Research Article

Global Stability of Humoral Immunity HIV Infection Models with Chronically Infected Cells and Discrete Delays

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We study the global stability of three HIV infection models with humoral immune response. We consider two types of infected cells: the first type is the short-lived infected cells and the second one is the long-lived chronically infected cells. In the three HIV infection models, we modeled the incidence rate by bilinear, saturation, and general forms. The models take into account two types of discrete-time delays to describe the time between the virus entering into an uninfected $CD4^+$ T cell and the emission of new active viruses. The existence and stability of all equilibria are completely established by two bifurcation parameters, R_0 and R_1 . The global asymptotic stability of the steady states has been proven using Lyapunov method. In case of the general incidence rate, we have presented a set of sufficient conditions which guarantee the global stability of model. We have presented an example and performed numerical simulations to confirm our theoretical results.

1. Introduction

During last decades, many researchers have developed and analyzed several mathematical models which describe human immunodeficiency virus (HIV) dynamics (see, e.g., [1–12]). HIV mainly targets the $CD4^+$ T cells, leading to Acquired Immunodeficiency Syndrome (AIDS). Most of the HIV mathematical models presented in the literature consider only one type of infected cells called short-lived infected cells. However, it was shown that there is another source for the virus which is called long-lived chronically infected cells. This type of cells generates smaller number of viruses than the short-lived infected cells, but it lives longer [4]. The basic HIV dynamics model with long-lived chronically infected cells presented in [4] is given by

$$\begin{aligned}\dot{T} &= \pi - (1 - \varepsilon)kTV - dT, \\ \dot{T}^* &= (1 - \xi)(1 - \varepsilon)kTV - bT^*, \\ \dot{C}^* &= \xi(1 - \varepsilon)kTV - aC^*, \\ \dot{V} &= N_T bT^* + N_C aC^* - cV.\end{aligned}\tag{1}$$

Here T, T^*, C^* , and V are the concentrations of the uninfected $CD4^+$ T cells, short-lived infected $CD4^+$ T cells, long-lived chronically infected $CD4^+$ T cells, and free virus particles, respectively. π represents birth rate constant of the uninfected $CD4^+$ T cells. k is the infection rate constant. Parameters d, b, a , and c are the death rate constants of uninfected $CD4^+$ T cells, short-lived infected $CD4^+$ T cells, long-lived chronically infected $CD4^+$ T cells, and free viruses, respectively. The fractions ξ and $(1 - \xi)$ with $0 < \xi < 1$ are the probabilities that, upon infection, an uninfected $CD4^+$ T cell will become either long-lived chronically infected or short-lived infected. N_T and N_C denote the average numbers of free virus particles produced in the lifetime of the short-lived infected and long-lived chronically infected cells, respectively. Model (1) incorporates reverse transcriptase inhibitor drugs with drug efficacy ε and $0 \leq \varepsilon < 1$.

In model (1), the immune response has not been modeled. The immune response plays an important role in controlling the diseases. In reality, the immune response needs indispensable components to do its job such as antibodies, cytokines, natural killer cells, and T cells. The antibody immune response is a part of the adaptive system in which

the body responds to pathogens by primarily using the antibodies which are generated by the B cells, while the other part is the Cytotoxic T Lymphocytes (CTL) immune response where the CTL attacks and kills the infected cells [3]. In malaria disease, the humoral immune response is more effective than the CTL immune response. In the virus dynamics literature, several models have considered the effect of CTL immune response [3, 13] or the humoral immune response [14–16]. Obaid and Elaiw [15] proposed the following model which takes into consideration the humoral immune response:

$$\begin{aligned}\dot{T} &= \pi - (1 - \varepsilon)kTV - dT, \\ \dot{T}^* &= (1 - \xi)(1 - \varepsilon)kTV - bT^*, \\ \dot{C}^* &= \xi(1 - \varepsilon)kTV - aC^*, \\ \dot{V} &= N_T bT^* + N_C aC^* - rVZ - cV, \\ \dot{Z} &= gVZ - \mu Z.\end{aligned}\quad (2)$$

Here, the variable Z represents the concentration of B cells. The HIV are attacked by the B cells at rate rVZ . The terms gVZ and μZ represent the proliferation and death rates of the B cells, respectively. In model (2), it is assumed that once the HIV contacts the $CD4^+$ T cell, it becomes infected producing new viruses. Actually, there exists an intracellular time delay between the time the HIV contacts an uninfected $CD4^+$ T cell and the time it becomes actively infected $CD4^+$ T cell [17]. In the literature, several papers have proposed various HIV models with time delays [17–21].

Our aim in this work is to propose three HIV dynamics models with two types of infected cells, two types of intracellular delays, and humoral immunity. Bilinear and saturated incidences have been proposed in the first and second model, respectively, while a general nonlinear incidence rate is proposed in the third model. For each model, we derive two bifurcation parameters, R_0 and R_1 , and establish the global stability using Lyapunov functional.

2. Model with Bilinear Incidence

We propose the following HIV infection model with humoral immunity, two types of infected cells and two types of intracellular delays:

$$\dot{T}(t) = \pi - (1 - \varepsilon)kT(t)V(t) - dT(t), \quad (3)$$

$$\begin{aligned}\dot{T}^*(t) &= e^{-m_1\tau_1}(1 - \xi)(1 - \varepsilon)kT(t - \tau_1)V(t - \tau_1) \\ &\quad - bT^*(t),\end{aligned}\quad (4)$$

$$\begin{aligned}\dot{C}^*(t) &= e^{-m_1\tau_1}\xi(1 - \varepsilon)kT(t - \tau_1)V(t - \tau_1) \\ &\quad - aC^*(t),\end{aligned}\quad (5)$$

$$\begin{aligned}\dot{V}(t) &= e^{-m_2\tau_2}N_T bT^*(t - \tau_2) + e^{-m_3\tau_3}N_C aC^*(t - \tau_3) \\ &\quad - rV(t)Z(t) - cV(t),\end{aligned}\quad (6)$$

$$\dot{Z}(t) = gV(t)Z(t) - \mu Z(t). \quad (7)$$

Parameter τ_1 represents the time between HIV contact with an uninfected $CD4^+$ T cell and the cell becoming infected but not yet producer cell. The factor $e^{-m_1\tau_1}$ represents the loss of $CD4^+$ T cells during the interval $[t - \tau_1, t]$. The parameters τ_2 and τ_3 represent the time necessary for producing new infectious viruses from the short-lived infected and long-lived chronically infected cells, respectively. The factors $e^{-m_2\tau_2}$ and $e^{-m_3\tau_3}$ represent the loss of short-lived and long-lived chronically infected cells during the intervals $[t - \tau_2, t]$ and $[t - \tau_3, t]$, respectively. Here, m_1 , m_2 , and m_3 are positive constants.

The initial conditions for system (2) are given by

$$\begin{aligned}T(u) &= \varphi_1(u), \\ T^*(u) &= \varphi_2(u), \\ C^*(u) &= \varphi_3(u), \\ V(u) &= \varphi_4(u), \\ Z(u) &= \varphi_5(u), \\ \varphi_j(u) &\geq 0, \quad u \in [-\tau, 0], \quad j = 1, 2, \dots, 5, \\ \varphi_j(0) &> 0, \quad j = 1, 2, \dots, 5,\end{aligned}\quad (8)$$

where $(\varphi_1(u), \varphi_2(u), \dots, \varphi_5(u)) \in C([-\tau, 0], R_+^5)$, where C is the Banach space of continuous functions and $\tau = \max\{\tau_1, \tau_2, \tau_3\}$. We note that system (3)–(7) with initial conditions (8) has a unique solution satisfying t [22].

2.1. Positive Invariance

Proposition 1. *The solution $(T(t), T^*(t), C^*(t), V(t), Z(t))$ of (3)–(7) with initial conditions (8) is nonnegative for $t \geq 0$ and ultimately bounded.*

Proof. Assume that $T(t) \leq 0$ on $[0, \rho]$ for some constant ρ and $T(t^*) = 0$, $t^* \in [0, \rho]$. From (3), we get $\dot{T}(t^*) = \pi > 0$ and hence $T(t) > 0$, for some $t \in (t^*, t^* + \varrho)$, and sufficiently small $\varrho > 0$. This leads to a contradiction; therefore, $T(t) > 0$, for all $t \geq 0$. From (4), (5), and (6), we have

$$\begin{aligned}T^*(t) &= \varphi_2(0)e^{-bt} + e^{-m_1\tau_1}(1 - \xi)(1 - \varepsilon) \\ &\quad \cdot k \int_0^t e^{-b(t-\eta)} T(\eta - \tau_1) V(\eta - \tau_1) d\eta, \\ C^*(t) &= \varphi_3(0)e^{-at} + e^{-m_1\tau_1}\xi(1 - \varepsilon) \\ &\quad \cdot k \int_0^t e^{-a(t-\eta)} T(\eta - \tau_1) V(\eta - \tau_1) d\eta, \\ V(t) &= \varphi_4(0)e^{-\int_0^t (c+rZ(\varsigma))d\varsigma} \\ &\quad + e^{-m_2\tau_2}N_T b \int_0^t e^{-\int_\eta^t (c+rZ(\varsigma))d\varsigma} T^*(\eta - \tau_2) d\eta \\ &\quad + e^{-m_3\tau_3}N_C a \int_0^t e^{-\int_\eta^t (c+rZ(\varsigma))d\varsigma} C^*(\eta - \tau_3) d\eta.\end{aligned}\quad (9)$$

This confirms that $T^*(t) \geq 0$, $C^*(t) \geq 0$, and $V(t) \geq 0$, for all $t \in [0, \tau]$. By a recursive argument, we get that $T^*(t) \geq 0$, $C^*(t) \geq 0$, and $V(t) \geq 0$, for all $t \geq 0$. Moreover, from (7), we obtain

$$Z(t) = \varphi_5(0) e^{-\int_0^t (\mu - gV(\varsigma)) d\varsigma}. \quad (10)$$

