C$ condition. In Section 4, we complete the proof of Theorem 1.

2. The Dual Variational Framework and Preliminary Results

The energy functional corresponding to problem (1) is defined as follows:

$$\begin{aligned} J(u) = & \frac{a}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{b}{4} \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^2 \\ & + a \int_{\mathbb{R}^N} u^2 |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x) u^2 dx \\ & - \int_{\mathbb{R}^N} G(x, u) dx. \end{aligned} \quad (13)$$

It should be pointed out that the main difficulty in treating this class of quasilinear equations in $\mathbb{R}^N$ is the lack of compactness and the second-order nonhomogeneous term $au\Delta u^2$ which prevents us from working directly in a classical function space. J is not defined in $H_V^1(\mathbb{R}^N)$; thus we may not apply directly the variational method to study (1). To move these obstacles, we make the change of variable (see [12, 14]); that is, we consider $v = f^{-1}(u)$, where f is defined by

$$f'(t) = \frac{1}{\sqrt{1 + 2f^2(t)}}, \quad \text{on } [0, +\infty), \quad (14)$$

$$f(t) = -f(-t), \quad \text{on } (-\infty, 0].$$

In order to prove our main result, we need some further properties of the function f .

Lemma 2. The function f has the following properties:

- (1) f is uniquely defined, C^∞ , and invertible;
- (2) $|f'(t)| \leq 1$ for all $t \in \mathbb{R}$;
- (3) $|f(t)| \leq |t|$ for all $t \in \mathbb{R}$;
- (4) $\lim_{t \rightarrow 0} (f(t)/t) = 1$; $\lim_{t \rightarrow +\infty} (f(t)/\sqrt{t}) = 2^{1/4}$;
- (5) $|f(t)| \leq 2^{1/4}|t|^{1/2}$ for all $t \in \mathbb{R}$;

(6) $(1/2)f(t) \leq tf'(t) \leq f(t)$, for all $t \geq 0$; $f(t) < tf'(t) < (1/2)f(t)$ for all $t < 0$;

(7) there exists a positive constant C such that

$$|f(t)| \geq \begin{cases} C|t|, & \text{if } |t| \leq 1, \\ C|t|^{1/2}, & \text{if } |t| \geq 1; \end{cases} \quad (15)$$

(8) $|f(t)f'(t)| \leq 1/\sqrt{2} < 1$ for all $t \in \mathbb{R}$;

(9) there exist positive constants C_1, C_2 such that

$$|t| \leq C_1 |f(t)| + C_2 |f(t)|^2 \quad \forall t \in \mathbb{R}; \quad (16)$$

(10) $f^2(t)$ is convex and $f^2(\lambda t) \leq \lambda^2 f^2(t)$ for all $t \in \mathbb{R}$ and $\lambda \geq 1$;

(11) $\forall \xi > 0, \exists C(\xi) > 0$, s. t. $f^2(\xi t) \leq C(\xi) f^2(t)$;

(12) $f(t)/t$ is decreasing for $t > 0$.

After this change of variables, the functional

$$\begin{aligned} I(v) &= J(f(v)) \\ &= \frac{a}{2} \int_{\mathbb{R}^N} |\nabla f(v)|^2 dx + a \int_{\mathbb{R}^N} f^2(v) |\nabla f(v)|^2 dx \\ &\quad + \frac{b}{4} \left(\int_{\mathbb{R}^N} |\nabla f(v)|^2 dx \right)^2 \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^N} V(x) f^2(v) dx - \int_{\mathbb{R}^N} G(x, f(v)) dx \\ &= \frac{a}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{b}{4} \left(\int_{\mathbb{R}^N} |\nabla f(v)|^2 dx \right)^2 \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^N} V(x) f^2(v) dx - \int_{\mathbb{R}^N} G(x, f(v)) dx \end{aligned} \quad (17)$$

is well defined on

$$E = \left\{ v \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x) f^2(v) dx < \infty \right\} \quad (18)$$

which is a Banach space endowed with the norm

$$\begin{aligned} \|v\| &= \|\nabla v\|_2 + \inf_{\lambda > 0} \frac{1}{\lambda} \left\{ 1 + \int_{\mathbb{R}^N} V(x) f^2(\lambda v) dx \right\} \\ &:= \|\nabla v\|_2 + \|v\|. \end{aligned} \quad (19)$$

A standard argument which is similar to that in [12] shows that if $v \in E$ is a critical point of the functional I , then $u = f(v) \in E$ and u is a weak solution of (1).

We have the following result with respect to the space E . Its proof can be found in [13, 14].

