Research Article

Existence of Solutions for Boundary Value Problem of a Caputo Fractional Difference Equation

Zhiping Liu,¹ Shugui Kang,² Huiqin Chen,² Jianmin Guo,² Yaqiong Cui,² and Caixia Guo²

¹Department of Mathematics, North University of China, Shanxi, Taiyuan 030051, China

²School of Mathematics and Computer Sciences, Shanxi Datong University, Datong, Shanxi 037009, China

Correspondence should be addressed to Shugui Kang; dtkangshugui@126.com

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We investigate the existence of solutions for a Caputo fractional difference equation boundary value problem. We use Schauder fixed point theorem to deduce the existence of solutions. The proofs are based upon the theory of discrete fractional calculus. We also provide some examples to illustrate our main results.

1. Introduction

The theory of fractional difference equations and their applications have been receiving intensive attention. In the last ten years, new research achievements kept emerging (see [1–16] and the references therein). In [1–3], the authors introduced fractional sum and difference operators, studied their behavior, and developed a complete theory governing their compositions. Abdeljawad [3] considered the initial value problem to Caputo fractional difference equation. The authors explored BVP of fractional difference equation, and they deduced the existence of one or more positive solutions in [4–7]. Abdeljawad and Baleanu [8] introduced the fractional differences and integration by parts. In [9] the authors studied the stability of discrete nonautonomous systems within the frame of the Caputo fractional difference by using Lyapunov's direct method. They discussed the conditions for uniform stability, uniform asymptotic stability, and uniform global stability.

In [10, 11], the authors studied multiple solutions to fractional difference boundary value problems by means of Krasnosel'skii theorem and Schauder fixed point theorem. They obtained sufficient conditions of the existence of two positive solutions for the boundary value problem of fractional difference equations depending on parameters in [12].

Chen et al. [13] presented the existence of at least one positive solution for Caputo fractional boundary value problems. In [14], Kang et al. discussed existence of positive solutions for a system of Caputo fractional difference equations depending on parameters on the basis of [13].

Recently, Wu and Baleanu introduced some applications of the Caputo fractional difference to discrete chaotic maps in [15, 16]. Thus, the fractional difference equation has recently attracted increasing attention from a growing number of researchers.

Following this trend, we investigate the following boundary value problem of fractional difference equation (FBVP):

$$\Delta_C^\nu y(t) = -f(t + \nu - 1, y(t + \nu - 1)), \quad (1)$$

$$y(\nu - 3) = \Delta y(b + \nu) = \Delta^2 y(\nu - 3) = 0,$$

where $t \in \{0, 1, \dots, b + 1\} := [0, b + 1]_{\mathbb{N}_0}$, $b \geq 5$, is an integer. $f : [\nu - 2, b + \nu]_{\mathbb{N}_{\nu-2}} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and f is not identically zero, $2 < \nu \leq 3$, and $\Delta_C^\nu y(t)$ is the standard Caputo difference.

The reason for studying (1) is that, in [13], Chen et al. studied similar FBVP (1) by using the cone fixed point theorem. In this note, we consider existence of solutions to

FBVP (1) by using Schauder theorem. Here, the nonnegative function f is not necessary, but f is nonnegative in [13]. We extend the domain of f .

This paper is organized as follows. Section 1 introduces the developing of fractional difference equations in a simple way. We present some necessary definitions and lemmas in Section 2. In Section 3, we prove existence of solution to FBVP (1). Finally, we provide some examples to illustrate our main results.

2. Preliminaries

We first introduce some theory about fractional sums and differences. The definitions and some basic results about fractional sums and differences can be seen in [1–7], so we omit their proof.

Definition 1 (see [3]). For any t and ν , one defines

$$t^\nu = \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)} \quad (2)$$

for which the right-hand side is defined. One appeals to the convention that if $t+1-\nu$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^\nu = 0$.

Definition 2 (see [3]). The ν th fractional sum of a function f is

$$\Delta^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{\nu-1} f(s), \quad (3)$$

$$G(t, s) = \frac{1}{\Gamma(\nu)} \begin{cases} (\nu-1)(t-\nu+3)(b+\nu-s-1)^{\nu-2} - (t-s-1)^{\nu-1}, & 0 \leq s < t-\nu+1 \leq b+1, \\ (\nu-1)(t-\nu+3)(b+\nu-s-1)^{\nu-2}, & 0 \leq t-\nu+1 \leq s \leq b+1. \end{cases} \quad (8)$$

Remark 5. Notice that $G(\nu-3, s) = 0$, $G(t, b+2) = 0$. G could be extended to $[\nu-3, b+\nu]_{\mathbb{N}_{\nu-2}} \times [0, b+2]_{\mathbb{N}_0}$, so we only discuss $(t, s) \in [\nu-2, b+\nu]_{\mathbb{N}_{\nu-2}} \times [0, b+1]_{\mathbb{N}_0}$.

