Research Article

Multiple Bifurcations and Chaos in a Discrete Prey-Predator System with Generalized Holling III Functional Response

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A prey-predator system with the strong Allee effect and generalized Holling type III functional response is presented and discretized. It is shown that the combined influences of Allee effect and step size have an important effect on the dynamics of the system. The existences of Flip and Neimark-Sacker bifurcations and strange attractors and chaotic bands are investigated by using the center manifold theorem and bifurcation theory and some numerical methods.

1. Introduction

Lotka-Volterra model as a paradigm of the fundamental population models has been developed and widely applied in ecological sciences and other fields. Some linear and nonlinear functional responses are employed to describe the phenomena of predation including the Beddington-DeAngelis type [1, 2], Crowley-Martin type [3], and Holling types [4] that are more realistic and proper for modeling most prey-predator interactions [1–8]. Recently, a modified prey-predator model with a generalized Holling response function of type III was addressed [8]. The system exhibits more richer dynamics including the Hopf bifurcation of codimensions 1 and 2, Bogdanov-Takens bifurcation of codimensions 2 and 3.

The researches of the Allee effect on biological population [9–23] imply that the Allee effect probably is a nonignorable factor in ecology, especially for the situation when population density (or size) is low (or small). That fitness of an individual in a small or sparse population decreases as the population density declines leads to Allee effect occurrence. Two most common continuous growth equations to express Allee effect for a single species are given [15–18]

$$\dot{x} = rx \left(1 - \frac{x}{K}\right) (x - m), \quad \dot{x} = r \left(1 - \frac{x}{K} - \frac{l}{x + b}\right) x, \quad (1)$$

which are called multiplicative Allee effect model [15, 16] and the additive Allee effect model [17, 18], respectively. Parameters m , l , and $b > 0$ are the Allee thresholds, where $lx/(x + b)$ is termed as the strong Allee effect item if $l > br$ and weak Allee effect item if $l < br$. Mathematically, the systems subjected to the Allee effect can be depicted by the bistability switch, which may lead to more complex dynamics. Thus, the consideration of prey-predator model subjected to an Allee effect is more realistic and has attracted renewed interest, which motivates us to make a modification to the system presented in [8] for further study:

$$\begin{aligned} \dot{x} &= x \left(r \left(1 - \frac{x}{K} \right) (x - m) - \frac{\beta xy}{a_0 x^2 + b_0 x + 1} \right), \\ \dot{y} &= y \left(-d_0 + \frac{c_0 x^2}{a_0 x^2 + b_0 x + 1} \right), \end{aligned} \quad (2)$$

where $\dot{} = d/dt$. x and y denote the prey and predator densities at time t , respectively. r , K , m , β , a_0 , c_0 , and d_0 are positive constants. b_0 is positive or negative. In this paper, we will focus on the case $b_0 > 0$. In particular, r and K represent the intrinsic growth rate and carrying capacity of prey in the absence of predation, respectively. $m > 0$ is so called the strong Allee threshold. The predator consumes the prey with functional response $\beta x^2 y / (a_0 x^2 + b_0 x + 1)$, known

as Holling type III response and contributes to its growth rate $c_0 x^2 y / (a_0 x^2 + b_0 x + 1)$. d_0 is the maximum death rate in the absence of prey. For simplicity, take nondimensional transformations $x = KX$, $y = Y/\beta$, $s = Kt$ to system (2) and rewrite X, Y, s as x, y, t ; we have

$$\begin{aligned}\dot{x} &= x \left(r(1-x)(x-A) - \frac{xy}{ax^2 + bx + 1} \right), \\ \dot{y} &= y \left(-d + \frac{cx^2}{ax^2 + bx + 1} \right),\end{aligned}\quad (3)$$

where $A = m/K$, $a = a_0 K^2$, $b = b_0 K$, $d = d_0/K$, and $c = c_0 K$.

Many studies of discrete-time models [19–38] were made and suggested that discrete-time models described by the difference equations are more appropriate than the continuous time models when populations have nonoverlapping generations or the number of populations is small. Moreover, dynamical patterns produced in discrete-time models are much richer than those observed in continuous time models.

Applying the forward Euler scheme, we obtain the discrete-time form of system (3) as follows:

$$\begin{aligned}x_{k+1} &= x_k + \theta x_k \left(r(1-x_k)(x_k-A) - \frac{x_k y_k}{ax_k^2 + bx_k + 1} \right) \\ &\triangleq f(x_k, y_k), \\ y_{k+1} &= y_k + \theta y_k \left(-d + \frac{cx_k^2}{ax_k^2 + bx_k + 1} \right) \triangleq g(x_k, y_k),\end{aligned}\quad (4)$$

where $\theta > 0$ is the step size.

The paper is organized as follows: in Section 2 we show the existence and stability of fixed points of (4). In Section 3 we investigate the existence of flip bifurcation and the Neimark-Sacker bifurcation. In Section 4 we show some numerical simulations to illustrate our main results. Finally, a brief discussion of (4) is given to sum up our analysis.

2. Existence and Stability of Fixed Points

To obtain the fixed points of system (4), it is sufficient to solve the following algebraic equations:

$$\begin{aligned}x \left(r(1-x)(x-A) - \frac{xy}{ax^2 + bx + 1} \right) &= 0, \\ y \left(-d + \frac{cx^2}{ax^2 + bx + 1} \right) &= 0.\end{aligned}\quad (5)$$

After calculating, we have the following statement.