Clearly, $Z(t) \geq 0$, $t \geq 0$. Now, we let $G_1(t) = e^{-m_1 \tau_1} T(t - \tau_1) + T^*(t) + C^*(t)$, $G_2(t) = V(t) + (r/g)Z(t)$, $s_1 = \min\{d, a, b\}$, $s_2 = \min\{c, \mu\}$, $L_1 = \pi/s_1$, $L_2 = (N_T b + N_C a)L_1/s_2$, and $L_3 = gL_2/r$; then,

$$\begin{aligned} \dot{G}_1(t) &= e^{-m_1 \tau_1} [\pi - dT(t - \tau_1)] - bT^*(t) - aC^*(t) \\ &\leq \pi e^{-m_1 \tau_1} \\ &\quad - s_1 (e^{-m_1 \tau_1} T(t - \tau_1) + T^*(t) + C^*(t)) \\ &= \pi e^{-m_1 \tau_1} - s_1 G_1(t) < \pi - s_1 G_1(t). \end{aligned} \quad (11)$$

Hence, $\limsup_{t \rightarrow \infty} G_1(t) \leq L_1$. Since $T(t) > 0$, $T^*(t) \geq 0$, and $C^*(t) \geq 0$, then $\limsup_{t \rightarrow \infty} T^*(t) \leq L_1$ and $\limsup_{t \rightarrow \infty} C^*(t) \leq L_1$. Also,

$$\begin{aligned} \dot{G}_2(t) &= e^{-m_2 \tau_2} N_T b T^*(t - \tau_2) \\ &\quad + e^{-m_3 \tau_3} N_C a C^*(t - \tau_3) - cV(t) - \frac{r\mu}{g} Z(t) \\ &\leq (e^{-m_2 \tau_2} N_T b + e^{-m_3 \tau_3} N_C a) L_1 \\ &\quad - s_2 \left(V(t) + \frac{r}{g} Z(t) \right) \\ &< (N_T b + N_C a) L_1 - s_2 G_2(t). \end{aligned} \quad (12)$$

Hence, $\limsup_{t \rightarrow \infty} G_2(t) \leq L_2$. Since $V(t)$ and $Z(t)$ are non-negative, then $\limsup_{t \rightarrow \infty} V(t) \leq L_2$ and $\limsup_{t \rightarrow \infty} Z(t) \leq L_3$. Therefore, $T(t)$, $T^*(t)$, $C^*(t)$, $V(t)$, and $Z(t)$ are ultimately bounded. $\square$

2.2. Steady States. System (3)–(7) always admits an uninfected steady state $S_0 = (\pi/d, 0, 0, 0, 0)$. Let $T_0 = \pi/d$. Now, we define the basic reproduction number for system (3)–(7) as

$$R_0 = \frac{(1 - \varepsilon) k T_0 [e^{-(m_1 \tau_1 + m_2 \tau_2)} (1 - \xi) N_T + e^{-(m_1 \tau_1 + m_3 \tau_3)} \xi N_C]}{c}. \quad (13)$$

The system has two other steady states, infected steady state without humoral immune response $S_1 = (T_1, T_1^*, C_1^*, V_1, 0)$

and infected steady state with humoral immune response $S_2 = (T_2, T_2^*, C_2^*, V_2, Z_2)$:

$$\begin{aligned} T_1 &= \frac{T_0}{R_0}, \\ T_1^* &= e^{-m_1 \tau_1} \frac{(1 - \xi) \pi (R_0 - 1)}{b R_0}, \\ C_1^* &= e^{-m_1 \tau_1} \frac{\xi \pi (R_0 - 1)}{a R_0}, \\ V_1 &= \frac{d}{(1 - \varepsilon) k} (R_0 - 1), \\ T_2 &= \frac{\pi g}{dg + (1 - \varepsilon) k \mu}, \\ T_2^* &= e^{-m_1 \tau_1} \frac{(1 - \xi) (1 - \varepsilon) k \pi \mu}{b (dg + (1 - \varepsilon) k \mu)}, \\ C_2^* &= e^{-m_1 \tau_1} \frac{\xi (1 - \varepsilon) k \pi \mu}{a (dg + (1 - \varepsilon) k \mu)}, \\ V_2 &= \frac{\mu}{g}, \\ Z_2 &= \frac{c}{r} \left(\frac{dg R_0}{dg + (1 - \varepsilon) k \mu} - 1 \right). \end{aligned} \quad (14)$$

We note that T_1, T_1^*, C_1^* , and V_1 are positive when $R_0 > 1$ and where T_2, T_2^*, C_2^* , and $V_2 > 0$ and $Z_2 > 0$ when $dg R_0 / (dg + (1 - \varepsilon) k \mu) > 1$. Now, we define humoral immune response reproduction number R_1 as

$$R_1 = \frac{R_0}{1 + (1 - \varepsilon) k \mu / dg}. \quad (15)$$

Clearly, $R_0 > R_1$. From above, we can state the following lemma.

Lemma 2. For system (3)–(7), one has the following:

- (i) If $R_0 \leq 1$, then the system has only one positive steady state S_0 .
- (ii) If $R_1 \leq 1 < R_0$, then the system has two positive steady states S_0 and S_1 .
- (iii) If $R_1 > 1$, then the system has three positive steady states S_0, S_1 , and S_2 .

2.3. Global Stability Analysis. We establish the global stability of all the steady states of system (3)–(7) employing the method of Lyapunov function. Let us define

$$H(\theta) = \theta - 1 - \ln \theta. \quad (16)$$

Theorem 3. For system (3)–(7), if $R_0 \leq 1$, then S_0 is GAS.

Proof. Define

$$\begin{aligned} U_0 = & T_0 H\left(\frac{T}{T_0}\right) + \eta_1 T^* + \eta_2 C^* + \eta_3 V + \eta_4 Z \\ & + \eta_5 \int_0^{\tau_1} T(t-\theta) V(t-\theta) d\theta \\ & + \eta_6 \int_0^{\tau_2} T^*(t-\theta) d\theta + \eta_7 \int_0^{\tau_3} C^*(t-\theta) d\theta, \end{aligned} \quad (17)$$

where η_i , $i = 1, \dots, 7$, are positive constants satisfying the following equations:

$$\begin{aligned} e^{-m_1 \tau_1} (1-\varepsilon) k ((1-\xi) \eta_1 + \xi \eta_2) &= \eta_5, \\ \eta_6 - b \eta_1 &= 0, \\ \eta_7 - a \eta_2 &= 0, \\ \eta_6 - \eta_3 e^{-m_2 \tau_2} N_T b &= 0, \\ \eta_7 - \eta_3 e^{-m_3 \tau_3} N_C a &= 0, \\ g \eta_4 - r \eta_3 &= 0, \\ \eta_5 &= (1-\varepsilon) k. \end{aligned} \quad (18)$$

The solution of (18) is given by

$$\begin{aligned} \eta_1 &= \frac{N_T e^{m_1 \tau_1 + m_3 \tau_3}}{e^{m_3 \tau_3} (1-\xi) N_T + e^{m_2 \tau_2} \xi N_C}, \\ \eta_2 &= \frac{N_C e^{m_1 \tau_1 + m_2 \tau_2}}{e^{m_3 \tau_3} (1-\xi) N_T + e^{m_2 \tau_2} \xi N_C}, \\ \eta_3 &= \frac{e^{m_1 \tau_1 + m_2 \tau_2 + m_3 \tau_3}}{e^{m_3 \tau_3} (1-\xi) N_T + e^{m_2 \tau_2} \xi N_C}, \\ \eta_4 &= \frac{r e^{m_1 \tau_1 + m_2 \tau_2 + m_3 \tau_3}}{g [e^{m_3 \tau_3} (1-\xi) N_T + e^{m_2 \tau_2} \xi N_C]}, \\ \eta_5 &= (1-\varepsilon) k, \\ \eta_6 &= \frac{b N_T e^{m_1 \tau_1 + m_3 \tau_3}}{e^{m_3 \tau_3} (1-\xi) N_T + e^{m_2 \tau_2} \xi N_C}, \\ \eta_7 &= \frac{a N_C e^{m_1 \tau_1 + m_2 \tau_2}}{e^{m_3 \tau_3} (1-\xi) N_T + e^{m_2 \tau_2} \xi N_C}. \end{aligned} \quad (19)$$

The values of η_i , $i = 1, \dots, 7$, will be used through the paper. Calculating the derivative of U_0 along the solutions of system (3)–(7) and applying $\pi = T_0 d$, we obtain

$$\begin{aligned} \frac{dU_0}{dt} = & \left(1 - \frac{T_0}{T}\right) (\pi - (1-\varepsilon) k T V - d T) \\ & + \eta_1 (e^{-m_1 \tau_1} (1-\xi) (1-\varepsilon) k T (t-\tau_1) V (t-\tau_1) \end{aligned}$$

$$\begin{aligned} & - b T^*) + \eta_2 (e^{-m_1 \tau_1} \xi (1-\varepsilon) k T (t-\tau_1) V (t-\tau_1) \\ & - a C^*) + \eta_3 (e^{-m_2 \tau_2} N_T b T^* (t-\tau_2) \\ & + e^{-m_3 \tau_3} N_C a C^* (t-\tau_3) - r V Z - c V) + \eta_4 (g V Z \\ & - \mu Z) + \eta_5 T V - \eta_5 T (t-\tau_1) V (t-\tau_1) + \eta_6 T^* \\ & - \eta_6 T^* (t-\tau_2) + \eta_7 C^* - \eta_7 C^* (t-\tau_3). \end{aligned} \quad (20)$$

Then,

$$\begin{aligned} \frac{dU_0}{dt} &= \left(1 - \frac{T_0}{T}\right) (\pi - d T) + (1-\varepsilon) k T_0 V - \eta_3 c V \\ &\quad - \eta_4 \mu Z \\ &= -d \frac{(T - T_0)^2}{T} + c \eta_3 (R_0 - 1) V - \eta_4 \mu Z. \end{aligned} \quad (21)$$

Therefore, if $R_0 \leq 1$, then $dU_0/dt \leq 0$, for all $T, V, Z > 0$. The solutions of system (3)–(7) limit M , the largest invariant subset of $\{dU_0/dt = 0\}$ [22]. We note that $dU_0/dt = 0$ if and only if $T(t) = T_0$, $V(t) = 0$, and $Z(t) = 0$. For each element of M , we have $V(t) = 0$ and $Z(t) = 0$; then, $\dot{V}(t) = 0$ and

$$\begin{aligned} 0 &= \dot{V}(t) \\ &= e^{-m_2 \tau_2} N_T b T^* (t-\tau_2) + e^{-m_3 \tau_3} N_C a C^* (t-\tau_3). \end{aligned} \quad (22)$$

Since $T^*(t), C^*(t) \geq 0$, then $T^*(t) = C^*(t) = 0$. Hence, $dU_0/dt = 0$ if and only if $T(t) = T_0$, $T^*(t) = 0$, $C^*(t) = 0$, $V(t) = 0$, and $Z(t) = 0$. It follows from LaSalle's invariance principle (LIP) that S_0 is GAS when $R_0 \leq 1$. $\square$

Theorem 4. For system (3)–(7), assume that $R_1 \leq 1 < R_0$; then, S_1 is GAS.