Lemma 6 (see [13]). *The Green function G has the following properties:*

- (i) $G(t, s) > 0$, $(t, s) \in [\nu-2, b+\nu]_{\mathbb{N}_{\nu-2}} \times [0, b+1]_{\mathbb{N}_0}$.
- (ii) $\max_{t \in [\nu-2, b+\nu]_{\mathbb{N}_{\nu-2}}} G(t, s) = G(b+\nu, s)$, $s \in [0, b+1]_{\mathbb{N}_0}$.

Let

$$\begin{aligned} \mathcal{K} &= \{y \mid y : [\nu-3, \nu+b]_{\mathbb{N}_{\nu-3}} \rightarrow \mathbb{R}, y(\nu-3) \\ &= \Delta y(\nu+b) = \Delta^2 y(\nu-3) = 0\}. \end{aligned} \quad (9)$$

for $\nu > 0$ and $t \in \{a+\nu, a+\nu+1, \dots\} := \mathbb{N}_{a+\nu}$. One also defines the ν th Caputo fractional difference for $\nu > 0$ by

$$\begin{aligned} \Delta_C^\nu f(t) &= \Delta^{-(n-\nu)} \Delta^n f(t) \\ &= \frac{1}{\Gamma(n-\nu)} \sum_{s=a}^{t-(n-\nu)} (t-s-1)^{n-\nu-1} \Delta^n f(s), \end{aligned} \quad (4)$$

where $n-1 < \nu \leq n$.

Lemma 3 (see [3]). *Assume that $\nu > 0$ and f is defined on domains $\mathbb{N}_a$, then*

$$\Delta_{a+(n-\nu)}^{-\nu} \Delta_C^\nu f(t) = f(t) - \sum_{k=0}^{n-1} c_k (t-a)^k, \quad (5)$$

where $c_k \in \mathbb{R}$, $k = 0, 1, \dots, n-1$, and $n-1 < \nu \leq n$.

Lemma 4 (see [13]). *Let $2 < \nu \leq 3$ and $g : [\nu-2, b+\nu]_{\mathbb{N}_{\nu-2}} \rightarrow \mathbb{R}$ be given. Then the solution of FBVP,*

$$\Delta_C^\nu y(t) = -g(t+\nu-1), \quad (6)$$

$$y(\nu-3) = \Delta y(b+\nu) = \Delta^2 y(\nu-3) = 0,$$

is given by

$$y(t) = \sum_{s=0}^{b+1} G(t, s) g(s+\nu-1), \quad (7)$$

where Green's function $G : [\nu-2, b+\nu]_{\mathbb{N}_{\nu-2}} \times [0, b+1]_{\mathbb{N}_0} \rightarrow \mathbb{R}$ is defined by

It is clear that $\mathcal{K}$ is a Banach space with the norm $\|y\| = \max_{t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}} |y(t)|$. Now consider the operator T defined by

$$T(y)(t) = \sum_{s=0}^{b+1} G(t, s) f(s+\nu-1, y(s+\nu-1)), \quad (10)$$

where $t \in [\nu-2, b+\nu]_{\mathbb{N}_{\nu-2}}$, $f : [\nu-2, b+\nu]_{\mathbb{N}_{\nu-2}} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous, and f is not identically zero. It is easy to see that $y = y(t)$ is a solution of FBVP (1) if and only if $y = y(t)$ is a fixed point of T .

Lemma 7 (see [17] (Schauder fixed point theorem)). *Suppose that $\mathcal{K}$ is a Banach space. Let $\mathcal{B}$ be a bounded closed convex set of $\mathcal{K}$, and let $T : \mathcal{B} \rightarrow \mathcal{B}$ be a complete continuous operator. Then T has at least one fixed point in $\mathcal{B}$.*

3. Main Results

In this section, we give the main result of this paper. We will prove this result by using Schauder fixed point theorem and provide some examples to illustrate our main results.

For the sake of convenience, we write out the conditions as follows:

(H₁) There exists a nonnegative function $a \in C([\nu - 2, b + \nu]_{\mathbb{N}_{\nu-2}})$ and a constant c such that $|f(t, y)| \leq a(t) + c|y|^\rho$, where $c \geq 0, 0 < \rho < 1$.