Lemma 1. *System (4) always has the trivial fixed point $E_0 = (0, 0)$ and the boundary fixed points $E_1 = (1, 0)$, $E_2 = (A, 0)$. If*

$d(a+b+1) < c < d(a+b/A+1/A^2)$, (4) has a unique positive fixed point $E_ = (x_*, y_*)$, where*

$$\begin{aligned}x_* &= \frac{bd + \sqrt{b^2 d^2 + 4d(c-ad)}}{2(c-ad)}, \\ y_* &= \frac{r(1-x_*)(x_*-A)(ax_*^2 + bx_* + 1)}{x_*}.\end{aligned}\quad (6)$$

To analyze dynamical properties of (4), we need to compute the Jacobian matrix of (4) at any fixed point $E = (x, y)$

$$M_E = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}, \quad (7)$$

where

$$\begin{aligned}f_x &= 1 - \theta r(3x^2 - 2x(A+1) + A) - \frac{\theta xy(bx+2)}{(ax^2 + bx + 1)^2}; \\ f_y &= -\frac{\theta x^2}{ax^2 + bx + 1}; \\ g_x &= \frac{\theta cxy(bx+2)}{(ax^2 + bx + 1)^2}; \\ g_y &= 1 + \theta \left(-d + \frac{cx^2}{ax^2 + bx + 1} \right).\end{aligned}\quad (8)$$

The corresponding characteristic polynomial of M_E can be written as

$$F(\lambda) = \lambda^2 + p\lambda + q, \quad (9)$$

where $p = -f_x - g_y$, $q = f_x g_y - f_y g_x$.

Let λ_1 and λ_2 be two roots of equation $F(\lambda) = 0$. Clearly, the local stability of the fixed point $E(x, y)$ is determined by the modules of eigenvalues $|\lambda_1|$ and $|\lambda_2|$; that is, if $|\lambda_1| < 1$ and $|\lambda_2| < 1$, then $E(x, y)$ is locally asymptotically stable and is termed a sink. If $|\lambda_1| > 1$ and $|\lambda_2| > 1$, then $E(x, y)$ is unstable and is termed a source. If $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$), then $E(x, y)$ is termed a saddle. If $|\lambda_1| = 1$ or $|\lambda_2| = 1$, then $E(x, y)$ is termed nonhyperbolic. These can be stated by the following lemma.

Lemma 2 (see [20]). *Suppose $F(1) > 0$; then*

- (i) $|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $F(-1) > 0$ and $q < 1$;
- (ii) $\lambda_1 = -1$ and $|\lambda_2| \neq 1$ if and only if $F(-1) = 0$ and $p \neq 0, 2$;
- (iii) $|\lambda_1| > 1$ and $|\lambda_2| > 1$ if and only if $F(-1) > 0$ and $q > 1$;
- (iv) $|\lambda_1| > 1$ and $|\lambda_2| < 1$ if and only if $F(-1) < 0$.

A simple computation show that two eigenvalues of M_{E_0} are $\lambda_1 = 1 - \theta rA$ and $\lambda_2 = 1 - \theta d$. Also note that $F(1) = \theta^2 rAd > 0$, $F(-1) = (\theta d - 2)(\theta rA - 2)$, $p = \theta(Ar + d) - 2$,

and $q = (\theta rA - 1)(\theta d - 1)$. Thus, from Lemma 2, we have the following.

Lemma 3. *The fixed point $E_0 = (0, 0)$ of model (4) always exists; moreover,*

- (i) E_0 is a source if $(\theta d - 2)(\theta rA - 2) > 0$, $\theta rAd > d + rA$;
- (ii) E_0 is a sink if $(\theta d - 2)(\theta rA - 2) > 0$, $\theta rAd < d + rA$;
- (iii) E_0 is a saddle if $(\theta d - 2)(\theta rA - 2) < 0$;
- (iv) E_0 is nonhyperbolic if $(\theta d - 2)(\theta rA - 2) = 0$.

Remark 4. Define

$$\begin{aligned} S_{E_0}^1 &= \left\{ (A, a, b, c, d, \theta) : \theta = \frac{2}{d}, A \neq \frac{d}{r} \right\}, \\ S_{E_0}^2 &= \left\{ (A, a, b, c, d, \theta) : A = \frac{2}{r\theta}, \theta \neq \frac{2}{d} \right\}. \end{aligned} \quad (10)$$

From Lemma 3, we see that if $(A, a, b, c, d, \theta) \in S_{E_0}^1$ or $S_{E_0}^2$, then one of the two eigenvalues of $F(\lambda) = 0$ is -1 and the other is neither 1 nor -1 , which implies that system (4) may undergo the flip bifurcation at the fixed point E_0 when parameters (A, a, b, c, d, θ) vary in a small neighborhood of $S_{E_0}^1$ or $S_{E_0}^2$.

Similarly, the eigenvalues of Jacobian M_{E_1} are $\lambda_1 = 1 - r\theta(1 - A)$ and $\lambda_2 = 1 + \theta(c - d(a + b + 1))/(a + b + 1)$. By Lemma 2, we obtain the following results.

Lemma 5. *For $E_1 = (1, 0)$, one has the following:*

- (i) E_1 is a sink if $0 < \theta < 2/r(1 - A)$, $(a + b + 1)(d - 2/\theta) < c < d(a + b + 1)$;
- (ii) E_1 is a source if one of the following two cases holds:
 - (1) $\theta > 2/r(1 - A)$, $0 < c < (a + b + 1)(d - 2/\theta)$ with $d > 2/\theta$;
 - (2) $\theta > 2/r(1 - A)$, $c > d(a + b + 1)$;
- (iii) E_1 is a saddle if one of the following three cases holds:
 - (1) $0 < \theta < 2/r(1 - A)$, $0 < c < (a + b + 1)(d - 2/\theta)$ with $d > 2/\theta$;
 - (2) $0 < \theta < 2/r(1 - A)$, $c > d(a + b + 1)$;
 - (3) $\theta > 2/r(1 - A)$, $(a + b + 1)(d - 2/\theta) < c < d(a + b + 1)$;
- (iv) E_1 is nonhyperbolic if one of the following three cases holds:
 - (1) $\theta = 2/r(1 - A)$;
 - (2) $c = (a + b + 1)(d - 2/\theta)$ with $\theta > 2/d$;
 - (3) $c = d(a + b + 1)$.