Proof. Define

$$\begin{aligned} U_1 = & T_1 H\left(\frac{T}{T_1}\right) + \eta_1 T_1^* H\left(\frac{T^*}{T_1^*}\right) + \eta_2 C_1^* H\left(\frac{C^*}{C_1^*}\right) \\ & + \eta_3 V_1 H\left(\frac{V}{V_1}\right) + \eta_4 Z \\ & + \eta_5 T_1 V_1 \int_0^{\tau_1} H\left(\frac{T(t-\theta) V(t-\theta)}{T_1 V_1}\right) d\theta \\ & + \eta_6 T_1^* \int_0^{\tau_2} H\left(\frac{T^*(t-\theta)}{T_1^*}\right) d\theta \\ & + \eta_7 C_1^* \int_0^{\tau_3} H\left(\frac{C^*(t-\theta)}{C_1^*}\right) d\theta. \end{aligned} \quad (23)$$

Then, dU_1/dt is given by

$$\begin{aligned}
 \frac{dU_1}{dt} = & \left(1 - \frac{T_1}{T}\right) (\pi - (1 - \varepsilon) kTV - dT) + \eta_1 \left(1 - \frac{T_1^*}{T^*}\right) (e^{-m_1\tau_1} (1 - \xi) (1 - \varepsilon) kT(t - \tau_1) V(t - \tau_1) \\
 & - bT^*) + \eta_2 \left(1 - \frac{C_1^*}{C^*}\right) \cdot (e^{-m_1\tau_1} \xi (1 - \varepsilon) kT(t - \tau_1) V(t - \tau_1) \\
 & - aC^*) + \eta_3 \left(1 - \frac{V_1}{V}\right) (e^{-m_2\tau_2} N_T bT^*(t - \tau_2) \\
 & + e^{-m_3\tau_3} N_C aC^*(t - \tau_3) - rVZ - cV) + \eta_4 (gVZ \\
 & - \mu Z) + \eta_5 \left[TV - T(t - \tau_1) V(t - \tau_1)\right. \\
 & \left.+ T_1 V_1 \ln\left(\frac{T(t - \tau_1) V(t - \tau_1)}{TV}\right)\right] + \eta_6 \left[T^* \right. \\
 & \left.- T^*(t - \tau_2) + T_1^* \ln\left(\frac{T^*(t - \tau_2)}{T^*}\right)\right] + \eta_7 \left[C^* \right. \\
 & \left.- C^*(t - \tau_3) + C_1^* \ln\left(\frac{C^*(t - \tau_3)}{C^*}\right)\right]. \quad (24)
 \end{aligned}$$

Equation (24) can be simplified as

$$\begin{aligned}
 \frac{dU_1}{dt} = & \left(1 - \frac{T_1}{T}\right) (\pi - dT) \\
 & - e^{m_1\tau_1} \eta_1 (1 - \xi) (1 - \varepsilon) kT(t - \tau_1) V(t - \tau_1) \frac{T_1^*}{T^*} \\
 & + \eta_1 bT_1^* \\
 & - e^{-m_1\tau_1} \eta_2 \xi (1 - \varepsilon) kT(t - \tau_1) V(t - \tau_1) \frac{C_1^*}{C^*} \\
 & + \eta_2 aC_1^* - b\eta_1 \frac{T^*(t - \tau_2) V_1}{V} \\
 & - a\eta_2 \frac{C^*(t - \tau_3) V_1}{V} + c\eta_3 V_1 + r\eta_3 V_1 Z - \mu\eta_4 Z \\
 & + \eta_5 T_1 V_1 \ln\left(\frac{T(t - \tau_1) V(t - \tau_1)}{TV}\right) \\
 & + \eta_1 bT_1^* \ln\left(\frac{T^*(t - \tau_2)}{T^*}\right) \\
 & + \eta_2 aC_1^* \ln\left(\frac{C^*(t - \tau_3)}{C^*}\right). \quad (25)
 \end{aligned}$$

Applying $\pi = dT_1 + (1 - \varepsilon)kT_1V_1$, we get

$$\begin{aligned}
 \frac{dU_1}{dt} = & \left(1 - \frac{T_1}{T}\right) (dT_1 - dT) + (1 - \varepsilon) \\
 & \cdot kT_1V_1 \left(1 - \frac{T_1}{T}\right) - e^{-m_1\tau_1} \eta_1 (1 - \xi) (1 - \varepsilon) \\
 & \cdot kT(t - \tau_1) V(t - \tau_1) \frac{T_1^*}{T^*} + \eta_1 bT_1^* \\
 & - e^{-m_1\tau_1} \eta_2 \xi (1 - \varepsilon) kT(t - \tau_1) V(t - \tau_1) \frac{C_1^*}{C^*} \\
 & + \eta_2 aC_1^* - b\eta_1 \frac{T^*(t - \tau_2) V_1}{V} - a\eta_2 \\
 & \cdot \frac{C^*(t - \tau_3) V_1}{V} + c\eta_3 V_1 + r\eta_3 V_1 Z - \mu\eta_4 Z \\
 & + T_1 V_1 e^{-m_1\tau_1} \eta_1 (1 - \xi) (1 - \varepsilon) k \\
 & \cdot \ln\left(\frac{T(t - \tau_1) V(t - \tau_1)}{TV}\right) \\
 & + T_1 V_1 e^{-m_1\tau_1} \eta_2 \xi (1 - \varepsilon) k \\
 & \cdot \ln\left(\frac{T(t - \tau_1) V(t - \tau_1)}{TV}\right) + \eta_1 bT_1^* \\
 & \cdot \ln\left(\frac{T^*(t - \tau_2)}{T^*}\right) + \eta_2 aC_1^* \\
 & \cdot \ln\left(\frac{C^*(t - \tau_3)}{C^*}\right). \quad (26)
 \end{aligned}$$

Using the steady state conditions for S_1 ,

$$\begin{aligned}
 (1 - \xi) (1 - \varepsilon) kT_1V_1 &= e^{m_1\tau_1} bT_1^*, \\
 \xi (1 - \varepsilon) kT_1V_1 &= e^{m_1\tau_1} aC_1^*, \\
 cV_1 &= e^{-m_2\tau_2} N_T bT_1^* + e^{-m_3\tau_3} N_C aC_1^*, \quad (27)
 \end{aligned}$$

we get $(1 - \varepsilon)kT_1V_1 = \eta_1 bT_1^* + \eta_2 aC_1^*$ and

$$\begin{aligned}
 \frac{dU_1}{dt} = & -d \frac{(T - T_1)^2}{T} + \eta_1 bT_1^* \left(1 - \frac{T_1}{T}\right) \\
 & + \eta_2 aC_1^* \left(1 - \frac{T_1}{T}\right) \\
 & - \eta_1 bT_1^* \frac{T(t - \tau_1) V(t - \tau_1) T_1^*}{T_1 V_1 T^*} + \eta_1 bT_1^* \\
 & - \eta_2 aC_1^* \frac{T(t - \tau_1) V(t - \tau_1) C_1^*}{T_1 V_1 C^*} + \eta_2 aC_1^* \\
 & - \eta_1 bT_1^* \frac{V_1 T^*(t - \tau_2)}{VT_1^*}
 \end{aligned}$$

$$\begin{aligned}
& -\eta_2 a C_1^* \frac{C^*(t-\tau_3) V_1}{V C_1^*} + \eta_1 b T_1^* + \eta_2 a C_1^* \\
& + r \eta_3 V_1 Z - \mu \eta_4 Z \\
& + \eta_1 b T_1^* \ln \left(\frac{T(t-\tau_1) V(t-\tau_1)}{TV} \right) \\
& + \eta_2 a C_1^* \ln \left(\frac{T(t-\tau_1) V(t-\tau_1)}{TV} \right) \\
& + \eta_1 b T_1^* \ln \left(\frac{T^*(t-\tau_2)}{T^*} \right) \\
& + \eta_2 a C_1^* \ln \left(\frac{C^*(t-\tau_3)}{C^*} \right).
\end{aligned} \tag{28}$$