Proposition 3. (1) There exists a positive C such that, for all $v \in E$,

$$\frac{\int_{\mathbb{R}^N} V(x) f^2(v) dx}{1 + \left(\int_{\mathbb{R}^N} V(x) f^2(v) dx \right)^{1/2}} \leq C \|v\|. \quad (20)$$

(2) If $v_n \rightarrow v$ in E , then

$$\begin{aligned} \int_{\mathbb{R}^N} V(x) |f^2(v_n) - f^2(v)| dx &\rightarrow 0, \\ \int_{\mathbb{R}^N} V(x) |f(v_n) - f(v)|^2 dx &\rightarrow 0. \end{aligned} \quad (21)$$

(3) If $v_n(x) \rightarrow v(x)$ a.e. $x \in \mathbb{R}^N$ and

$$\int_{\mathbb{R}^N} V(x) f^2(v_n) dx \rightarrow \int_{\mathbb{R}^N} V(x) f^2(v) dx, \quad (22)$$

then

$$\begin{aligned} \|v_n - v\| &\rightarrow 0, \text{ i.e. } \inf_{\lambda > 0} \frac{1}{\lambda} \left[1 + \int_{\mathbb{R}^N} V(x) f^2(\lambda(v_n - v)) dx \right] \\ &\rightarrow 0. \end{aligned} \quad (23)$$

(4) If $v_n \rightarrow v$ in E , then $f(v_n) \rightarrow f(v)$ in $L^r(\mathbb{R}^N)$, for $2 \leq r \leq 2^*$.

(5) Under assumption (V), the embedding $E \hookrightarrow L^s(\mathbb{R}^N)$ is compact for $2 \leq s < 2^*$, and the embedding $E \hookrightarrow L^s(\mathbb{R}^N)$ is continuous for $2 \leq s \leq 2^*$.

(6) The embedding $E \hookrightarrow H^1(\mathbb{R}^N)$ is continuous. Moreover, $C_0^\infty(\mathbb{R}^N)$ is dense in E .

Lemma 4. Assume that conditions (V), (g_2) , and (g_3) hold; then, one has for I the following assertions:

(1) $I : E \rightarrow \mathbb{R}$ is continuous.

(2) For every $v \in E$ and $\varphi \in E \cap L^\infty(\mathbb{R}^N)$, the derivation of I in the direction φ at v exists and will be denoted by

$$\begin{aligned} \langle I'(v), \varphi \rangle &= a \int_{\mathbb{R}^N} \nabla v \nabla \varphi dx \\ &\quad + b \left(\int_{\mathbb{R}^N} |f'(v)|^2 |\nabla v|^2 dx \right) \\ &\quad \cdot \left[\int_{\mathbb{R}^N} |f'(v)|^2 \nabla v \nabla \varphi dx \right. \\ &\quad + \int_{\mathbb{R}^N} f'(v) f''(v) |\nabla v|^2 \varphi dx \left. \right] \\ &\quad + \int_{\mathbb{R}^N} V(x) f(v) f'(v) \varphi dx \\ &\quad - \int_{\mathbb{R}^N} g(x, f(v)) f'(v) \varphi dx. \end{aligned} \quad (24)$$

(3) The map $\langle v, \varphi \rangle \rightarrow \langle I'(v), \varphi \rangle$ satisfies the following:

- (i) $\langle I'(v), \varphi \rangle$ is linear in $\varphi \in E \cap L^\infty(\mathbb{R}^N)$;
- (ii) $\langle I'(v), \varphi \rangle$ is continuous in v ; that is, if $v_n \rightarrow v$ in E , then $\langle I'(v_n), \varphi \rangle \rightarrow \langle I'(v), \varphi \rangle$ as $n \rightarrow \infty$.

Remark 5. Consider the following:

$$\forall \varphi \in E \cap L^\infty(\mathbb{R}^N),$$

$$\begin{aligned} \langle I'(v), \varphi \rangle &= a \int_{\mathbb{R}^N} \nabla v \nabla \varphi \, dx + \int_{\mathbb{R}^N} V(x) f(v) f'(v) \varphi \, dx \\ &\quad - \int_{\mathbb{R}^N} g(x, f(v)) f'(v) \varphi \, dx + b \left(\int_{\mathbb{R}^N} \frac{|\nabla v|^2}{1 + 2f^2(v)} \, dx \right) \\ &\quad \cdot \left(\int_{\mathbb{R}^N} \frac{\nabla v \nabla \varphi \cdot (1 + 2f^2(v)) - 2|\nabla v|^2 f(v) f'(v) \varphi}{[1 + 2f^2(v)]^2} \, dx \right). \end{aligned} \quad (25)$$

Proof. Let $\{v_n\} \subset E$ be a sequence such that $v_n \rightarrow v$ in E . By Proposition 3, we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} V(x) f^2(v_n) \, dx = \int_{\mathbb{R}^N} V(x) f^2(v) \, dx. \quad (26)$$

Moreover, $v_n \rightarrow v$ in $L^s(\mathbb{R}^N)$ for $2 \leq s \leq 2^*$, $\nabla v_n \rightarrow \nabla v$ in $L^2(\mathbb{R}^N)$, and, up to subsequence, $v_n(x) \rightarrow v(x)$, $\nabla v_n(x) \rightarrow \nabla v(x)$ a.e. $x \in \mathbb{R}^N$.