(H₂) $|f(t, y)| \leq c|y|^\rho$, where $c > 0, \rho > 1$.

Theorem 8. Let $f : [\nu - 2, b + \nu]_{\mathbb{N}_{\nu-2}} \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Suppose that one of conditions (H₁) and (H₂) is satisfied. Then FBVP (1) has at least one solution.

Proof. First, suppose that condition (H₁) is satisfied. Let

$$\mathcal{B} = \{y(t) \mid y(t) \in \mathcal{K}, \|y\| \leq R, t \in [\nu - 2, b + \nu]_{\mathbb{N}_{\nu-2}}\}, \quad (11)$$

where

$$R \geq \max \left\{ 2 \sum_{s=0}^{b+1} G(b + \nu, s) |a(s + \nu - 1)|, \left[2c \frac{\Gamma(b + \nu + 1)}{\Gamma(b + 1)\Gamma(\nu)} \left(\frac{1}{\nu} + \frac{b + 3}{b + 1} \right) \right]^{1/(1-\rho)} \right\}. \quad (12)$$

Obviously $\mathcal{B}$ is the ball in the Banach space.

Now we prove that $T : \mathcal{B} \rightarrow \mathcal{B}$. For any $y \in \mathcal{B}$, then

$$\begin{aligned} |(Ty)(t)| &\leq \sum_{s=0}^{b+1} |G(t, s) [a(s + \nu - 1) + c|y(s + \nu - 1)|^\rho]| \\ &\leq \sum_{s=0}^{b+1} |G(t, s) a(s + \nu - 1)| + \sum_{s=0}^{b+1} c \|y\|^\rho \cdot |G(t, s)| \\ &\leq \sum_{s=0}^{b+1} G(b + \nu, s) |a(s + \nu - 1)| + cR^\rho \left(\sum_{s=0}^{t-\nu} |G(t, s)| \right. \\ &\quad \left. + \sum_{s=t-\nu+1}^{b+1} |G(t, s)| \right) \leq \sum_{s=0}^{b+1} G(b + \nu, s) |a(s + \nu - 1)| \\ &\quad + \frac{cR^\rho}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t - s - 1)^{\nu-1} + \frac{cR^\rho}{\Gamma(\nu)} \sum_{s=0}^{b+1} (\nu - 1) \end{aligned}$$

$$\begin{aligned} &\cdot (t - \nu + 3) (b + \nu - s - 1)^{\nu-2} \leq \sum_{s=0}^{b+1} G(b + \nu, s) \\ &\cdot |a(s + \nu - 1)| + cR^\rho \max_{t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}} \left(\frac{1}{\Gamma(\nu)} \right. \\ &\cdot \sum_{s=0}^{t-\nu} (t - s - 1)^{\nu-1} \left. \right) + cR^\rho \max_{t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}} \left(\frac{t - \nu + 3}{\Gamma(\nu - 1)} \right. \\ &\cdot \sum_{s=0}^{b+1} (b + \nu - s - 1)^{\nu-2} \left. \right). \end{aligned} \quad (13)$$

As

$$\begin{aligned} \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t - s - 1)^{\nu-1} &= \frac{1}{\Gamma(\nu)} \left[-\frac{1}{\nu} (t - s)^\nu \right]_{s=0}^{t-\nu} \\ &= \frac{t^\nu}{\Gamma(\nu + 1)} \leq \frac{\Gamma(b + \nu + 1)}{\Gamma(b + 1)\Gamma(\nu + 1)}, \\ \frac{t - \nu + 3}{\Gamma(\nu - 1)} \sum_{s=0}^{b+1} (b + \nu - s - 1)^{\nu-2} & \\ &\leq \frac{b + 3}{\Gamma(\nu - 1)} \left[-\frac{1}{\nu - 1} (b + \nu - s)^{\nu-1} \right]_{s=0}^{b+2} \\ &= \frac{(b + 3)\Gamma(b + \nu + 1)}{\Gamma(b + 2)\Gamma(\nu)}, \end{aligned} \quad (14)$$

it follows that

$$\begin{aligned} |(Ty)(t)| &\leq \sum_{s=0}^{b+1} G(b + \nu, s) |a(s + \nu - 1)| \\ &\quad + cR^\rho \frac{\Gamma(b + \nu + 1)}{\Gamma(b + 1)\Gamma(\nu + 1)} \\ &\quad + cR^\rho \frac{(b + 3)\Gamma(b + \nu + 1)}{\Gamma(b + 2)\Gamma(\nu)} \\ &= \sum_{s=0}^{b+1} G(b + \nu, s) |a(s + \nu - 1)| \\ &\quad + cR^\rho \frac{\Gamma(b + \nu + 1)}{\Gamma(b + 1)\Gamma(\nu)} \left[\frac{1}{\nu} + \frac{b + 3}{b + 1} \right] \\ &\leq \frac{R}{2} + \frac{R}{2} = R. \end{aligned} \quad (15)$$