Remark 6. Lemma 5(iv) implies that there are two parameter surfaces defined, respectively, by

$$\begin{aligned} S_{E_1}^1 &= \left\{ (A, a, b, c, d, \theta) : \theta = \frac{2}{r(1 - A)}, \right. \\ &\quad \left. c \neq (a + b + 1) \left(d - \frac{2}{\theta} \right), c \neq d(a + b + 1) \right\}, \\ S_{E_1}^2 &= \left\{ (A, a, b, c, d, \theta) : c = (a + b + 1) \left(d - \frac{2}{\theta} \right), \right. \\ &\quad \left. \theta \neq \frac{2}{r(1 - A)}, d > \frac{2}{\theta} \right\}, \end{aligned} \quad (11)$$

such that one of the eigenvalues of M_{E_1} is -1 and the other is neither 1 nor -1 when $(A, a, b, c, d, \theta) \in S_{E_1}^1$ or $S_{E_1}^2$. Thus, system (4) may undergo a flip bifurcation when (A, a, b, c, d, θ) changes in a small neighborhood of $S_{E_1}^1$ or $S_{E_1}^2$.

After calculating, we also obtain the eigenvalues of the Jacobian M_{E_2} are $\lambda_1 = 1 + \theta Ar(1 - A)$ and $\lambda_2 = 1 + \theta(-d + cA^2/(aA^2 + bA + 1))$. Note that $0 < A < 1$ so $\lambda_1 > 1$. Obviously, if $|\lambda_2| > 1$, then E_2 is a source. If $|\lambda_2| < 1$, then E_2 is a saddle. If $|\lambda_2| = 1$, then E_2 is nonhyperbolic. By calculating, one can obtain the following.

Lemma 7. *The dynamical behaviors of the fixed point $E_2 = (A, 0)$ can be stated as follows.*

- (i) It is a source if $c > d(aA^2 + bA + 1)/A^2$ or $0 < c < (d\theta - 2)(aA^2 + bA + 1)/A^2\theta$ with $\theta > 2/d$.
- (ii) It is a saddle if $(d\theta - 2)(aA^2 + bA + 1)/A^2\theta < c < d(aA^2 + bA + 1)/A^2$.
- (iii) It is nonhyperbolic if $c = d(aA^2 + bA + 1)/A^2$ or $c = (d\theta - 2)(aA^2 + bA + 1)/A^2\theta$ with $\theta > 2/d$.

Now, we discuss local stability of the fixed point $E_*(x_*, y_*)$. After calculating, we get

$$\begin{aligned} p &= -f_x(x_*, y_*) - g_y(x_*, y_*) = -2 - \theta(rC - B); \\ q &= 1 + \theta(rC - B) + dB\theta^2, \end{aligned} \quad (12)$$

where $B = x_*y_*(bx_* + 2)/(ax_*^2 + bx_* + 1)^2 > 0$ and $C = -3x_*^2 + 2(1 + A)x_* - A$. The discriminant of equation $F(\lambda) = 0$ can be calculated $\Delta = \theta^2((rC - B)^2 - 4dB)$. Moreover, $F(1) = dB\theta^2 > 0$ and

$$F(-1) = dB\theta^2 + 2(rC - B)\theta + 4. \quad (13)$$

Regarding q and $F(-1)$ as two functions of θ , then the discriminants of equations $q(\theta) = 0$ and $F(-1)(\theta) = 0$ are $(rC - B)^2 - 4dB$ and $4((rC - B)^2 - 4dB)$, respectively.

Clearly, the sign of Δ is determined by $(rC - B)^2 - 4dB$. Therefore we have the following.

Lemma 8. *Assume $d(a + b + 1) < c < d(a + b + 1)/A^2$, then*

- (i) E_* is a source if one of the following three cases holds:
 - (1) $B \leq rC$;

- (2) $B > rC$, $(rC - B)^2 \leq 4dB$, $\theta > (B - rC)/dB$;
 (3) $B > rC$, $(rC - B)^2 > 4dB$, $\theta > (B - rC + \sqrt{(B - rC)^2 - 4dB})/dB$;
 (ii) E_* is a sink if one of the following two cases holds:
 (1) $B > rC$, $(rC - B)^2 \leq 4dB$, $0 < \theta < (B - rC)/dB$;
 (2) $B > rC$, $(rC - B)^2 > 4dB$, $0 < \theta < (B - rC - \sqrt{(B - rC)^2 - 4dB})/dB$;
 (iii) E_* is a saddle if $(B - rC - \sqrt{(B - rC)^2 - 4dB})/dB < \theta < (B - rC + \sqrt{(B - rC)^2 - 4dB})/dB$ and $B > rC$, $(rC - B)^2 > 4dB$;
 (iv) E_* is nonhyperbolic if one of the following three cases holds:
 (1) $B > rC$, $(rC - B)^2 \leq 4dB$, $\theta = (B - rC)/dB$;
 (2) $B > rC$, $(rC - B)^2 > 4dB$, $\theta = (B - rC - \sqrt{(B - rC)^2 - 4dB})/dB$;
 (3) $B > rC$, $(rC - B)^2 > 4dB$, $\theta = (B - rC + \sqrt{(B - rC)^2 - 4dB})/dB$.

Proof. That $F(1) > 0$ always holds. Set λ_1 and λ_2 are eigenvalues of matrix M_{E_*} . We will consider two cases to prove Lemma 8.

Case 1. Assume $B \leq rC$, then $F(-1) > 0$ and $q > 1$ which yield $|\lambda_1| > 1$ and $|\lambda_2| > 1$.

Case 2. Assume $B > rC$. One can see that if $(rC - B)^2 \leq 4dB$, then $q \geq 0$, $F(-1) \geq 0$; moreover $F(-1) = 0$ if and only if $(rC - B)^2 = 4dB$ and $\theta = (B - rC)/dB$. By comparing q and 1, we obtain that if $\theta > (B - rC)/dB$, then $F(-1) > 0$ and $q > 1$ which lead to the fact that E_* is a source. If $0 < \theta < (B - rC)/dB$ the $F(-1) > 0$ and $q < 1$ which lead to the fact that E_* is a sink. If $\theta = (B - rC)/dB$ the $F(-1) = 0$ and $q = 1$ which lead to $|\lambda_1| = 1$ and $|\lambda_2| = 1$, that is, E_* is nonhyperbolic.