$\forall \varepsilon > 0$, by (g_2) and (g_3) , there exists $C_\varepsilon > 0$ such that

$$\begin{aligned} |g(x, t)| &\leq \varepsilon |t| + C_\varepsilon |t|^{p-1}, \\ |G(x, t)| &\leq \varepsilon t^2 + C_\varepsilon |t|^p, \end{aligned} \quad (27)$$

$$\forall (x, t) \in \mathbb{R}^N \times \mathbb{R}.$$

Thus, by Lemma 2 and (27),

$$\begin{aligned} |G(x, f(v_n)) - G(x, f(v))| \\ \leq \varepsilon (|v_n|^2 + |v|^2) + C \left(|v_n|^{p/2} + |v|^{p/2} \right). \end{aligned} \quad (28)$$

By Lemma A.1 of [2] and Lebesgue's theorem, one has

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} G(x, f(v_n)) \, dx = \int_{\mathbb{R}^N} G(x, f(v)) \, dx. \quad (29)$$

By Lemma 2 and Lebesgue's theorem,

$$\begin{aligned} &\left| \left(\int_{\mathbb{R}^N} |\nabla f(v_n)|^2 \, dx \right)^2 - \left(\int_{\mathbb{R}^N} |\nabla f(v)|^2 \, dx \right)^2 \right| \\ &\leq C \left| \int_{\mathbb{R}^N} |\nabla f(v_n)|^2 \, dx - \int_{\mathbb{R}^N} |\nabla f(v)|^2 \, dx \right| \\ &= C \left| \int_{\mathbb{R}^N} |f'(v_n)|^2 (|\nabla v_n|^2 - |\nabla v|^2) \, dx \right. \\ &\quad \left. + \int_{\mathbb{R}^N} (|f'(v_n)|^2 - |f'(v)|^2) |\nabla v|^2 \, dx \right| \leq o_n(1); \end{aligned} \quad (30)$$

thus

$$\lim_{n \rightarrow \infty} \left(\int_{\mathbb{R}^N} |\nabla f(v_n)|^2 \, dx \right)^2 = \left(\int_{\mathbb{R}^N} |\nabla f(v)|^2 \, dx \right)^2. \quad (31)$$

Consequently, $I(v_n) \rightarrow I(v)$ and I is continuous in E . Similarly, it can be proved that $\langle I'(v), \varphi \rangle$ is continuous in v for each $\varphi \in E \cap L^\infty(\mathbb{R}^N)$. The proof of (24) is standard by using conditions (V) , (g_2) , and (g_3) . It is obvious that $\langle I'(v), \varphi \rangle$ is linear in $\varphi \in E \cap L^\infty(\mathbb{R}^N)$. The proof is completed. $\square$

3. Nonsmooth Critical Framework

Let us begin by recalling some notions and results of nonsmooth critical point theory (see [21, 22]).

In the following, X will denote a metric space endowed with the metric d .

Definition 6. Let $f : X \rightarrow \mathbb{R}$ be a continuous function and let $u \in X$. One denotes by $|df|(u)$ the supremum of the σ 's in $[0, +\infty)$ such that there exist $\delta > 0$ and a continuous map

$$H : B(u, \delta) \times [0, \delta] \rightarrow X \quad (32)$$

satisfying

$$\begin{aligned} d(H(v, t), v) &\leq t, \\ f(H(v, t)) &\leq f(v) - \sigma t \end{aligned} \quad (33)$$

for all $(v, t) \in B(u, \delta) \times [0, \delta]$, where $B(u, \delta)$ is the open ball of center $u \in X$ and of radius δ . The extended real number $|df|(u)$ is called the weak slope of f at u .

If X is a Finsler manifold of class C^1 and $f \in C^1$, it turns out that $|df|(u) = \|f'(u)\|$.

The function $|df| : X \rightarrow [0, +\infty]$ is lower semicontinuous.

Definition 7. Let $f : X \rightarrow \mathbb{R}$ be a continuous function. A point $u \in X$ is called a critical point of f if $|df|(u) = 0$. A real number c is called a critical value of f if there exists a $u \in X$ such that $f(u) = c$ and $|df|(u) = 0$.

Definition 8. Let $f : X \rightarrow \mathbb{R}$ be a continuous function and let $c \in \mathbb{R}$. One says that f satisfies the $(PS)_c$ condition if each sequence $\{u_n\} \subset X$ with $f(u_n) \rightarrow c$ and $|df|(u_n) \rightarrow 0$ has a convergent subsequence.

We can state a generalized version of the symmetric Mountain Pass Theorem for the case of continuous functionals.