Consider the following equalities:

$$\begin{aligned}
& \ln \left(\frac{T(t-\tau_1) V(t-\tau_1)}{TV} \right) \\
& = \ln \left(\frac{T_j}{T} \right) + \ln \left(\frac{T(t-\tau_1) V(t-\tau_1) T_j^*}{T_j V_j T^*} \right) \\
& \quad + \ln \left(\frac{V_j T^*}{V T_j^*} \right), \\
& \ln \left(\frac{T(t-\tau_1) V(t-\tau_1)}{TV} \right) \\
& = \ln \left(\frac{T_j}{T} \right) + \ln \left(\frac{T(t-\tau_1) V(t-\tau_1) C_j^*}{T_j V_j C^*} \right) \\
& \quad + \ln \left(\frac{V_j C^*}{V C_j^*} \right), \\
& \ln \left(\frac{T^*(t-\tau_2)}{T^*} \right) \\
& = \ln \left(\frac{T^*(t-\tau_2) V_j}{V T_j^*} \right) + \ln \left(\frac{V T_j^*}{V_j T^*} \right), \\
& \ln \left(\frac{C^*(t-\tau_3)}{C^*} \right) \\
& = \ln \left(\frac{C^*(t-\tau_3) V_j}{V C_j^*} \right) + \ln \left(\frac{V C_j^*}{V_j C^*} \right),
\end{aligned} \tag{29}$$

$$j = 1, 2.$$

Using (29) in case of $j = 1$, we obtain

$$\begin{aligned}
\frac{dU_1}{dt} & = -d \frac{(T-T_1)^2}{T} - \eta_1 b T_1^* \left[\frac{T_1}{T} - 1 - \ln \left(\frac{T_1}{T} \right) \right] \\
& \quad - \eta_2 a C_1^* \left[\frac{T_1}{T} - 1 - \ln \left(\frac{T_1}{T} \right) \right]
\end{aligned}$$

$$\begin{aligned}
& -\eta_1 b T_1^* \left[\frac{T(t-\tau_1) V(t-\tau_1) T_1^*}{T_1 V_1 T^*} - 1 \right. \\
& \quad \left. - \ln \left(\frac{T(t-\tau_1) V(t-\tau_1) T_1^*}{T_1 V_1 T^*} \right) \right] \\
& - \eta_2 a C_1^* \left[\frac{T(t-\tau_1) V(t-\tau_1) C_1^*}{T_1 V_1 C^*} - 1 \right. \\
& \quad \left. - \ln \left(\frac{T(t-\tau_1) V(t-\tau_1) C_1^*}{T_1 V_1 C^*} \right) \right] \\
& - \eta_1 b T_1^* \left[\frac{T^*(t-\tau_2) V_1}{V T_1^*} - 1 \right. \\
& \quad \left. - \ln \left(\frac{T^*(t-\tau_2) V_1}{V T_1^*} \right) \right] - \eta_2 a C_1^* \left[\frac{C^*(t-\tau_3) V_1}{V C_1^*} \right. \\
& \quad \left. - 1 - \ln \left(\frac{C^*(t-\tau_3) V_1}{V C_1^*} \right) \right] + r \eta_3 \left(V_1 - \frac{\mu}{g} \right) Z \\
& = -d \frac{(T-T_1)^2}{T} - \eta_1 b T_1^* \left[H \left(\frac{T_1}{T} \right) \right. \\
& \quad + H \left(\frac{T(t-\tau_1) V(t-\tau_1) T_1^*}{T_1 V_1 T^*} \right) \\
& \quad + H \left(\frac{T^*(t-\tau_2) V_1}{V T_1^*} \right) \left. \right] - \eta_2 a C_1^* \left[H \left(\frac{T_1}{T} \right) \right. \\
& \quad + H \left(\frac{T(t-\tau_1) V(t-\tau_1) C_1^*}{T_1 V_1 C^*} \right) \\
& \quad + H \left(\frac{C^*(t-\tau_3) V_1}{V C_1^*} \right) \left. \right] + r \eta_3 \left(\frac{dg + \mu(1-\varepsilon)k}{g(1-\varepsilon)k} \right) \\
& \quad \cdot [R_1 - 1] Z.
\end{aligned} \tag{30}$$

We have if $R_0 > 1$, then $T_1, T_1^*, C_1^*, V_1 > 0$. Since $R_1 \leq 1$, then for all $T, T^*, C^*, V, Z > 0$, we have $dU_1/dt \leq 0$. We note that $dU_1/dt = 0$ at S_1 . Then, from LIP, S_1 is GAS. $\square$

Theorem 5. For system (3)–(7), assume that $R_1 > 1$; then, S_2 is GAS.

Proof. We consider

$$\begin{aligned}
U_2 & = T_2 H \left(\frac{T}{T_2} \right) + \eta_1 T_2^* H \left(\frac{T^*}{T_2^*} \right) + \eta_2 C_2^* H \left(\frac{C^*}{C_2^*} \right) \\
& \quad + \eta_3 V_2 H \left(\frac{V}{V_2} \right) + \eta_4 Z_2 H \left(\frac{Z}{Z_2} \right) \\
& \quad + \eta_5 T_2 V_2 \int_0^{\tau_1} H \left(\frac{T(t-\theta) V(t-\theta)}{T_2 V_2} \right) d\theta
\end{aligned}$$

$$\begin{aligned}
& + \eta_6 T_2^* \int_0^{\tau_2} H \left(\frac{T^*(t-\theta)}{T_2^*} \right) d\theta \\
& + \eta_7 C_2^* \int_0^{\tau_3} H \left(\frac{C^*(t-\theta)}{C_2^*} \right) d\theta.
\end{aligned} \tag{31}$$

Function U_2 along the trajectories of system (3)–(7) satisfies

$$\begin{aligned}
\frac{dU_2}{dt} = & \left(1 - \frac{T_2}{T} \right) (\pi - (1-\varepsilon)kTV - dT) + \eta_1 \left(1 - \frac{T_2^*}{T^*} \right) (e^{-m_1\tau_1} (1-\xi) (1-\varepsilon)kT(t-\tau_1)V(t-\tau_1) \\
& - bT^*) + \eta_2 \left(1 - \frac{C_2^*}{C^*} \right) \cdot (e^{-m_1\tau_1}\xi(1-\varepsilon)kT(t-\tau_1)V(t-\tau_1) \\
& - aC^*) + \eta_3 \left(1 - \frac{V_2}{V} \right) (e^{-m_2\tau_2}N_T bT^*(t-\tau_2) \\
& + e^{-m_3\tau_3}N_C aC^*(t-\tau_3) - rVZ - cV) + \eta_4 \left(1 - \frac{Z_2}{Z} \right) (gVZ - \mu Z) + \eta_5 \left[TV \right. \\
& - T(t-\tau_1)V(t-\tau_1) \\
& + T_2V_2 \ln \left(\frac{T(t-\tau_1)V(t-\tau_1)}{TV} \right) \Big] + \eta_6 \left[T^* \right. \\
& - T^*(t-\tau_2) + T_2^* \ln \left(\frac{T^*(t-\tau_2)}{T^*} \right) \Big] + \eta_7 \left[C^* \right. \\
& - C^*(t-\tau_3) + C_2^* \ln \left(\frac{C^*(t-\tau_3)}{C^*} \right) \Big].
\end{aligned} \tag{32}$$

Using the steady state conditions for S_2 ,

$$\begin{aligned}
\pi &= dT_2 + (1-\varepsilon)kT_2V_2, \\
(1-\xi)(1-\varepsilon)kT_2V_2 &= e^{m_1\tau_1}bT_2^*, \\
\xi(1-\varepsilon)kT_2V_2 &= e^{m_1\tau_1}aC_2^*, \\
cV_2 &= e^{-m_2\tau_2}N_T bT_2^* + e^{-m_3\tau_3}N_C aC_2^* \\
&\quad - rV_2Z_2,
\end{aligned} \tag{33}$$

we get $(1-\varepsilon)kT_2V_2 = \eta_1 bT_2^* + \eta_2 aC_2^*$ and

$$\begin{aligned}
\frac{dU_2}{dt} = & -d \frac{(T-T_2)^2}{T} + \eta_1 bT_2^* \left(1 - \frac{T_2}{T} \right) \\
& + \eta_2 aC_2^* \left(1 - \frac{T_2}{T} \right) \\
& - \eta_1 bT_2^* \frac{T(t-\tau_1)V(t-\tau_1)T_2^*}{T_2V_2T^*} + \eta_1 bT_2^*
\end{aligned}$$

$$\begin{aligned}
& - \eta_2 aC_2^* \frac{T(t-\tau_1)V(t-\tau_1)C_2^*}{T_2V_2C^*} + \eta_2 aC_2^* \\
& - \eta_1 bT_2^* \frac{V_2T^*(t-\tau_2)}{VT_2^*} - \eta_2 aC_2^* \frac{V_2C^*(t-\tau_3)}{VC_2^*} \\
& + \eta_1 bT_2^* + \eta_2 aC_2^* \\
& + \eta_1 bT_2^* \ln \left(\frac{T(t-\tau_1)V(t-\tau_1)}{TV} \right) \\
& + \eta_2 aC_2^* \ln \left(\frac{T(t-\tau_1)V(t-\tau_1)}{TV} \right) \\
& + \eta_1 bT_2^* \ln \left(\frac{T^*(t-\tau_2)}{T^*} \right) \\
& + \eta_2 aC_2^* \ln \left(\frac{C^*(t-\tau_3)}{C^*} \right).
\end{aligned} \tag{34}$$

Using (29) in case of $j = 2$, we obtain

$$\begin{aligned}
\frac{dU_2}{dt} = & -d \frac{(T-T_2)^2}{T} - \eta_1 bT_2^* \left[H \left(\frac{T_2}{T} \right) \right. \\
& + H \left(\frac{T(t-\tau_1)V(t-\tau_1)T_2^*}{T_2V_2T^*} \right) \\
& + H \left(\frac{T^*(t-\tau_2)V_2}{VT_2^*} \right) \Big] - \eta_2 aC_2^* \left[H \left(\frac{T_2}{T} \right) \right. \\
& + H \left(\frac{T(t-\tau_1)V(t-\tau_1)C_2^*}{T_2V_2C^*} \right) \\
& + H \left(\frac{C^*(t-\tau_3)V_2}{VC_2^*} \right) \Big].
\end{aligned} \tag{35}$$

Since $R_1 > 1$, then T_2, T_2^*, C_2^*, V_2 , and $Z_2 > 0$. It is observed that $dU_2/dt = 0$ if and only if $T = T_2, T^* = T_2^*, C^* = C_2^*$, and $V = V_2$. Therefore, if $V = V_2$, then $\dot{V} = 0$ and (6) becomes

$$0 = e^{-m_2\tau_2}N_T bT_2^* + e^{-m_3\tau_3}N_C aC_2^* - cV_2 - rV_2Z_2, \tag{36}$$

which gives $Z = Z_2$. Hence, dU_2/dt is equal to zero at S_2 . The global stability of S_2 follows from LIP. $\square$