In Case 2, the fact that the assumption $(rC - B)^2 > 4dB$ leads to that the equation $F(-1) = 0$ has two distinct positive roots $\theta_1 = (B - rC - \sqrt{(B - rC)^2 - 4dB})/dB$ and $\theta_2 = (B - rC + \sqrt{(B - rC)^2 - 4dB})/dB$ so that $F(-1) > 0$ when $0 < \theta < \theta_1$ or $\theta > \theta_2$, $F(-1) < 0$ when $\theta_1 < \theta < \theta_2$. Also note that $q > 1$ if $\theta > (B - rC)/dB$, $q < 1$ if $0 < \theta < (B - rC)/dB$. Thus, we derive that $F(-1) > 0$ and $q > 1$ if $\theta > \theta_2$, $F(-1) > 0$ and $q < 1$ if $0 < \theta < \theta_1$, $F(-1) = 0$ if $\theta = \theta_1$ or θ_2 . Following above discussion and Lemma 2, the proof of Lemma 8 is complete. $\square$

From above analysis and Lemma 8, we have two parameter surfaces:

$$H_* = \left\{ (A, a, b, c, d, \theta) : \theta = \frac{B - rC}{dB}, \right. \\ \left. d(a + b + 1) < c < d \left(a + \frac{b}{A} + \frac{1}{A^2} \right), \right. \\ \left. 0 < B - rC < 2\sqrt{dB} \right\};$$

$$S_{E_*} = \left\{ (A, a, b, c, d, \theta) : \theta = \theta_1, \right.$$

$$d(a + b + 1) < c < d \left(a + \frac{b}{A} + \frac{1}{A^2} \right),$$

$$B - rC > 2\sqrt{dB}, \theta(B - rC) \neq 2, 4 \left. \right\}.$$

(14)

From Lemma 8, we can see that if $(A, a, b, c, d, \theta) \in H_*$, then two eigenvalues of M_{E_*} are a pair of conjugate complex numbers and satisfy $|\lambda_1| = |\lambda_2| = 1$. Thus, when (A, a, b, c, d, θ) vary in a small neighborhood of H_* model (4) undergoes a Neimark-Sacker bifurcation from the fixed point E_* . Similarly, if $(A, a, b, c, d, \theta) \in S_{E_*}$, then one of two eigenvalues of M_{E_*} is -1 and the other is neither 1 nor -1 ; that is, when (A, a, b, c, d, θ) vary in a small neighborhood of S_{E_*} , a flip bifurcation from the fixed point E_* can occur.

3. Analysis of Bifurcation

On the basis of the analysis in Section 2, by choosing parameter θ as a bifurcation parameter, we mainly investigate the flip bifurcation and Neimark-Sacker bifurcation of positive fixed point $E(x_*, y_*)$ by using the center manifold theorem and bifurcation theory of [39–41] in this section.

Giving a perturbation θ_* ($|\theta_*| \ll 1$) to parameter θ we consider a perturbation of model (4) as follows:

$$x_{k+1} = x_k + (\theta + \theta_*) x_k \\ \times \left(r(1 - x_k)(x_k - A) - \frac{x_k y_k}{ax_k^2 + bx_k + 1} \right), \quad (15)$$

$$y_{k+1} = y_k + (\theta + \theta_*) y_k \left(-d + \frac{cx_k^2}{ax_k^2 + bx_k + 1} \right).$$

Let $E(x, y)$ be a feasible fixed point of model (4). We translate $E(x, y)$ to the origin by using transformations $X_k = x_k - x$, $Y_k = y_k - y$. Then we have (rewrite X_k, Y_k as x_k, y_k)

$$x_{k+1} = x_k + (\theta + \theta_*) (x_k + x) \\ \times \left(r(1 - x_k - x)(x_k + x - A) \right. \\ \left. - \frac{(x_k + x)(y_k + y)}{a(x_k + x)^2 + b(x_k + x) + 1} \right), \quad (16)$$

$$y_{k+1} = y_k + (\theta + \theta_*) (y_k + y) \\ \times \left(-d + \frac{c(x_k + x)^2}{a(x_k + x)^2 + b(x_k + x) + 1} \right).$$

Expanding model (16) as Taylor series at $(x_k, y_k, \theta_*) = (0, 0, 0)$ to the second order, then we have

$$\begin{aligned} x_{k+1} &= a_{100}x_k + a_{010}y_k + a_{200}x_k^2 + a_{110}x_ky_k \\ &\quad + a_{101}x_k\theta_* + a_{011}y_k\theta_* + O(|x_k, y_k, \theta_*|^4); \\ y_{k+1} &= b_{100}x_k + b_{010}y_k + b_{200}x_k^2 + b_{110}x_ky_k \\ &\quad + b_{101}x_k\theta_* + b_{011}y_k\theta_* + O(|x_k, y_k, \theta_*|^4), \end{aligned} \quad (17)$$

where

$$\begin{aligned} a_{100} &= 1 + \theta r (2x - A - 3x^2 + 2xA) - \frac{\theta xy (bx + 2)}{(ax^2 + bx + 1)^2}; \\ a_{010} &= -\frac{x^2\theta}{ax^2 + bx + 1}; \\ a_{200} &= \theta r (1 + A - 3x) + \frac{\theta y (3ax^2 + ax^3b - 1)}{(ax^2 + bx + 1)^3}; \\ a_{110} &= -\frac{\theta x (bx + 2)}{(ax^2 + bx + 1)^2}; \\ a_{101} &= \frac{a_{100} - 1}{\theta}; \quad a_{011} = -\frac{x^2}{ax^2 + bx + 1}; \\ b_{100} &= \frac{\theta cxy (bx + 2)}{(ax^2 + bx + 1)^2}; \\ b_{010} &= 1 + \theta \left(-d + \frac{cx^2}{ax^2 + bx + 1} \right); \\ b_{200} &= -\frac{\theta yc (abx^3 + 3ax^2 - 1)}{(ax^2 + bx + 1)^3}; \\ b_{110} &= \frac{\theta cx (bx + 2)}{(ax^2 + bx + 1)^2}; \quad b_{101} = \frac{cxy (bx + 2)}{(ax^2 + bx + 1)^2}; \\ b_{011} &= -d + \frac{cx^2}{ax^2 + bx + 1}. \end{aligned} \quad (18)$$

The flip bifurcation of model (4) at E_* will be investigated firstly when (A, a, b, c, d, θ) varies in the small neighborhood of S_{E_*} . Similar arguments can be applied to the other cases $S_{E_0}^1, S_{E_0}^2, S_{E_1}^1, S_{E_1}^2$, and S_{E_2} .