Theorem 9 (see [21]). Let E be an infinite-dimensional Banach space and let $f : E \rightarrow \mathbb{R}$ be continuous, even, and satisfying $(PS)_c$ condition for every $c \in \mathbb{R}$. Assume the following:

- (i) There exist $\rho > 0$, $\beta > f(0)$, and a subspace $V \subset E$ of finite codimension such that

$$\forall u \in V : \|u\| = \rho \implies f(u) \geq \beta. \quad (34)$$

- (ii) For every finite-dimensional subspace $W \subset E$, there exists $R > 0$ such that

$$\forall u \in W : \|u\| \geq R \implies f(u) \leq f(0). \quad (35)$$

Then there exists a sequence $\{c_n\}$ of critical values of f with $c_n \rightarrow \infty$ as $n \rightarrow \infty$.

Now consider the functional I given in the previous section.

Lemma 10. Consider $|dI|(v) \geq \sup\{\langle I'(v), \varphi \rangle : \varphi \in E \cap L^\infty(\mathbb{R}^N), \|\varphi\| = 1\}$.

Proof. We use a similar argument in the proof of Theorem 1.5 in [21]. Let $v \in E$ and

$$\begin{aligned} \Psi(v) \\ = \sup \{ \langle I'(v), \varphi \rangle : \varphi \in E \cap L^\infty(\mathbb{R}^N), \|\varphi\| = 1 \}. \end{aligned} \quad (36)$$

If $\Psi(v) = 0$, the conclusion is obvious. Otherwise, take $\sigma > 0$ such that $\Psi(v) > \sigma$. Then there exists a point $\varphi \in E \cap L^\infty(\mathbb{R}^N)$ with $\|\varphi\| = 1$ and $\langle I'(v), \varphi \rangle > \sigma$. By Lemma 4, there exist

$\delta_0 > 0$ such that $\langle I'(u), \varphi \rangle > \sigma$ for every $u \in B(v, \delta_0)$. Let $\delta = \delta_0/2$ and $H(u, t) = u - t\varphi$. It is obvious that $H : B(u, \delta) \times [0, \delta] \rightarrow E$ is continuous,

$$\begin{aligned} \|H(u, t) - u\| &\leq t, \\ I(H(u, t)) &= I(H(u, 0)) \\ &\quad - t \int_0^1 \langle I'(H(u, st)), \varphi \rangle ds \\ &\leq I(u) - \sigma t \end{aligned} \quad (37)$$

for $(u, t) \in B(v, \delta) \times [0, \delta]$. Hence, by Definition 6, $|dI|(v) \geq \sigma$ and $|dI|(v) \geq \Psi(v)$ by the arbitrariness of σ . The proof is completed. $\square$

Lemma 11. Under assumptions (g_2) , (g_3) , and (V) , if $v \in E$ satisfies $\langle I'(v), \varphi \rangle = 0$, $\forall \varphi \in E \cap L^\infty(\mathbb{R}^N)$, then $v \in L^\infty(\mathbb{R}^N)$.

Proof. By (25), we have

$$\begin{aligned} &a \int_{\mathbb{R}^N} \nabla v \cdot \nabla \varphi \, dx + b \left(\int_{\mathbb{R}^N} \frac{|\nabla v|^2}{1 + 2f^2(v)} \, dx \right) \\ &\quad \cdot \left(\int_{\mathbb{R}^N} \frac{\nabla v \cdot \nabla \varphi \cdot (1 + 2f^2(v)) - 2|\nabla v|^2 f(v) f'(v) \varphi}{[1 + 2f^2(v)]^2} \, dx \right) \\ &= \int_{\mathbb{R}^N} g(x, f(v)) f'(v) \varphi \, dx \\ &\quad - \int_{\mathbb{R}^N} V(x) f(v) f'(v) \varphi \, dx \end{aligned} \quad (38)$$

for all $\varphi \in E \cap L^\infty(\mathbb{R}^N)$.

For any $T > 0$, let $v^T = v$ if $|v| \leq T$ and $v^T = \text{sign}(v)T$ if $|v| \geq T$. For any $r > 0$, take $\varphi = v|v^T|^{2r}$ in the above equality; then it can be deduced from (38) that

$$\begin{aligned} &a \int_{\mathbb{R}^N} \nabla v \cdot \nabla (v|v^T|^{2r}) \, dx \\ &\quad + b \left(\int_{\mathbb{R}^N} \frac{|\nabla v|^2}{1 + 2f^2(v)} \, dx \right) \left(\int_{\mathbb{R}^N} \frac{\nabla v \cdot \nabla (v|v^T|^{2r}) (1 + 2f^2(v)) - 2|\nabla v|^2 f(v) f'(v) \cdot v|v^T|^{2r}}{[1 + 2f^2(v)]^2} \, dx \right) \\ &= \int_{\mathbb{R}^N} g(x, f(v)) f'(v) \cdot v|v^T|^{2r} \, dx - \int_{\mathbb{R}^N} V(x) f(v) f'(v) \cdot v|v^T|^{2r} \, dx. \end{aligned} \quad (39)$$