Hence, $\|Ty\| \leq R$. Namely, $T : \mathcal{B} \rightarrow \mathcal{B}$.

Second, let condition (H₂) be valid. Choose

$$0 < R \leq \left[\frac{\Gamma(\nu)\Gamma(b + 1)}{c\Gamma(b + \nu + 1)} \left(\frac{1}{\nu} + \frac{b + 3}{b + 1} \right)^{-1} \right]^{1/(\rho-1)}. \quad (16)$$

Repeating the course of the above, we obtain $|T(y)| \leq R$. Consequently, we get $T : \mathcal{B} \rightarrow \mathcal{B}$. By means of the

continuity of G and f , it is easy to see that operator T is continuous. Next, we show that T is a completely continuous operator. For this, we take

$$M = \max_{t \in [\nu-2, b+\nu]_{\mathbb{N}_{\nu-2}}} |f(t, y(t))|. \quad (17)$$

For any $y \in \mathcal{B}$, let $t, \tau \in [\nu-2, b+\nu]_{\mathbb{N}_{\nu-2}}$ such that $t < \tau$; then

$$\begin{aligned} |(Ty)(t) - (Ty)(\tau)| &= \sum_{s=0}^{b+1} |(G(t, s) - G(\tau, s)) \\ &\quad \cdot f(s + \nu - 1, y(s + \nu - 1))| \\ &\leq M \left[\sum_{s=0}^{t-\nu} |G(t, s) - G(\tau, s)| \right. \\ &\quad + \sum_{s=t-\nu+1}^{\tau-\nu} |G(t, s) - G(\tau, s)| \\ &\quad + \sum_{s=\tau-\nu+1}^{b+1} |G(t, s) - G(\tau, s)| \Big] = \frac{M}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} |(\nu-1) \\ &\quad \cdot (t - \nu + 3)(b + \nu - s - 1)^{\nu-2} - (t - s - 1)^{\nu-1} \\ &\quad - (\nu-1)(\tau - \nu + 3)(b + \nu - s - 1)^{\nu-2} \\ &\quad + (\tau - s - 1)^{\nu-1}| + \frac{M}{\Gamma(\nu)} \sum_{s=t-\nu+1}^{\tau-\nu} |(\nu-1)(t - \nu + 3) \\ &\quad \cdot (b + \nu - s - 1)^{\nu-2} - (\nu-1)(\tau - \nu + 3) \\ &\quad \cdot (b + \nu - s - 1)^{\nu-2}| + \frac{M}{\Gamma(\nu)} \sum_{s=\tau-\nu+1}^{b+1} |(\nu-1) \\ &\quad \cdot (t - \nu + 3)(b + \nu - s - 1)^{\nu-2} - (\nu-1)(\tau - \nu + 3) \\ &\quad \cdot (b + \nu - s - 1)^{\nu-2}| \leq \frac{M}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} |(\nu-1) \\ &\quad \cdot (b + \nu - s - 1)^{\nu-2} (t - \tau) + (\tau - s - 1)^{\nu-1} \\ &\quad - (t - s - 1)^{\nu-1}| + \frac{M}{\Gamma(\nu)} \sum_{s=t-\nu+1}^{\tau-\nu} |(\nu-1) \\ &\quad \cdot (b + \nu - s - 1)^{\nu-2} (t - \tau) + (\tau - s - 1)^{\nu-1}| + \frac{M}{\Gamma(\nu)} \\ &\quad \cdot \sum_{s=\tau-\nu+1}^{b+1} |(\nu-1)(b + \nu - s - 1)^{\nu-2} (t - \tau)| \\ &= \frac{M}{\Gamma(\nu)} \left[\sum_{s=0}^{b+1} (\nu-1)(b + \nu - s - 1)^{\nu-2} |t - \tau| \right. \\ &\quad \left. + \sum_{s=0}^{\tau-\nu} (\tau - s - 1)^{\nu-1} - \sum_{s=0}^{t-\nu} (t - s - 1)^{\nu-1} \right] \end{aligned}$$