3. HIV Dynamics Model with Saturated Incidence

We present an HIV infection model with saturated incidence:

$$\begin{aligned}
\dot{T}(t) &= \pi - \frac{(1-\varepsilon)kT(t)V(t)}{1+\beta V(t)} - dT(t), \\
\dot{T}^*(t) &= e^{-m_1\tau_1} \frac{(1-\xi)(1-\varepsilon)kT(t-\tau_1)V(t-\tau_1)}{1+\beta V(t-\tau_1)}
\end{aligned}$$

$$\begin{aligned}
& -bT^*(t), \\
\dot{C}^*(t) &= e^{-m_1\tau_1} \frac{\xi(1-\varepsilon)kT(t-\tau_1)V(t-\tau_1)}{1+\beta V(t-\tau_1)} \\
& -aC^*(t), \\
\dot{V}(t) &= e^{-m_2\tau_2} N_T bT^*(t-\tau_2) \\
& + e^{-m_3\tau_3} N_C aC^*(t-\tau_3) - rV(t)Z(t) \\
& -cV(t), \\
\dot{Z}(t) &= gV(t)Z(t) - \mu Z(t),
\end{aligned} \tag{37}$$

where $\beta > 0$ is the saturation incidence rate constant. Similar to the previous section, one can show that the solutions of the model are nonnegative and bounded.

3.1. Steady States. System (37) always admits an uninfected steady state $S_0 = (T_0, 0, 0, 0, 0)$, where $T_0 = \pi/d$. Now, we define the basic reproduction number for system (37) as

$$\begin{aligned}
R_0 &= \frac{(1-\varepsilon)kT_0 [e^{-(m_1\tau_1+m_2\tau_2)}(1-\xi)N_T + e^{-(m_1\tau_1+m_3\tau_3)}\xi N_C]}{c}.
\end{aligned} \tag{38}$$

The system has two other steady states $S_1(T_1, T_1^*, C_1^*, V_1, 0)$ and $S_2(T_2, T_2^*, C_2^*, V_2, Z_2)$, where

$$\begin{aligned}
T_1 &= \frac{\beta\pi [e^{-(m_1\tau_1+m_2\tau_2)}(1-\xi)N_T + e^{-(m_1\tau_1+m_3\tau_3)}\xi N_C] + c}{((1-\varepsilon)k + d\beta) [e^{-(m_1\tau_1+m_2\tau_2)}(1-\xi)N_T + e^{-(m_1\tau_1+m_3\tau_3)}\xi N_C]}, \\
T_1^* &= e^{-m_1\tau_1} \frac{(1-\xi)cd}{b((1-\varepsilon)k + d\beta) [e^{-(m_1\tau_1+m_2\tau_2)}(1-\xi)N_T + e^{-(m_1\tau_1+m_3\tau_3)}\xi N_C]} (R_0 - 1), \\
C_1^* &= e^{-m_1\tau_1} \frac{\xi cd}{a((1-\varepsilon)k + d\beta) [e^{-(m_1\tau_1+m_2\tau_2)}(1-\xi)N_T + e^{-(m_1\tau_1+m_3\tau_3)}\xi N_C]} (R_0 - 1), \\
V_1 &= \frac{d}{(1-\varepsilon)k + d\beta} (R_0 - 1), \\
T_2 &= \frac{\pi(g + \beta\mu)}{gd + (1-\varepsilon)k\mu + d\beta\mu}, \\
T_2^* &= e^{-m_1\tau_1} \frac{(1-\xi)(1-\varepsilon)k\pi\mu}{b(gd + (1-\varepsilon)k\mu + d\beta\mu)}, \\
C_2^* &= e^{-m_1\tau_1} \frac{\xi(1-\varepsilon)k\pi\mu}{a(gd + (1-\varepsilon)k\mu + d\beta\mu)}, \\
V_2 &= \frac{\mu}{g}, \\
Z_2 &= \frac{c}{r} \left(\frac{dgR_0}{dg + (1-\varepsilon)k\mu + d\beta\mu} - 1 \right).
\end{aligned} \tag{39}$$

We note that T_1, T_1^*, C_1^* , and V_1 are positive when $R_0 > 1$. And $T_2, T_2^*, C_2^*, V_2 > 0$ and $Z_2 > 0$ when $dgR_0/(dg + (1-\varepsilon)k\mu) > 1$. Now, we define another threshold parameter R_1 as

$$R_1 = \frac{R_0}{1 + ((1-\varepsilon)k\mu + d\beta\mu)/dg}. \tag{40}$$

Clearly, $R_0 > R_1$. From above, we have the following result.

Lemma 6. For system (37), one has the following:

- (i) If $R_0 \leq 1$, then the system has only one positive steady state S_0 .

- (ii) If $R_1 \leq 1 < R_0$, then the system has two positive steady states S_0 and S_1 .

- (iii) If $R_1 > 1$, then the system has three positive steady states S_0, S_1 , and S_2 .

3.2. Global Stability Analysis. In this subsection, we investigate the global stability of system (37) by constructing suitable Lyapunov functionals and applying LaSalle invariance principle.

Theorem 7. For system (37), if $R_0 \leq 1$, then S_0 is GAS.

Proof. Define

$$\begin{aligned} U_0 = & T_0 H\left(\frac{T}{T_0}\right) + \eta_1 T^* + \eta_2 C^* + \eta_3 V + \eta_4 Z \\ & + \eta_5 \int_0^{\tau_1} \frac{T(t-\theta)V(t-\theta)}{1+\beta V(t-\theta)} d\theta \\ & + \eta_6 \int_0^{\tau_2} T^*(t-\theta) d\theta + \eta_7 \int_0^{\tau_3} C^*(t-\theta) d\theta. \end{aligned} \quad (41)$$

Calculating the derivative of U_0 along the solutions of system (37), we obtain

$$\begin{aligned} \frac{dU_0}{dt} = & \left(1 - \frac{T_0}{T}\right) \left(\pi - \frac{(1-\varepsilon)kTV}{1+\beta V} - dT\right) \\ & + \eta_1 \left(e^{-m_1 \tau_1} \frac{(1-\xi)(1-\varepsilon)kT(t-\tau_1)V(t-\tau_1)}{1+\beta V(t-\tau_1)} - bT^*\right) \\ & + \eta_2 \left(e^{-m_1 \tau_1} \frac{\xi(1-\varepsilon)kT(t-\tau_1)V(t-\tau_1)}{1+\beta V(t-\tau_1)} - aC^*\right) \\ & + \eta_3 (e^{-m_2 \tau_2} N_T bT^*(t-\tau_2) \\ & + e^{-m_3 \tau_3} N_C aC^*(t-\tau_3) - rVZ - cV) + \eta_4 (gVZ \\ & - \mu Z) + \eta_5 \frac{TV}{1+\beta V} - \eta_5 \frac{T(t-\tau_1)V(t-\tau_1)}{1+\beta V(t-\tau_1)} \\ & + \eta_6 T^* - \eta_6 T^*(t-\tau_2) + \eta_7 C^* - \eta_7 C^*(t-\tau_3). \end{aligned} \quad (42)$$

Collecting terms of (42), we obtain

$$\begin{aligned} \frac{dU_0}{dt} = & \left(1 - \frac{T_0}{T}\right) (\pi - dT) + \frac{(1-\varepsilon)kT_0V}{1+\beta V} - \eta_3 cV \\ & - \eta_4 \mu Z \\ = & - \left[d \frac{(T-T_0)^2}{T} + \eta_3 \frac{c\beta R_0 V^2}{1+\beta V} + \eta_4 \mu Z \right] \\ & + c\eta_3 (R_0 - 1)V. \end{aligned} \quad (43)$$

It follows that S_0 is GAS when $R_0 \leq 1$. $\square$

Theorem 8. For system (37), assume that $R_1 \leq 1 < R_0$; then, S_1 is GAS.

Proof. Consider

$$\begin{aligned} U_1 = & T_1 H\left(\frac{T}{T_1}\right) + \eta_1 T_1^* H\left(\frac{T^*}{T_1^*}\right) + \eta_2 C_1^* H\left(\frac{C^*}{C_1^*}\right) \\ & + \eta_3 V_1 H\left(\frac{V}{V_1}\right) + \eta_4 Z + \eta_5 \frac{T_1 V_1}{1+\beta V_1} \\ & \cdot \int_0^{\tau_1} H\left(\frac{T(t-\theta)V(t-\theta)(1+\beta V_1)}{T_1 V_1 (1+\beta V(t-\theta))}\right) d\theta \\ & + \eta_6 T_1^* \int_0^{\tau_2} H\left(\frac{T^*(t-\theta)}{T_1^*}\right) d\theta \\ & + \eta_7 C_1^* \int_0^{\tau_3} H\left(\frac{C^*(t-\theta)}{C_1^*}\right) d\theta. \end{aligned} \quad (44)$$