By Lemma 2 and the Sobolev embedding theorem,

$$\begin{aligned} &a \int_{\mathbb{R}^N} \nabla v \cdot \nabla (v|v^T|^{2r}) \, dx \\ &\quad + b \left(\int_{\mathbb{R}^N} \frac{|\nabla v|^2}{1 + 2f^2(v)} \, dx \right) \left(\int_{\mathbb{R}^N} \frac{\nabla v \cdot \nabla (v|v^T|^{2r}) (1 + 2f^2(v)) - 2|\nabla v|^2 f(v) f'(v) \cdot v|v^T|^{2r}}{[1 + 2f^2(v)]^2} \, dx \right) \\ &\geq a \int_{\mathbb{R}^N} \nabla v \cdot \nabla (v|v^T|^{2r}) \, dx = a \int_{\mathbb{R}^N} [|\nabla v|^2 |v^T|^{2r} + 2r |v|^{2r} |\nabla v^T|^2] \, dx \end{aligned}$$

$$\begin{aligned}
&\geq \frac{a}{r+2} \int_{\mathbb{R}^N} \left[|\nabla v|^2 |v^T|^{2r} + (2r+r^2) |v|^{2r} |\nabla v^T|^2 \right] dx = \frac{a}{r+2} \int_{\mathbb{R}^N} |\nabla (v |v^T|^r)|^2 dx \\
&\geq \frac{C_1}{r+1} \left(\int_{\mathbb{R}^N} |v \cdot |v^T|^r|^{2^*} dx \right)^{2/2^*},
\end{aligned} \tag{40}$$

where C_1 is positive constant.

For any $0 < \varepsilon \leq V_0$, by (27), (V), Lemma 2, Hölder inequality, and Sobolev inequality, for simplicity, taking $q = p/2$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} g(x, f(v)) f'(v) \cdot v |v^T|^{2r} dx \\
&\quad - \int_{\mathbb{R}^N} V(x) f(v) f'(v) \cdot v |v^T|^{2r} dx \\
&\leq \int_{\mathbb{R}^N} (\varepsilon |f(v)| + C_\varepsilon |f(v)|^{p-1}) |f'(v) v| \\
&\quad \cdot |v^T|^{2r} dx - \int_{\mathbb{R}^N} V_0 |f(v) f'(v) v| \cdot |v^T|^{2r} dx \\
&\leq C \int_{\mathbb{R}^N} |f(v)|^p \cdot |v^T|^{2r} dx \leq C \int_{\mathbb{R}^N} |v|^{p/2} \\
&\quad \cdot |v^T|^{2r} dx \\
&\leq C \left[\int_{\mathbb{R}^N} (|v|^2 |v^T|^{2r})^{2^*/(2^*+2-q)} dx \right]^{(2^*+2-q)/2^*} \\
&\quad \cdot \left[\int_{\mathbb{R}^N} |v|^{2^*} dx \right]^{(q-2)/2^*} \\
&\leq C_2 \left[\int_{\mathbb{R}^N} (|v|^2 |v^T|^{2r})^{2^*/(2^*+2-q)} dx \right]^{(2^*+2-q)/2^*},
\end{aligned} \tag{41}$$

where C and C_2 are positive constants.

Set $d = 2^*/(2^* + 2 - q)$. Because $2 < q = p/2 < 2^*$, $1 < d < 2^*/2$. Moreover, by (39), (40), and (41), we have

$$\begin{aligned}
&\left(\int_{\mathbb{R}^N} |v|^{2^*} |v^T|^{2^*r} dx \right)^{1/2^*(r+1)} \\
&\leq [C(r+1)]^{1/2(r+1)} \left(\int_{\mathbb{R}^N} |v|^{2d} |v^T|^{2dr} dx \right)^{1/2d(r+1)}.
\end{aligned} \tag{42}$$

Taking the limit $T \rightarrow \infty$ in (42),

$$\begin{aligned}
&\left(\int_{\mathbb{R}^N} |v|^{2^*(r+1)} dx \right)^{1/2^*(r+1)} \\
&\leq [C(r+1)]^{1/2(r+1)} \left(\int_{\mathbb{R}^N} |v|^{2d(r+1)} dx \right)^{1/2d(r+1)}.
\end{aligned} \tag{43}$$

Set $r_0 = 0$ and $2d(r_{k+1}+1) = 2^*(r_k+1)$, $\forall k \in \mathbb{N}$. Then $r_k \rightarrow \infty$ as $k \rightarrow \infty$ and

$$\begin{aligned}
&\left(\int_{\mathbb{R}^N} |v|^{2d(r_k+1)} dx \right)^{1/2d(r_k+1)} \\
&\leq \prod_{i=1}^k [C(r_i+1)]^{1/2(r_i+1)} \left(\int_{\mathbb{R}^N} |v|^{2^*} dx \right)^{1/2^*}.
\end{aligned} \tag{44}$$