Let $(A, a, b, c, d, \theta) \in S_{E_*}$ arbitrarily. From the discussion in Section 2, we can see model (4) has a unique positive fixed point E_* ; its eigenvalues are $\lambda_1 = -1$ and $-1 < \lambda_2 < 0$. In this case, $x = x_*$, $y = y_*$, and $\theta = \theta_1$. Also note that $p = -a_{100} - b_{010} = -a_{100} - 1$ and $\lambda_1 + \lambda_2 = -p$ so $\lambda_2 = 2 + a_{100}$. Moreover, there is a transformation

$$\begin{pmatrix} x_k \\ y_k \end{pmatrix} = \begin{pmatrix} a_{010} & a_{010} \\ -1 - a_{100} & \lambda_2 - a_{100} \end{pmatrix} \begin{pmatrix} u_k \\ v_k \end{pmatrix} \quad (19)$$

so that model (17) becomes

$$\begin{aligned} u_{k+1} &= -u_k + f_1(x_k, y_k, \theta_*) + O(|x_k, y_k, \theta_*|^4), \\ v_{k+1} &= \lambda_2 v_k + f_2(x_k, y_k, \theta_*) + O(|x_k, y_k, \theta_*|^4), \end{aligned} \quad (20)$$

where

$$\begin{aligned} f_1(x_k, y_k, \theta_*) &= \frac{2a_{200} - a_{010}b_{200}}{a_{010}(1 + \lambda_2)}x_k^2 + \frac{2a_{110} - a_{010}b_{110}}{a_{010}(1 + \lambda_2)}x_ky_k \\ &\quad + \frac{2a_{101} - a_{010}b_{101}}{a_{010}(1 + \lambda_2)}x_k\theta_* + \frac{2a_{011}}{a_{010}(1 + \lambda_2)}y_k\theta_*, \\ f_2(x_k, y_k, \theta_*) &= \frac{a_{010}b_{200} - pa_{200}}{a_{010}(1 + \lambda_2)}x_k^2 + \frac{a_{010}b_{110} - pa_{110}}{a_{010}(1 + \lambda_2)}x_ky_k \\ &\quad + \frac{a_{010}b_{101} - pa_{101}}{a_{010}(1 + \lambda_2)}x_k\theta_* - \frac{pa_{011}}{a_{010}(1 + \lambda_2)}y_k\theta_*. \end{aligned} \quad (21)$$

By the center manifold theorem [41], we can get the approximate representation of the center manifold $W^c(0, 0, 0)$ of model (20) at the fixed point $(0, 0)$ in a small neighborhood of θ_1 as follows:

$$\begin{aligned} W^c(0, 0, 0) &= \{(u_k, v_k, \theta_*) \in \mathbb{R}^3 : \\ &\quad v_k = \eta_0 u_k^2 + \eta_1 u_k \theta_* + O(|u_k, \theta_*|^3)\}, \end{aligned} \quad (22)$$

where

$$\begin{aligned} \eta_0 &= \frac{a_{010}(a_{010}b_{200} - pa_{200})}{1 - \lambda_2^2} + \frac{p(a_{010}b_{110} - pa_{110})}{1 - \lambda_2^2}; \\ \eta_1 &= -\frac{a_{010}b_{101} - pa_{101}}{(1 + \lambda_2)^2} + \frac{p^2 a_{011}}{a_{010}(1 + \lambda_2)^2}. \end{aligned} \quad (23)$$

Furthermore, on the center manifold $W^c(0, 0, 0)$, we have

$$\begin{aligned} x_k &= a_{010}(u_k + \eta_0 u_k^2 + \eta_1 u_k \theta_*) + O(|u_k, \theta_*|^3); \\ y_k &= pu_k + 2(\eta_0 u_k^2 + \eta_1 u_k \theta_*) + O(|u_k, \theta_*|^3); \\ x_k^2 &= a_{010}^2(3u_k^2 + 2\eta_0 u_k^3 + 2\eta_1 u_k^2 \theta_*) + O(|u_k, \theta_*|^4); \\ x_k y_k &= -a_{010}u_k^2(-2\eta_0 u_k - u_k \eta_0 p - p - 2\eta_1 \theta_* - \eta_1 \theta_* p) \\ &\quad + O(|u_k, \theta_*|^4); \\ x_k \theta_* &= a_{010}(u_k + \eta_0 u_k^2 + \eta_1 u_k \theta_*)\theta_* + O(|u_k, \theta_*|^4); \\ y_k \theta_* &= (pu_k + 2\eta_0 u_k^2 + 2\eta_1 u_k \theta_*)\theta_* + O(|u_k, \theta_*|^4). \end{aligned} \quad (24)$$

Therefore, the map F which is model (4) restricted to the center manifold $W^c(0, 0, 0)$ takes the form

$$\begin{aligned} F(u_k) &= -u_k + h_1 u_k^2 + h_2 u_k \theta_* + h_3 u_k^3 \\ &\quad + h_4 u_k^2 \theta_* + h_5 u_k \theta_*^2 + O(|u_k, \theta_*|^4), \end{aligned} \quad (25)$$

where

$$\begin{aligned}
 h_1 &= \frac{p(2a_{110} - a_{010}b_{110})}{1 + \lambda_2} + \frac{a_{010}(2a_{200} - a_{010}b_{200})}{1 + \lambda_2}; \\
 h_2 &= \frac{2pa_{011}}{a_{010}(1 + \lambda_2)} + \frac{2a_{101} - a_{010}b_{101}}{1 + \lambda_2}; \\
 h_3 &= \frac{2\eta_0(1 + p)(2a_{110} - a_{010}b_{110})}{1 + \lambda_2} \\
 &\quad + \frac{2\eta_0a_{010}(2a_{200} - a_{010}b_{200})}{1 + \lambda_2}; \\
 h_4 &= \frac{\eta_1(2 + p)(2a_{110} - a_{010}b_{110})}{1 + \lambda_2} + \frac{4\eta_0a_{011}}{a_{010}(1 + \lambda_2)} \\
 &\quad + \frac{2\eta_1a_{010}(2a_{200} - a_{010}b_{200})}{1 + \lambda_2} + \frac{\eta_0(2a_{101} - a_{010}b_{101})}{1 + \lambda_2}; \\
 h_5 &= \frac{4\eta_1a_{011}}{a_{010}(1 + \lambda_2)} + \frac{\eta_1(2a_{101} - a_{010}b_{101})}{1 + \lambda_2}.
 \end{aligned} \tag{26}$$