Then, dU_1/dt is given by

$$\begin{aligned} \frac{dU_1}{dt} = & \left(1 - \frac{T_1}{T}\right) \left(\pi - dT - \frac{(1-\varepsilon)kTV}{(1+\beta V)}\right) + \eta_1 \left(1 - \frac{T_1^*}{T^*}\right) \\ & \cdot \left(e^{-m_1 \tau_1} \frac{(1-\xi)(1-\varepsilon)kT(t-\tau_1)V(t-\tau_1)}{1+\beta V(t-\tau_1)} - bT^*\right) \\ & + \eta_2 \left(1 - \frac{C_1^*}{C^*}\right) \cdot \left(e^{-m_1 \tau_1} \frac{\xi(1-\varepsilon)kT(t-\tau_1)V(t-\tau_1)}{1+\beta V(t-\tau_1)} - aC^*\right) \\ & + \eta_3 \left(1 - \frac{V_1}{V}\right) (e^{-m_2 \tau_2} N_T bT^*(t-\tau_2) \\ & + e^{-m_3 \tau_3} N_C aC^*(t-\tau_3) - rVZ - cV) + \eta_4 (gVZ \\ & - \mu Z) + \eta_5 \left[\frac{TV}{1+\beta V} - \frac{T(t-\tau_1)V(t-\tau_1)}{(1+\beta V(t-\tau_1))}\right] \\ & + \frac{T_1 V_1}{1+\beta V_1} \ln\left(\frac{T(t-\tau_1)V(t-\tau_1)(1+\beta V)}{TV(1+\beta V(t-\tau_1))}\right) \\ & + \eta_6 \left[T^* - T^*(t-\tau_2) + T_1^* \ln\left(\frac{T^*(t-\tau_2)}{T^*}\right)\right] \\ & + \eta_7 \left[C^* - C^*(t-\tau_3) + C_1^* \ln\left(\frac{C^*(t-\tau_3)}{C^*}\right)\right]. \end{aligned} \quad (45)$$

Simplifying (45), we get

$$\begin{aligned} \frac{dU_1}{dt} = & \left(1 - \frac{T_1}{T}\right) (\pi - dT) + \frac{(1-\varepsilon)kT_1V}{1+\beta V} - e^{m_1 \tau_1} \eta_1 \\ & \cdot \frac{(1-\xi)(1-\varepsilon)kT(t-\tau_1)V(t-\tau_1)}{1+\beta V(t-\tau_1)} \frac{T_1^*}{T^*} \end{aligned}$$

$$\begin{aligned}
& + \eta_1 b T_1^* - e^{-m_1 \tau_1} \eta_2 \\
& \cdot \frac{\xi (1 - \varepsilon) k T (t - \tau_1) V (t - \tau_1) C_1^*}{1 + \beta V (t - \tau_1)} + \eta_2 a C_1^* \\
& - b \eta_1 \frac{T^* (t - \tau_2) V_1}{V} - a \eta_2 \frac{C^* (t - \tau_3) V_1}{V} \\
& - c \eta_3 V + c \eta_3 V_1 + r \eta_3 V_1 Z - \mu \eta_4 Z + \eta_5 \\
& \cdot \frac{T_1 V_1}{(1 + \beta V_1)} \\
& \cdot \ln \left(\frac{T (t - \tau_1) V (t - \tau_1) (1 + \beta V)}{TV (1 + \beta V (t - \tau_1))} \right) \\
& + \eta_1 b T_1^* \ln \left(\frac{T^* (t - \tau_2)}{T^*} \right) + \eta_2 a C_1^* \\
& \cdot \ln \left(\frac{C^* (t - \tau_3)}{C^*} \right).
\end{aligned} \tag{46}$$

Applying $\pi = dT_1 + (1 - \varepsilon)kT_1V_1/(1 + \beta V_1)$, we get

$$\begin{aligned}
\frac{dU_1}{dt} & = \left(1 - \frac{T_1}{T}\right) (dT_1 - dT) \\
& + \frac{(1 - \varepsilon)kT_1V_1}{1 + \beta V_1} \left(1 - \frac{T_1}{T}\right) + \frac{(1 - \varepsilon)kT_1V}{1 + \beta V} \\
& - e^{-m_1 \tau_1} \eta_1 \\
& \cdot \frac{(1 - \xi)(1 - \varepsilon)kT(t - \tau_1)V(t - \tau_1)T_1^*}{1 + \beta V(t - \tau_1)} \frac{1}{T^*} \\
& + \eta_1 b T_1^* - e^{-m_1 \tau_1} \eta_2 \\
& \cdot \frac{\xi(1 - \varepsilon)kT(t - \tau_1)V(t - \tau_1)C_1^*}{1 + \beta V(t - \tau_1)} + \eta_2 a C_1^* \\
& - b \eta_1 \frac{T^* (t - \tau_2) V_1}{V} - a \eta_2 \frac{C^* (t - \tau_3) V_1}{V} \\
& - c \eta_3 V + c \eta_3 V_1 + r \eta_3 V_1 Z - \mu \eta_4 Z \\
& + \frac{T_1 V_1 e^{-m_1 \tau_1} \eta_1 (1 - \xi)(1 - \varepsilon)k}{1 + \beta V_1} \\
& \cdot \ln \left(\frac{T(t - \tau_1)V(t - \tau_1)(1 + \beta V)}{TV(1 + \beta V(t - \tau_1))} \right) \\
& + \frac{T_1 V_1 e^{-m_1 \tau_1} \eta_2 \xi(1 - \varepsilon)k}{1 + \beta V_1} \\
& \cdot \ln \left(\frac{T(t - \tau_1)V(t - \tau_1)(1 + \beta V)}{TV(1 + \beta V(t - \tau_1))} \right) \\
& + \eta_1 b T_1^* \ln \left(\frac{T^* (t - \tau_2)}{T^*} \right) + \eta_2 a C_1^* \\
& \cdot \ln \left(\frac{C^* (t - \tau_3)}{C^*} \right).
\end{aligned} \tag{47}$$

Using the steady state conditions for S_1 ,

$$\begin{aligned}
\frac{(1 - \xi)(1 - \varepsilon)kT_1V_1}{1 + \beta V_1} & = e^{m_1 \tau_1} b T_1^*, \\
\frac{\xi(1 - \varepsilon)kT_1V_1}{1 + \beta V_1} & = e^{m_1 \tau_1} a C_1^*, \\
cV_1 & = e^{-m_2 \tau_2} N_T b T_1^* \\
& + e^{-m_3 \tau_3} N_C a C_1^*,
\end{aligned} \tag{48}$$

we obtain $(1 - \varepsilon)kT_1V_1/(1 + \beta V_1) = \eta_1 b T_1^* + \eta_2 a C_1^*$ and

$$\begin{aligned}
\frac{dU_1}{dt} & = -d \frac{(T - T_1)^2}{T} + \frac{(1 - \varepsilon)kT_1V_1}{1 + \beta V_1} \frac{(1 + \beta V_1)V}{(1 + \beta V)V_1} \\
& + \eta_1 b T_1^* \left(1 - \frac{T_1}{T}\right) + \eta_2 a C_1^* \left(1 - \frac{T_1}{T}\right) \\
& - \eta_1 b T_1^* \frac{T(t - \tau_1)V(t - \tau_1)T_1^*(1 + \beta V_1)}{T_1 V_1 T^* (1 + \beta V(t - \tau_1))} \\
& + \eta_1 b T_1^* \\
& - \eta_2 a C_1^* \frac{T(t - \tau_1)V(t - \tau_1)C_1^*(1 + \beta V_1)}{T_1 V_1 C^* (1 + \beta V(t - \tau_1))} \\
& + \eta_2 a C_1^* - \eta_1 b T_1^* \frac{V_1 T^* (t - \tau_2)}{V T_1^*} \\
& - \eta_2 a C_1^* \frac{C^* (t - \tau_3) V_1}{V C_1^*} + \eta_1 b T_1^* + \eta_2 a C_1^* \\
& - \frac{(1 - \varepsilon)kT_1V_1}{1 + \beta V_1} \frac{V}{V_1} + r \eta_3 V_1 Z - \mu \eta_4 Z \\
& + \eta_1 b T_1^* \ln \left(\frac{T(t - \tau_1)V(t - \tau_1)(1 + \beta V)}{TV(1 + \beta V(t - \tau_1))} \right) \\
& + \eta_2 a C_1^* \ln \left(\frac{T(t - \tau_1)V(t - \tau_1)(1 + \beta V)}{TV(1 + \beta V(t - \tau_1))} \right) \\
& + \eta_1 b T_1^* \ln \left(\frac{T^* (t - \tau_2)}{T^*} \right) \\
& + \eta_2 a C_1^* \ln \left(\frac{C^* (t - \tau_3)}{C^*} \right).
\end{aligned} \tag{49}$$

Consider the following equalities:

$$\begin{aligned}
& \ln \left(\frac{T(t - \tau_1)V(t - \tau_1)(1 + \beta V)}{TV(1 + \beta V(t - \tau_1))} \right) \\
& = \ln \left(\frac{T_j}{T} \right)
\end{aligned}$$

$$\begin{aligned}
& + \ln \left(\frac{T(t-\tau_1)V(t-\tau_1)(1+\beta V_1)T_j^*}{T_j V_j (1+\beta V(t-\tau_1))T^*} \right) \\
& + \ln \left(\frac{V_j T_j^*}{V T_j^*} \right) + \ln \left(\frac{1+\beta V}{1+\beta V_j} \right), \\
& \ln \left(\frac{T(t-\tau_1)V(t-\tau_1)(1+\beta V)}{TV(1+\beta V(t-\tau_1))} \right) \\
& = \ln \left(\frac{T_j}{T} \right) \\
& + \ln \left(\frac{T(t-\tau_1)V(t-\tau_1)(1+\beta V_j)C_j^*}{T_j V_j (1+\beta V(t-\tau_1))C^*} \right) \\
& + \ln \left(\frac{V_j C_j^*}{V C_j^*} \right) + \ln \left(\frac{1+\beta V}{1+\beta V_j} \right), \\
& \ln \left(\frac{T^*(t-\tau_2)}{T^*} \right) \\
& = \ln \left(\frac{T^*(t-\tau_2)V_j}{V T_j^*} \right) + \ln \left(\frac{V T_j^*}{V_j T^*} \right), \\
& \ln \left(\frac{C^*(t-\tau_3)}{C^*} \right) \\
& = \ln \left(\frac{C^*(t-\tau_3)V_j}{V C_j^*} \right) + \ln \left(\frac{V C_j^*}{V_j C^*} \right), \\
& j = 1, 2. \tag{50}
\end{aligned}$$