Because

$$\begin{aligned}
&\prod_{i=1}^k [C(r_i+1)]^{1/2(r_i+1)} \\
&= \exp \left\{ \sum_{i=1}^k \frac{\ln C ((2^*/2) d^{-1})^i}{2 ((2^*/2) d^{-1})^i} \right\} \\
&= \exp \left\{ \sum_{i=1}^k \left[\frac{\ln C}{2 ((2^*/2) d^{-1})^i} + \frac{i \ln ((2^*/2) d^{-1})}{2 ((2^*/2) d^{-1})^i} \right] \right\}
\end{aligned} \tag{45}$$

is convergent as $k \rightarrow \infty$, let $C_k = \prod_{i=1}^k [C(r_i+1)]^{1/2(r_i+1)}$; then $C_k \rightarrow C_\infty > 0$ as $k \rightarrow \infty$. Hence

$$\|v\|_{L^{2d(r_k+1)}(\mathbb{R}^N)} \leq C_k \|v\|_{L^{2^*}(\mathbb{R}^N)}. \tag{46}$$

Let $k \rightarrow \infty$; then, we have

$$\|v\|_{L^\infty(\mathbb{R}^N)} \leq C_\infty \|v\|_{L^{2^*}(\mathbb{R}^N)} \leq C, \tag{47}$$

where the positive constant C is independent of k . The proof is completed. $\square$

Corollary 12. If $v \in E$ is a critical point of I , then $v \in L^\infty(\mathbb{R}^N)$.

Proof. The proof is completed by Lemmas 10 and 11. $\square$

Lemma 13. $\forall c \in \mathbb{R}$, under assumptions (g_2) , (g_3) , and (g_4) , the functional I satisfies $(PS)_c$ condition.

Proof. Let $\{v_n\} \subset E$ be a sequence with $I(v_n) \rightarrow c$ and $|dI|(v_n) \rightarrow 0$. Then, it follows from Lemma 10 that

$$\begin{aligned}
&\delta_n = \sup \{ \langle I'(v_n), \varphi \rangle : \varphi \in E \cap L^\infty(\mathbb{R}^N), \|\varphi\| = 1 \} \\
&\rightarrow 0.
\end{aligned} \tag{48}$$

Moreover,

$$|\langle I'(v_n), \varphi \rangle| \leq \delta_n \|\varphi\|, \quad \forall \varphi \in E \cap L^\infty(\mathbb{R}^N). \tag{49}$$

Note that

$$\begin{aligned} \langle I'(v_n), \varphi \rangle &= a \int_{\mathbb{R}^N} \nabla v_n \nabla \varphi \, dx + \int_{\mathbb{R}^N} V(x) f(v_n) f'(v_n) \varphi \, dx - \int_{\mathbb{R}^N} g(x, f(v_n)) f'(v_n) \varphi \, dx \\ &\quad + b \left(\int_{\mathbb{R}^N} \frac{|\nabla v_n|^2}{1 + 2f^2(v_n)} \, dx \right) \left(\int_{\mathbb{R}^N} \frac{\nabla v_n \nabla \varphi \cdot (1 + 2f^2(v_n)) - 2|\nabla v_n|^2 f(v_n) f'(v_n) \varphi}{[1 + 2f^2(v_n)]^2} \, dx \right). \end{aligned} \quad (50)$$

Letting $\varphi = w_n = f(v_n)/f'(v_n) = \sqrt{1 + 2f^2(v_n)} f(v_n)$ and

$$|\nabla w_n| = \left(1 + \frac{2f^2(v_n)}{1 + 2f^2(v_n)} \right) |\nabla v_n| \leq 2 |\nabla v_n|, \quad (51)$$

we deduce that $\|w_n\| \leq C \|v_n\|$ for some positive constant C . From (49) and (50), we have

$$\begin{aligned} C\delta_n \|v_n\| &\geq a \int_{\mathbb{R}^N} \left(1 + \frac{2f^2(v_n)}{1 + 2f^2(v_n)} \right) |\nabla v_n|^2 \, dx \\ &\quad + b \left(\int_{\mathbb{R}^N} \frac{|\nabla v_n|^2}{1 + 2f^2(v_n)} \, dx \right)^2 \\ &\quad + \int_{\mathbb{R}^N} V(x) f^2(v_n) \, dx \\ &\quad - \int_{\mathbb{R}^N} g(x, f(v_n)) f(v_n) \, dx \end{aligned} \quad (52)$$

for each n . Since $I(v_n) \rightarrow c$, by (g_4) and Lemma 2, $\forall \lambda > 0$, we have