In order to undergo a flip bifurcation for map (25), we require that two discriminatory quantities α_1 and α_2 are not zero, where

$$\begin{aligned}
 \alpha_1 &= \left(\frac{\partial^2 F}{\partial u_k \partial \theta_*} + \frac{1}{2} \frac{\partial F}{\partial \theta_*} \frac{\partial^2 F}{\partial u_k^2} \right) \bigg|_{(0,0)} = h_2 \neq 0; \\
 \alpha_2 &= \left(\frac{1}{6} \frac{\partial^3 F}{\partial u_k^3} + \left(\frac{1}{2} \frac{\partial^2 F}{\partial u_k^2} \right)^2 \right) \bigg|_{(0,0)} = h_3 + h_1^2.
 \end{aligned} \tag{27}$$

From above analysis and the theorem in [40], we have the following.

Theorem 9. *If $\alpha_2 \neq 0$, then model (4) undergoes a flip bifurcation at equilibrium $E_*(x_*, y_*)$ when the parameter θ varies in the small neighborhood of θ_1 . Moreover, if $\alpha_2 > 0$ (resp. $\alpha_2 < 0$), then the period 2 points that bifurcate from E_* are stable (resp., unstable).*

Next, we discuss the Neimark-Sacker bifurcation of E_* when parameters (A, a, b, c, d, θ) vary in the small neighborhood of H_* . Taking $\theta = (B - rC)/dB$ (for simplicity, denoting $(B - rC)/dB$ by θ_0) and choosing θ^* as a perturbation of bifurcation parameter, a perturbed form of model (4) is obtained as follows:

$$\begin{aligned}
 x_{k+1} &= x_k + (\theta_0 + \theta^*) x_k \\
 &\quad \times \left(r(1 - x_k)(x_k - A) - \frac{x_k y_k}{ax_k^2 + bx_k + 1} \right), \\
 y_{k+1} &= y_k + (\theta_0 + \theta^*) y_k \left(-d + \frac{cx_k^2}{ax_k^2 + bx_k + 1} \right),
 \end{aligned} \tag{28}$$

where $|\theta^*| \ll 1$ which is a small perturbation.

We transform the unique positive fixed point E_* into the origin by using $X_k = x_k - x_*$, $Y_k = y_k - y_*$:

$$\begin{aligned}
 X_{k+1} &= a_{10}X_k + a_{01}Y_k + a_{11}X_kY_k + a_{20}X_k^2 \\
 &\quad + a_{02}Y_k^2 + a_{30}X_k^3 + a_{21}X_k^2Y_k + (|X_k, Y_k|^3), \\
 Y_{k+1} &= b_{10}X_k + b_{01}Y_k + b_{11}X_kY_k + b_{20}X_k^2 \\
 &\quad + b_{02}Y_k^2 + b_{30}X_k^3 + b_{21}X_k^2Y_k + (|X_k, Y_k|^3),
 \end{aligned} \tag{29}$$

where

$$\begin{aligned}
 a_{10} &= 1 + (\theta_0 + \theta^*)r(2x_* - A - 3x_*^2 + 2x_*A) \\
 &\quad - \frac{(\theta_0 + \theta^*)x_*y_*(bx_* + 2)}{(ax_*^2 + bx_* + 1)^2}; \\
 a_{01} &= -\frac{d(\theta_0 + \theta^*)}{c}; \quad a_{11} = -\frac{(\theta_0 + \theta^*)x_*(bx_* + 2)}{(ax_*^2 + bx_* + 1)^2}; \\
 a_{20} &= (\theta_0 + \theta^*)r(1 + A - 3x_*) \\
 &\quad + \frac{(\theta_0 + \theta^*)y_*(3ax_*^2 + ax_*^3b - 1)}{(ax_*^2 + bx_* + 1)^3}; \\
 a_{30} &= -\frac{(\theta_0 + \theta^*)y_*(ax_*^2 - 1)(ax_*^2b + 4ax_* + b)}{(ax_*^2 + bx_* + 1)^4} \\
 &\quad - (\theta_0 + \theta^*)r; \\
 a_{21} &= \frac{(\theta_0 + \theta^*)(abx_*^3 + 3ax_*^2 - 1)}{(ax_*^2 + bx_* + 1)^3}; \\
 b_{10} &= \frac{(\theta_0 + \theta^*)cx_*y_*(bx_* + 2)}{(ax_*^2 + bx_* + 1)^2}; \\
 b_{01} &= 1; \quad b_{20} = -\frac{(\theta_0 + \theta^*)y_*c(abx_*^3 + 3ax_*^2 - 1)}{(ax_*^2 + bx_* + 1)^3}; \\
 b_{11} &= \frac{(\theta_0 + \theta^*)cx_*(bx_* + 2)}{(ax_*^2 + bx_* + 1)^2}; \quad b_{30} = -ca_{30}; \\
 b_{21} &= -ca_{21}.
 \end{aligned} \tag{30}$$

The characteristic equation associated with linearization system of model (29) at $(0, 0)$ is

$$\lambda^2 + p(\theta^*)\lambda + q(\theta^*) = 0, \tag{31}$$

where

$$\begin{aligned}
 p(\theta^*) &= -2 - (\theta_0 + \theta^*)(rC - B), \\
 q(\theta^*) &= 1 + (\theta_0 + \theta^*)(rC - B) + dB(\theta_0 + \theta^*)^2.
 \end{aligned} \tag{32}$$