Using (50), when $j = 1$, we get

$$\begin{aligned}
\frac{dU_1}{dt} & = -d \frac{(T-T_1)^2}{T} + \frac{(1-\varepsilon)kT_1 V_1}{1+\beta V_1} \left[\frac{(1+\beta V_1)V}{(1+\beta V)V_1} \right. \\
& \quad \left. - \frac{V}{V_1} - 1 + \frac{1+\beta V}{1+\beta V_1} \right] - \eta_1 b T_1^* \left[\frac{T_1}{T} - 1 \right. \\
& \quad \left. - \ln \left(\frac{T_1}{T} \right) \right] - \eta_2 a C_1^* \left[\frac{T_1}{T} - 1 - \ln \left(\frac{T_1}{T} \right) \right] \\
& \quad - \eta_1 b T_1^* \left[\frac{T(t-\tau_1)V(t-\tau_1)(1+\beta V_1)T_1^*}{T_1 V_1 (1+\beta V(t-\tau_1))T^*} - 1 \right. \\
& \quad \left. - \ln \left(\frac{T(t-\tau_1)V(t-\tau_1)(1+\beta V_1)T_1^*}{T_1 V_1 (1+\beta V(t-\tau_1))T^*} \right) \right] \\
& \quad - \eta_2 a C_1^* \left[\frac{T(t-\tau_1)V(t-\tau_1)(1+\beta V_1)C_1^*}{T_1 V_1 (1+\beta V(t-\tau_1))C^*} - 1 \right. \\
& \quad \left. - \ln \left(\frac{T(t-\tau_1)V(t-\tau_1)(1+\beta V_1)C_1^*}{T_1 V_1 (1+\beta V(t-\tau_1))C^*} \right) \right]
\end{aligned}$$

$$\begin{aligned}
& - \eta_1 b T_1^* \left[\frac{T^*(t-\tau_2)V_1}{V T_1^*} - 1 \right. \\
& \quad \left. - \ln \left(\frac{T^*(t-\tau_2)V_1}{V T_1^*} \right) \right] - \eta_2 a C_1^* \left[\frac{C^*(t-\tau_3)V_1}{V C_1^*} \right. \\
& \quad \left. - 1 - \ln \left(\frac{C^*(t-\tau_3)V_1}{V C_1^*} \right) \right] - \eta_1 b T_1^* \left[\frac{1+\beta V}{1+\beta V_1} \right. \\
& \quad \left. - 1 - \ln \left(\frac{1+\beta V}{1+\beta V_1} \right) \right] - \eta_2 a C_1^* \left[\frac{1+\beta V}{1+\beta V_1} - 1 \right. \\
& \quad \left. - \ln \left(\frac{1+\beta V}{1+\beta V_1} \right) \right] + r \eta_3 \left(V_1 - \frac{\mu}{g} \right) Z \\
& = -d \frac{(T-T_1)^2}{T} \\
& \quad - \frac{(1-\varepsilon)kT_1 V_1}{1+\beta V_1} \left[\frac{\beta(V-V_1)^2}{(1+\beta V_1)(1+\beta V)V_1} \right] \\
& \quad - \eta_1 b T_1^* \left[H \left(\frac{T_1}{T} \right) \right. \\
& \quad \left. + H \left(\frac{T(t-\tau_1)V(t-\tau_1)(1+\beta V_1)T_1^*}{T_1 V_1 (1+\beta V(t-\tau_1))T^*} \right) \right. \\
& \quad \left. + H \left(\frac{T^*(t-\tau_2)V_1}{V T_1^*} \right) + H \left(\frac{1+\beta V}{1+\beta V_1} \right) \right] \\
& \quad - \eta_2 a C_1^* \left[H \left(\frac{T_1}{T} \right) \right. \\
& \quad \left. + H \left(\frac{T(t-\tau_1)V(t-\tau_1)(1+\beta V_1)C_1^*}{T_1 V_1 (1+\beta V(t-\tau_1))C^*} \right) \right. \\
& \quad \left. + H \left(\frac{C^*(t-\tau_3)V_1}{V C_1^*} \right) + H \left(\frac{1+\beta V}{1+\beta V_1} \right) \right] \\
& \quad + r \eta_3 \left(\frac{dg + \mu(1-\varepsilon)k + d\beta\mu}{g(1-\varepsilon)k + d\beta g} \right) [R_1 - 1] Z. \tag{51}
\end{aligned}$$

We have if $R_1 \leq 1 < R_0$, then $dU_1/dt \leq 0$, where equality occurs at S_1 . LIP implies global stability of S_1 . $\square$

Theorem 9. For system (37), assume that $R_1 > 1$; then, S_2 is GAS.

Proof. We consider

$$\begin{aligned}
U_2 & = T_2 H \left(\frac{T}{T_2} \right) + \eta_1 T_2^* H \left(\frac{T^*}{T_2^*} \right) + \eta_2 C_2^* H \left(\frac{C^*}{C_2^*} \right) \\
& \quad + \eta_3 V_2 H \left(\frac{V}{V_2} \right) + \eta_4 Z_2 H \left(\frac{Z}{Z_2} \right) + \eta_5 \frac{T_2 V_2}{1+\beta V_2} \\
& \quad \cdot \int_0^{\tau_1} H \left(\frac{T(t-\theta)V(t-\theta)(1+\beta V_2)}{T_2 V_2 (1+\beta V(t-\theta))} \right) d\theta
\end{aligned}$$

$$\begin{aligned}
& + \eta_6 T_2^* \int_0^{\tau_2} H \left(\frac{T^*(t-\theta)}{T_2^*} \right) d\theta \\
& + \eta_7 C_2^* \int_0^{\tau_3} H \left(\frac{C^*(t-\theta)}{C_2^*} \right) d\theta.
\end{aligned} \tag{52}$$

Calculate dU_2/dt along the trajectories of system (37) as

$$\begin{aligned}
\frac{dU_2}{dt} = & \left(1 - \frac{T_2}{T} \right) \left(\pi - dT - \frac{(1-\varepsilon)kTV}{1+\beta V} \right) + \eta_1 \left(1 - \frac{T_2^*}{T^*} \right) \\
& \cdot \left(\frac{e^{-m_1\tau_1}(1-\xi)(1-\varepsilon)kT(t-\tau_1)V(t-\tau_1)}{1+\beta V(t-\tau_1)} - bT^* \right) + \eta_2 \left(1 - \frac{C_2^*}{C^*} \right) \\
& \cdot \left(\frac{e^{-m_1\tau_1}\xi(1-\varepsilon)kT(t-\tau_1)V(t-\tau_1)}{1+\beta V(t-\tau_1)} - aC^* \right) + \eta_3 \left(1 - \frac{V_2}{V} \right) (e^{-m_2\tau_2}N_T bT^*(t-\tau_2) \\
& + e^{-m_3\tau_3}N_C aC^*(t-\tau_3) - rVZ - cV) + \eta_4 \left(1 - \frac{Z_2}{Z} \right) (gVZ - \mu Z) + \eta_5 \left[\frac{TV}{1+\beta V} \right. \\
& \left. - \frac{T(t-\tau_1)V(t-\tau_1)}{1+\beta V(t-\tau_1)} + \frac{T_2V_2}{1+\beta V_2} \ln \left(\frac{T(t-\tau_1)V(t-\tau_1)(1+\beta V)}{TV(1+\beta V(t-\tau_1))} \right) \right] \\
& + \eta_6 \left[T^* - T^*(t-\tau_2) + T_2^* \ln \left(\frac{T^*(t-\tau_2)}{T^*} \right) \right] \\
& + \eta_7 \left[C^* - C^*(t-\tau_3) + C_2^* \ln \left(\frac{C^*(t-\tau_3)}{C^*} \right) \right].
\end{aligned} \tag{53}$$

Using the steady state conditions for S_2 ,

$$\begin{aligned}
\pi &= dT_2 + \frac{(1-\varepsilon)kT_2V_2}{1+\beta V_2}, \\
\frac{(1-\xi)(1-\varepsilon)kT_2V_2}{1+\beta V_2} &= e^{m_1\tau_1}bT_2^*, \\
\frac{\xi(1-\varepsilon)kT_2V_2}{1+\beta V_2} &= e^{m_1\tau_1}aC_2^*, \\
cV_2 &= e^{-m_2\tau_2}N_T bT_2^* + e^{-m_3\tau_3}N_C aC_2^* \\
&\quad - rV_2Z_2,
\end{aligned} \tag{54}$$

we get $(1-\varepsilon)kT_2V_2/(1+\beta V_2) = \eta_1 bT_2^* + \eta_2 aC_2^*$, and

$$\begin{aligned}
\frac{dU_2}{dt} = & -d \frac{(T-T_2)^2}{T} + \frac{(1-\varepsilon)kT_2V_2}{1+\beta V_2} \frac{(1+\beta V_2)V}{(1+\beta V)V_2} \\
& + \eta_1 bT_2^* \left(1 - \frac{T_2}{T} \right) + \eta_2 aC_2^* \left(1 - \frac{T_2}{T} \right) \\
& - \eta_1 bT_2^* \frac{T(t-\tau_1)V(t-\tau_1)T_2^*(1+\beta V_2)}{T_2V_2T^*(1+\beta V(t-\tau_1))} \\
& + \eta_1 bT_2^* \\
& - \eta_2 aC_2^* \frac{T(t-\tau_1)V(t-\tau_1)C_2^*(1+\beta V_2)}{T_2V_2C^*(1+\beta V(t-\tau_1))} \\
& + \eta_2 aC_2^* - \eta_1 bT_2^* \frac{T^*(t-\tau_2)V_2}{VT_2^*} \\
& - \eta_2 aC_2^* \frac{C^*(t-\tau_3)V_2}{VC_2^*} + \eta_1 bT_2^* + \eta_2 aC_2^* \\
& - \frac{(1-\varepsilon)kT_2V_2}{1+\beta V_2} \frac{V}{V_2} \\
& + \eta_1 bT_2^* \ln \left(\frac{T(t-\tau_1)V(t-\tau_1)(1+\beta V)}{TV(1+\beta V(t-\tau_1))} \right) \\
& + \eta_2 aC_2^* \ln \left(\frac{T(t-\tau_1)V(t-\tau_1)(1+\beta V)}{TV(1+\beta V(t-\tau_1))} \right) \\
& + \eta_1 bT_2^* \ln \left(\frac{T^*(t-\tau_2)}{T^*} \right) \\
& + \eta_2 aC_2^* \ln \left(\frac{C^*(t-\tau_3)}{C^*} \right).
\end{aligned} \tag{55}$$