$$\begin{aligned} c + o_n(1) + C\delta_n \|v_n\| &\geq I(v_n) - \frac{1}{\mu} \langle I'(v_n), w_n \rangle \\ &= a \int_{\mathbb{R}^N} \left[\frac{1}{2} - \frac{1}{\mu} \left(1 + \frac{2f^2(v_n)}{1 + 2f^2(v_n)} \right) \right] |\nabla v_n|^2 \, dx \\ &\quad + \left(\frac{1}{4} - \frac{1}{\mu} \right) b \left(\int_{\mathbb{R}^N} |f'(v_n)|^2 |\nabla v_n|^2 \, dx \right)^2 + \left(\frac{1}{2} - \frac{1}{\mu} \right) \int_{\mathbb{R}^N} V(x) f^2(v_n) \, dx \\ &\quad + \int_{\mathbb{R}^N} \left[\frac{1}{\mu} g(x, f(v_n)) f(v_n) - G(x, f(v_n)) \right] \, dx \quad (53) \\ &\geq a \left(\frac{1}{2} - \frac{2}{\mu} \right) \int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx + \left(\frac{1}{2} - \frac{1}{\mu} \right) \int_{\mathbb{R}^N} V(x) \\ &\quad \cdot f^2(v_n) \, dx \geq C_1 \left[\int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx + 1 \right. \\ &\quad \left. + \int_{\mathbb{R}^N} V(x) f^2(v_n) \, dx \right] - C_1 \\ &\quad + \inf_{\lambda > 0} \frac{1}{\lambda} \left\{ 1 + \int_{\mathbb{R}^N} V(x) f^2(\lambda v) \, dx \right\} - C_1, \end{aligned}$$

where $C_1 > 0$. Their estimates imply that v_n is bounded in E .

Hence, up to a subsequence, $v_n \rightharpoonup v$ in E . Because $\delta_n \rightarrow 0$, from (49) and (50),

$$\begin{aligned} o_n(1) &+ \int_{\mathbb{R}^N} g(x, f(v_n)) f(v_n) \, dx \\ &= a \int_{\mathbb{R}^N} \left(1 + \frac{2f^2(v_n)}{1 + 2f^2(v_n)} \right) |\nabla v_n|^2 \, dx \\ &\quad + b \left(\int_{\mathbb{R}^N} |f'(v_n)|^2 |\nabla v_n|^2 \, dx \right)^2 \\ &\quad + \int_{\mathbb{R}^N} V(x) f^2(v_n) \, dx. \end{aligned} \quad (54)$$

Because $\langle I'(v), \varphi \rangle = 0$ for each $\varphi \in E \cap L^\infty(\mathbb{R}^N)$, set $w = f(v)/f'(v)$; it follows from Lemma 11 that $\langle I'(v), w \rangle = 0$. Consequently,

$$\begin{aligned} &\int_{\mathbb{R}^N} g(x, f(v)) f(v) \, dx \\ &= a \int_{\mathbb{R}^N} \left(1 + \frac{2f^2(v)}{1 + 2f^2(v)} \right) |\nabla v|^2 \, dx \\ &\quad + b \left(\int_{\mathbb{R}^N} |f'(v)|^2 |\nabla v|^2 \, dx \right)^2 \\ &\quad + \int_{\mathbb{R}^N} V(x) f^2(v) \, dx. \end{aligned} \quad (55)$$

By Proposition 3, we get $v_n \rightarrow v$ in $L^s(\mathbb{R}^N)$ for $2 \leq s < 2^*$. Using (27), Lemma 2, and Lebesgue's theorem, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^N} g(x, f(v_n)) f(v_n) \, dx \\ &\longrightarrow \int_{\mathbb{R}^N} g(x, f(v)) f(v) \, dx. \end{aligned} \quad (56)$$

By assumption (V), Lemma 2, the Fatou Lemma, and the Lebesgue dominated convergence theorem, one has

$$\begin{aligned}
\liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx &\geq \int_{\mathbb{R}^N} |\nabla v|^2 dx, \\
\liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} \frac{2f^2(v_n)}{1+2f^2(v_n)} |\nabla v_n|^2 dx \\
&\geq \int_{\mathbb{R}^N} \frac{2f^2(v)}{1+2f^2(v)} |\nabla v|^2 dx, \\
\liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} |f'(v_n)|^2 |\nabla v_n|^2 dx \\
&\geq \int_{\mathbb{R}^N} |f'(v)|^2 |\nabla v|^2 dx, \\
\liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} V(x) f^2(v_n) dx &\geq \int_{\mathbb{R}^N} V(x) f^2(v) dx.
\end{aligned} \tag{57}$$

These imply that

$$\int_{\mathbb{R}^N} |\nabla v_n|^2 dx \longrightarrow \int_{\mathbb{R}^N} |\nabla v|^2 dx, \tag{58}$$

$$\int_{\mathbb{R}^N} V(x) f^2(v_n) dx \longrightarrow \int_{\mathbb{R}^N} V(x) f^2(v) dx. \tag{59}$$

Notice that $\nabla v_n \rightharpoonup \nabla v$ in $L^2(\mathbb{R}^N)$; then, we conclude that $\nabla v_n \rightarrow \nabla v$ in $L^2(\mathbb{R}^N)$. By Proposition 3 and (59), we have

$$\inf_{\lambda > 0} \frac{1}{\lambda} \left[1 + \int_{\mathbb{R}^N} V(x) f^2(\lambda(v_n - v)) dx \right] \longrightarrow 0. \tag{60}$$

We obtain $v_n \rightarrow v$ in E ; that is, the functional I satisfies $(PS)_C$ condition. $\square$

4. Proof of the Main Result

The following lemma implies that I possesses the symmetric Mountain Pass Geometry.