$$\begin{aligned} &= \frac{M}{\Gamma(\nu)} \left[\left[-(b + \nu - s)^{\nu-1} \right]_{s=0}^{b+1} |t - \tau| \right. \\ &\quad \left. + \left[-\frac{1}{\nu} (\tau - s)^{\nu} \right]_{s=0}^{\tau-\nu+1} - \left[-\frac{1}{\nu} (t - s)^{\nu} \right]_{s=0}^{t-\nu+1} \right] \\ &= \frac{M}{\Gamma(\nu)} \left[\left[-(\nu-2)^{\nu-1} + (b + \nu)^{\nu-1} \right] |t - \tau| + \frac{1}{\nu} (\tau^{\nu} \right. \\ &\quad \left. - t^{\nu}) \right] = \frac{M}{\Gamma(\nu)} \left[(b + \nu)^{\nu-1} |t - \tau| + \frac{1}{\nu} (\tau^{\nu} - t^{\nu}) \right]. \end{aligned} \quad (18)$$

Since functions t^{ν} , t are uniformly continuous on interval $[\nu-2, b+\nu]_{\mathbb{N}_{\nu-2}}$, we conclude that $T(\mathcal{B})$ is an equicontinuous set. Obviously, it is uniformly bounded since $T(\mathcal{B}) \subset \mathcal{B}$. Thus, we know T is completely continuous.

Consequently, it follows at once by Schauder fixed point theorem that T has a fixed point y ; namely, y is a solution of (1). The theorem is proved. $\square$

Remark 9. In this paper, f is only continuous function, without nonnegative assumptions on function f .

Remark 10. If $\rho = 1$ in (H_1) , we need condition $c(\Gamma(\nu + b + 1)/\Gamma(b + 1)\Gamma(\nu))[1/\nu + (b + 3)/(b + 1)] \leq 1/2$. At this moment, choose

$$R \geq 2 \sum_{s=0}^{b+1} G(s + \nu - 1, s) |a(s + \nu - 1)|. \quad (19)$$

If $\rho = 1$ in (H_2) , we only need condition $c(\Gamma(\nu + b + 1)/\Gamma(b + 1)\Gamma(\nu))[1/\nu + (b + 3)/(b + 1)] \leq 1$. Then the conclusion of Theorem 8 remains true.

Example 11. Consider the following Caputo fractional difference boundary value problem:

$$\Delta_C^{9/4} y(t) = -\sin \pi \left(t + \frac{5}{4} \right) \left(1 + \frac{y^{1/2}(t + 5/4)}{y^2(t + 5/4) + 50} \right), \quad (20)$$

$$y\left(-\frac{3}{4}\right) = \Delta y\left(\frac{45}{4}\right) = \Delta^2 y\left(-\frac{3}{4}\right) = 0,$$

where $\nu = 9/4$, $b = 9$, $f(t, y) = \sin \pi t \cdot (1 + y^{1/2}/(y^2 + 50))$. Since

$$|f(t, y)| \leq 1 + \frac{1}{50} y^{1/2}, \quad (21)$$

then $a(t) = 1$, $c = 1/50$, and $\rho = 1/2$. At this moment, we take $R = 270.918$. The condition of (H_1) in Theorem 8 is satisfied. Applying Theorem 8, FBVP (20) has at least one solution y , and $|y| \leq R$.

Example 12. Consider the following Caputo fractional difference boundary value problem:

$$\Delta_C^{5/2} y(t) = -\frac{2 \sin \pi (t + 3/2)}{2t + 323} y^2 \left(t + \frac{3}{2} \right), \quad (22)$$

$$y\left(-\frac{1}{2}\right) = \Delta y\left(\frac{23}{2}\right) = \Delta^2 y\left(-\frac{1}{2}\right) = 0,$$

where $t \in [0, 10]_{\mathbb{N}_0}$ and $\nu = 5/2$, $b = 9$, and $f(t, y) = (\sin \pi t / (t + 160)) y^2$. As

$$|f(t, y)| \leq \frac{1}{160} y^2, \quad (23)$$

then $a(t) = 0$, $c = 1/160$, and $\rho = 2 > 1$. At this moment, we take $R = 0.3525$. The condition of (H_2) in Theorem 8 is satisfied. Applying Theorem 8, FBVP (22) has at least one solution y , and $|y| \leq R$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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