When parameters $(A, a, b, c, d, \theta) \in H_*$ and θ^* vary in a small neighborhood of $\theta^* = 0$, the roots of the characteristic

equation (31) are a pair of complex conjugate numbers λ_1 and λ_2 denoted by

$$\lambda_{1,2} = 1 + \frac{(\theta_0 + \theta^*)}{2} \left(rC - B \pm i\sqrt{4dB - (rC - B)^2} \right), \quad (33)$$

$$|\lambda_{1,2}| = \sqrt{q(\theta^*)} = \sqrt{1 + (\theta_0 + \theta^*)(rC - B) + dB(\theta_0 + \theta^*)^2}. \quad (34)$$

One can see that

$$\left. \frac{d|\lambda_{1,2}|}{d\theta^*} \right|_{\theta^*=0} = \frac{B - rC}{2} > 0. \quad (35)$$

On the other hand, it is required that $\lambda_{1,2}^k \neq 1$, $m = 1, 2, 3, 4$ when $\theta^* = 0$, which is equivalent to $p(0) \neq \pm 2, 0, -1$. Also note that $p(0)^2 - 4q(0) < 0$ and $q(0) = 1$ when $(A, a, b, c, d, \theta) \in H_*$. Thus, $p(0)^2 < 4$, which indicates $p(0) \neq \pm 2$. We only need to require that $p(0) \neq 0, 1$; that is,

$$(rC - B)^2 \neq 2dB, 3dB, \quad (36)$$

which implies that $\lambda_{1,2}$ defined by (33) do not lie in the intersection of the unit circle with the coordinate axes when $\theta^* = 0$.

Next we study the normal form of model (29) when $\theta^* = 0$.

Let $\alpha = \text{Re}(\lambda_1)$, $\beta = \text{Im}(\lambda_1)$, and

$$T = \begin{pmatrix} a_{01} & 0 \\ \alpha - a_{10} & -\beta \end{pmatrix} = \begin{pmatrix} a_{01} & 0 \\ -\frac{\theta_0(rC - B)}{2} & -\frac{\theta_0\sqrt{4dB - (rC - B)^2}}{2} \end{pmatrix} \quad (37)$$

and perform the transformations $X_k = a_{01}u_k$, $Y_k = (\alpha - a_{10})u_k - \beta v_k$ to (29); we can obtain

$$\begin{aligned} u_{k+1} &= \alpha u_k - \beta v_k + G_1(u_k, v_k), \\ v_{k+1} &= \beta u_k + \alpha v_k + G_2(u_k, v_k), \end{aligned} \quad (38)$$

where

$$\begin{aligned} G_1(u_k, v_k) &= \left(a_{20}a_{01} - \frac{a_{11}(a_{10} - b_{01})}{2} + \frac{a_{02}(a_{10} - b_{01})^2}{4a_{01}} \right) u_k^2 \\ &+ \frac{a_{02}\beta^2}{a_{01}} v_k^2 - \frac{\beta[a_{11}a_{01} + a_{02}(b_{01} - a_{10})]}{2a_{01}} u_k v_k \\ &+ \frac{a_{01}[2a_{30}a_{01} - a_{21}(a_{10} - b_{01})]}{2} u_k^3 \\ &- a_{01}a_{21}\beta u_k^2 v_k + O(|u_k, v_k|^4); \end{aligned}$$

$$\begin{aligned} G_2(u_k, v_k) &= -\frac{1}{8a_{01}\beta} \\ &\times \left((a_{10} - b_{01}) \right. \\ &\times [a_{02}(a_{10} - b_{01})^2 - 2a_{01}(a_{11} - b_{02})(a_{10} - b_{01}) \\ &\quad \left. + 4a_{01}^2(a_{20} - b_{11}) + 8b_{20}a_{01}^3 \right) u_k^2 \\ &+ \frac{1}{2a_{01}} \left(a_{01}(a_{11} - 2b_{02})(a_{10} - b_{01}) \right. \\ &\quad \left. - a_{02}(a_{10} - b_{01})^2 + 2b_{11}a_{01}^2 \right) u_k v_k \\ &- \frac{\beta[2a_{01}b_{02} + a_{02}(a_{10} - b_{01})]}{2a_{01}} v_k^2 \\ &- \frac{a_{01}}{4\beta} [2a_{01}(a_{30} - b_{21})(a_{10} - b_{01}) \\ &\quad - a_{21}(a_{10} - b_{01})^2 + 4b_{30}a_{01}^2] u_k^3 \\ &+ \frac{[a_{21}(a_{10} - b_{01}) + 2b_{21}a_{01}]a_{01}}{2} u_k^2 v_k. \end{aligned} \quad (39)$$

Let

$$\begin{aligned} \xi_{11} &= \frac{1}{4} \left[\frac{\partial^2 G_1}{\partial u_k^2} + \frac{\partial^2 G_1}{\partial v_k^2} + i \left(\frac{\partial^2 G_2}{\partial u_k^2} - \frac{\partial^2 G_2}{\partial v_k^2} \right) \right] \Big|_{(u_k, v_k)=(0,0)}, \\ \xi_{20} &= \frac{1}{8} \left[\frac{\partial^2 G_1}{\partial u_k^2} - \frac{\partial^2 G_1}{\partial v_k^2} + 2 \frac{\partial^2 G_2}{\partial u_k \partial v_k} \right. \\ &\quad \left. + i \left(\frac{\partial^2 G_2}{\partial u_k^2} - \frac{\partial^2 G_2}{\partial v_k^2} - 2 \frac{\partial^2 G_1}{\partial u_k \partial v_k} \right) \right] \Big|_{(u_k, v_k)=(0,0)}, \\ \xi_{02} &= \frac{1}{8} \left[\frac{\partial^2 G_1}{\partial u_k^2} - \frac{\partial^2 G_1}{\partial v_k^2} - 2 \frac{\partial^2 G_2}{\partial u_k \partial v_k} \right. \\ &\quad \left. + i \left(\frac{\partial^2 G_2}{\partial u_k^2} - \frac{\partial^2 G_2}{\partial v_k^2} + 2 \frac{\partial^2 G_1}{\partial u_k \partial v_k} \right) \right] \Big|_{(u_k, v_k)=(0,0)}, \\ \xi_{21} &= \frac{1}{16} \left[\frac{\partial^3 G_1}{\partial u_k^3} + \frac{\partial^3 G_1}{\partial u_k \partial v_k^2} + \frac{\partial^3 G_2}{\partial u_k^2 \partial v_k} + \frac{\partial^3 G_2}{\partial v_k^3} \right. \\ &\quad \left. + i \left(\frac{\partial^3 G_2}{\partial u_k^3} + \frac{\partial^3 G_2}{\partial u_k \partial v_k^2} \right. \right. \\ &\quad \left. \left. - \frac{\partial^3 G_1}{\partial u_k^2 \partial v_k} - \frac{\partial^3 G_1}{\partial v_k^3} \right) \right] \Big|_{(u_k, v_k)=(0,0)}. \end{aligned} \quad (40)$$