Lemma 14. *Suppose that (V), (g_2) , and (g_3) are satisfied. Then the functional I satisfies condition (i) of Theorem 9.*

Proof. Let $S(\rho) = \{v \in E : \|v\| = \rho\}$, $0 < \rho < 4$. If $\|v\| \geq (1/2)\rho$, by Lemma 2, we have

$$\begin{aligned}
\frac{1}{2}\rho &\leq \inf_{\lambda > 0} \frac{1}{\lambda} \left\{ 1 + \int_{\mathbb{R}^N} V(x) f^2(\lambda v) dx \right\} \\
&\leq \frac{1}{4}\rho + \frac{1}{4}\rho \int_{\mathbb{R}^N} V(x) f^2\left(\frac{4}{\rho}v\right) dx \\
&\leq \frac{1}{4}\rho + \frac{4}{\rho} \int_{\mathbb{R}^N} V(x) f^2(v) dx,
\end{aligned} \tag{61}$$

which implies that

$$\int_{\mathbb{R}^N} (|\nabla v|^2 + V(x) f^2(v)) dx \geq \frac{1}{16}\rho^2 \tag{62}$$

for $\|v\| \geq (1/2)\rho$. Note that $\|\nabla v\|_2 \geq (1/2)\rho$ if $v \in S(\rho)$ with $\|v\| < (1/2)\rho$. Hence

$$\inf_{v \in S(\rho)} \int_{\mathbb{R}^N} (|\nabla v|^2 + V(x) f^2(v)) dx \geq \frac{1}{16}\rho^2. \tag{63}$$

By (V), (27), Lemma 2, and Proposition 3, let $0 < \varepsilon \leq (1/4)V_0$; then, we have

$$\begin{aligned}
I(v) &\geq \frac{a}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x) f^2(v) dx \\
&\quad - \int_{\mathbb{R}^N} (\varepsilon f^2(v) + C_\varepsilon |v|^p) dx \\
&\geq \frac{a}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{4} \int_{\mathbb{R}^N} V(x) f^2(v) dx \\
&\quad - C \int_{\mathbb{R}^N} |v|^{p/2} dx \\
&\geq \frac{1}{16} \min \left\{ \frac{a}{2}, \frac{1}{4} \right\} \rho^2 - C\rho^{p/2}
\end{aligned} \tag{64}$$

for $v \in S(\rho)$ and $\rho < 1$. Hence condition (i) in Theorem 9 holds for small $\rho > 0$. $\square$

Lemma 15. *Assume that (V), (g_2) , and (g_4) hold; then the functional I satisfies condition (ii) of Theorem 9.*

Proof. By Lemma 2, for $m > 4$, $\exists C > 0$, such that

$$|t|^{m/2} \leq t^2 + C |f(t)|^m, \quad \forall t \in \mathbb{R}. \tag{65}$$

By (27) and (g_4) , $\exists C_1 > 0$ and $C_2 > 0$; one has

$$G(x, s) \geq C_1 |s|^\mu - C_2 s^2, \quad \forall (x, s) \in \mathbb{R}^N \times \mathbb{R}. \tag{66}$$

By Lemma 2, we have

$$\begin{aligned}
G(x, f(s)) &\geq C_1 |f(s)|^\mu - C_2 f^2(s) \\
&\geq C_3 |s|^{\mu/2} - C_4 s^2
\end{aligned} \tag{67}$$

$\forall (x, s) \in \mathbb{R}^N \times \mathbb{R}$, $C_3 > 0$, $C_4 > 0$. Let W be any finite-dimensional subspace of E , $\forall v \in W$; by (65), (67), and Lemma 2,

$$\begin{aligned}
I(v) &\leq C \left(\int_{\mathbb{R}^N} |\nabla v|^2 dx + \int_{\mathbb{R}^N} V(x) v^2 dx \right) \\
&\quad + C \int_{\mathbb{R}^N} |v|^2 dx - \int_{\mathbb{R}^N} G(x, f(v)) dx \\
&\leq C \int_{\mathbb{R}^N} (|\nabla v|^2 + V(x) v^2) dx + C \int_{\mathbb{R}^N} |v|^2 dx \\
&\quad - C_3 \int_{\mathbb{R}^N} |v|^{\mu/2} dx + C_4 \int_{\mathbb{R}^N} v^2 dx.
\end{aligned} \tag{68}$$

Since all norms are equivalent in a finite-dimensional space W , $I(v) \rightarrow -\infty$ as $\|v\| \rightarrow \infty$. So the conclusion follows. $\square$

Proof of Theorem 1. The functional I is evidently even. Therefore, Theorem 9 implies that I has a critical sequence $\{v_n\} \subset E$ such that $I(v_n) \rightarrow \infty$ as $n \rightarrow \infty$. Note that $I(v_n) = I(f(v_n))$; the proof is completed. $\square$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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