FIGURE 1: (a) The bifurcation diagram of system (4) with θ varying in $(0.2, 2)$. The system undergoes a Neimark-Sacker bifurcation when $\theta \approx 1.290556794$ (denoted by red point). (b) A stable fixed point E_* $(0.5928, 0.5537)$ exists when $\theta = 1.2$ which occurs before the bifurcation. (c) An invariant closed curve around the fixed point E_* created after the bifurcation which exists for $\theta = 1.3$. (d) A chaotic attractor occurs when $\theta = 6.6$.

By [39], we know that if the following discriminatory quantity holds, then model (38) undergoes a Neimark-Sacker bifurcation.

$$\mu_* = -\operatorname{Re} \left(\frac{(1-2\lambda)\bar{\lambda}^2}{1-\lambda} \xi_{20}\xi_{11} \right) - \frac{|\xi_{11}|^2}{2} \quad (41)$$

$$- |\xi_{02}|^2 + \operatorname{Re}(\bar{\lambda}\xi_{21}) \neq 0,$$

where

$$\lambda = \alpha + i\beta,$$

$$|\xi_{11}|^2 = \frac{1}{16} \left[\left(\frac{\partial^2 G_1}{\partial u_k^2} + \frac{\partial^2 G_1}{\partial v_k^2} \right)^2 + \left(\frac{\partial^2 G_2}{\partial u_k^2} + \frac{\partial^2 G_2}{\partial v_k^2} \right)^2 \right] \Big|_{(u_k, v_k)=(0,0)},$$

$$|\xi_{02}|^2 = \frac{1}{64} \left[\left(\frac{\partial^2 G_1}{\partial u_k^2} - \frac{\partial^2 G_1}{\partial v_k^2} - 2 \frac{\partial^2 G_2}{\partial u_k \partial v_k} \right)^2 + \left(\frac{\partial^2 G_2}{\partial u_k^2} - \frac{\partial^2 G_2}{\partial v_k^2} + 2 \frac{\partial^2 G_1}{\partial u_k \partial v_k} \right)^2 \right] \Big|_{(u_k, v_k)=(0,0)}. \quad (42)$$

Form above analysis and the theorem in [39], we have the following.

Theorem 10. Let $(rC - B)^2 \neq 2dB, 3dB$ and (41) hold. Then system (4) undergoes Neimark-Sacker bifurcation at the positive fixed point $E_*(x_*, y_*)$ when the parameter θ varies in the small neighborhood of θ_0 . Moreover, if $\mu_* < 0$

FIGURE 2: (a) The bifurcation diagram of system (4) with θ varying in $(0.3, 0.6)$. The system undergoes a flip bifurcation when $\theta \approx 0.456$. (b) With θ increasing, system (4) undergoes period-1 point (red line corresponding to $\theta = 0.36$), period-2 points (blue dot lines corresponding to $\theta = 0.46$), period-4 points (green dot lines corresponding to $\theta = 0.5$), and period-8 points (black dot lines corresponding to $\theta = 0.518$).

(resp., $\mu_* > 0$), then an attracting (resp., repelling) invariant closed curve bifurcates from the fixed point E_* for $\theta > \theta_0$ (resp., $\theta < \theta_0$).

4. Numerical Simulations

To confirm the above theoretical analysis, in this section we present the bifurcation diagrams and phase portraits for system (4). θ is chosen as the bifurcation parameter to verify the existence of flip bifurcation and Neimark-Sacker bifurcation.

Firstly, let $r = 0.5$, $A = 0.2$, $a = 0.4$, $b = 5$, $d = 0.1$, $c = 1.168$, and $\theta = 1.290556794$. One can verify that all conditions in H_* are satisfied. That is, if θ varies in a small neighborhood of $|\theta - 1.290556794| < \epsilon$ ($0 < \epsilon \ll 1$) and the other parameters are kept fixed, then system (4) undergoes a Neimark-Sacker bifurcation, which is illustrated in Figure 1.

Let $r = 10$, $A = 0.5$, $a = 3$, $b = 9$, $d = 0.1$, and $c = 6.62$. By $\theta_1 = (B - rC - \sqrt{(B - rC)^2 - 4dB})/dB$, we obtain $\theta_1 \approx 0.456$; that is, $(r, A, a, b, d, c, \theta) \in S_{E_*}$ when $\theta = \theta_1$, which leads to $\alpha_1 \neq 0$ and $\alpha_2 > 0$. Moreover, E_* is stable when $\theta < \theta_1$ and loses its stability when θ reaches and passes θ_1 , which leads to a cascade of the period doubling with θ increasing (see Figure 2).

5. Conclusion

In this paper, a discrete-time prey-predator model with generalized Holling type III functional response and Allee effect has been presented. The stability of fixed points and the existence of flip bifurcation and Neimark-Sacker bifurcation are investigated. Our analysis shows that the strong Allee effect can change topological structure and dynamics of the system. The combined influence of Allee effect and the other parameters induce system (4) to yield more rich dynamical

behaviors including chaotic bands and chaotic attractors which do not exist in the continuous case. These results reported in this paper could be very useful for the biologists who devote to study the discrete-time prey-predator models.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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