Using (50), when $j = 2$, we get

$$\begin{aligned}
\frac{dU_2}{dt} = & -d \frac{(T-T_2)^2}{T} + \frac{(1-\varepsilon)kT_2V_2}{1+\beta V_2} \left[\frac{(1+\beta V_2)V}{(1+\beta V)V_2} \right. \\
& \left. - \frac{V}{V_2} - 1 + \frac{1+\beta V}{1+\beta V_2} \right] - \eta_1 bT_2^* \left[\frac{T_2}{T} - 1 \right. \\
& \left. - \ln \left(\frac{T_2}{T} \right) \right] - \eta_2 aC_2^* \left[\frac{T_2}{T} - 1 - \ln \left(\frac{T_2}{T} \right) \right] \\
& - \eta_1 bT_2^* \left[\frac{T(t-\tau_1)V(t-\tau_1)(1+\beta V)T_2^*}{T_2V_2T^*(1+\beta V(t-\tau_1))} - 1 \right. \\
& \left. - \ln \left(\frac{T(t-\tau_1)V(t-\tau_1)(1+\beta V)T_2^*}{T_2V_2T^*(1+\beta V(t-\tau_1))} \right) \right] \\
& - \eta_2 aC_2^* \left[\frac{T(t-\tau_1)V(t-\tau_1)(1+\beta V)C_2^*}{T_2V_2C^*(1+\beta V(t-\tau_1))} - 1 \right. \\
& \left. - \ln \left(\frac{T(t-\tau_1)V(t-\tau_1)(1+\beta V)C_2^*}{T_2V_2C^*(1+\beta V(t-\tau_1))} \right) \right]
\end{aligned}$$

$$\begin{aligned}
& -\eta_1 b T_2^* \left[\frac{T^*(t - \tau_2) V_2}{V T_2^*} - 1 \right. \\
& \left. - \ln \left(\frac{T^*(t - \tau_2) V_2}{V T_2^*} \right) \right] - \eta_2 a C_2^* \left[\frac{C^*(t - \tau_3) V_2}{V C_2^*} \right. \\
& \left. - 1 - \ln \left(\frac{C^*(t - \tau_3) V_2}{V C_2^*} \right) \right] - \eta_1 b T_2^* \left[\frac{1 + \beta V}{1 + \beta V_2} \right. \\
& \left. - 1 - \ln \left(\frac{1 + \beta V}{1 + \beta V_2} \right) \right] - \eta_2 a C_2^* \left[\frac{1 + \beta V}{1 + \beta V_2} - 1 \right. \\
& \left. - \ln \left(\frac{1 + \beta V}{1 + \beta V_2} \right) \right] \\
& = -d \frac{(T - T_2)^2}{T} \\
& - \frac{(1 - \varepsilon) k T_2 V_2}{1 + \beta V_2} \left[\frac{\beta (V - V_2)^2}{(1 + \beta V_2)(1 + \beta V) V_2} \right] \\
& - \eta_1 b T_2^* \left[H \left(\frac{T_2}{T} \right) \right. \\
& + H \left(\frac{T(t - \tau_1) V(t - \tau_1) (1 + \beta V) T_2^*}{T_2 V_2 T^* (1 + \beta V(t - \tau_1))} \right) \\
& + H \left(\frac{T^*(t - \tau_2) V_2}{V T_2^*} \right) + H \left(\frac{1 + \beta V}{1 + \beta V_2} \right) \left. \right] \\
& - \eta_2 a C_2^* \left[H \left(\frac{T_2}{T} \right) \right. \\
& + H \left(\frac{T(t - \tau_1) V(t - \tau_1) (1 + \beta V) C_2^*}{T_2 V_2 C^* (1 + \beta V(t - \tau_1))} \right) \\
& + H \left(\frac{C^*(t - \tau_3) V_2}{V C_2^*} \right) + H \left(\frac{1 + \beta V}{1 + \beta V_2} \right) \left. \right]. \quad (56)
\end{aligned}$$

Similar to the proof of Theorem 5, one can show that S_2 is GAS. $\square$

4. Model with General Incidence

In the following model, we assume the incidence rate is given by a general function of the concentration of $CD4^+$ T cells and viruses:

$$\dot{T}(t) = \pi - \Phi(T(t), V(t)) - dT, \quad (57)$$

$$\begin{aligned}
\dot{T}^*(t) &= e^{-m_1 \tau_1} (1 - \xi) \Phi(T(t - \tau_1), V(t - \tau_1)) \\
&\quad - b T^*(t), \quad (58)
\end{aligned}$$

$$\dot{C}^*(t) = e^{-m_1 \tau_1} \xi \Phi(T(t - \tau_1), V(t - \tau_1)) - a C^*(t), \quad (59)$$

$$\begin{aligned}
\dot{V}(t) &= e^{-m_2 \tau_2} N_T b T^*(t - \tau_2) \\
&\quad + e^{-m_3 \tau_3} N_C a C^*(t - \tau_3) - r V(t) Z(t) \\
&\quad - c V(t), \quad (60)
\end{aligned}$$

$$\dot{Z}(t) = g V(t) Z(t) - \mu Z(t). \quad (61)$$

The incidence rate is given by $\Phi(T(t), V(t))$ which is assumed to be continuously differentiable; moreover, it satisfies the following conditions.

Condition C1. (i) Consider $\Phi(T, V) > 0$ and $\Phi(0, V) = \Phi(T, 0) = 0$, for all $T, V \in (0, \infty)$. (ii) Consider $\partial \Phi(T, V) / \partial T > 0$, $\partial \Phi(T, V) / \partial V > 0$, and $\partial \Phi(T, 0) / \partial V > 0$, for all $T, V \in (0, \infty)$.

Condition C2. (i) Consider $\Phi(T, V) \leq V(\partial \Phi(T, 0) / \partial V)$, for all $T, V \in (0, \infty)$. (ii) Consider $(d/dT)(\partial \Phi(T, 0) / \partial V) > 0$.

The nonnegativity and boundedness of the solutions of model (57)–(61) can be shown as given in Section 2.

4.1. Steady States

Lemma 10. Suppose that Conditions C1 and C2 are satisfied; then, there exist two bifurcation parameters R_0 and R_1 with $R_0 > R_1 > 0$. Moreover,

- (i) if $R_0 \leq 1$, then the system has only one positive steady state S_0 ,
- (ii) if $R_1 \leq 1 < R_0$, then the system has two positive steady states S_0 and S_1 ,
- (iii) if $R_1 > 1$, then the system has three positive steady states S_0 , S_1 , and S_2 .

Proof. Let

$$\pi - \Phi(T, V) - dT = 0, \quad (62)$$

$$e^{-m_1 \tau_1} (1 - \xi) \Phi(T, V) - b T^* = 0, \quad (63)$$

$$e^{-m_1 \tau_1} \xi \Phi(T, V) - a C^* = 0, \quad (64)$$

$$e^{-m_2 \tau_2} N_T b T^* + e^{-m_3 \tau_3} N_C a C^* - r V Z - c V = 0, \quad (65)$$

$$g V Z - \mu Z = 0. \quad (66)$$

Equation (66) admits two solutions, $Z = 0$ and $V = \mu/g$. Substituting $Z = 0$ into (63) and (64), we obtain T^* and C^* as

$$\begin{aligned}
T^* &= \frac{e^{-m_1 \tau_1} (1 - \xi) \Phi(T, V)}{b}, \\
C^* &= \frac{e^{-m_1 \tau_1} \xi \Phi(T, V)}{a}. \quad (67)
\end{aligned}$$

Substituting (67) into (65), we get

$$\begin{aligned}
& \left[e^{-(m_1 \tau_1 + m_2 \tau_2)} (1 - \xi) N_T + e^{-(m_1 \tau_1 + m_3 \tau_3)} \xi N_C \right] \Phi(T, V) \\
& - c V = 0. \quad (68)
\end{aligned}$$

Using Condition C1, we have $V = 0$ is a solution of (68). Therefore, $T^* = C^* = 0$ and $T = T_0$ which leads to $S_0 = (T_0, 0, 0, 0, 0)$. If $V \neq 0$, then, from (62) and (68), we obtain

$$V = \frac{\gamma\Phi(T, V)}{c} = \frac{\gamma(\pi - dT)}{c} \implies T = T_0 - \frac{cV}{d\gamma}, \quad (69)$$

where $\gamma = [e^{-(m_1\tau_1+m_2\tau_2)}(1-\xi)N_T + e^{-(m_1\tau_1+m_3\tau_3)}\xi N_C]$. Then, (68) becomes

$$\gamma\Phi\left(T_0 - \frac{c}{d\gamma}V, V\right) - cV = 0. \quad (70)$$

Let us define a function Ψ_1 as

$$\Psi_1(V) = \gamma\Phi\left(T_0 - \frac{c}{d\gamma}V, V\right) - cV = 0. \quad (71)$$

Condition C1 implies that $\Psi_1(0) = 0$, and when $V = V^* = \pi\gamma/c > 0$, then $\Psi_1(V^*) = -cV^* < 0$. We have

$$\Psi_1'(0) = \gamma\left[-\frac{c}{d\gamma}\frac{\partial\Phi(T_0, 0)}{\partial T} + \frac{\partial\Phi(T_0, 0)}{\partial V}\right] - c. \quad (72)$$

From Condition C1, we have $\partial\Phi(T_0, 0)/\partial T = 0$; then,

$$\Psi_1'(0) = \gamma\frac{\partial\Phi(T_0, 0)}{\partial V} - c = c\left(\frac{\gamma}{c}\frac{\partial\Phi(T_0, 0)}{\partial V} - 1\right). \quad (73)$$

Therefore, if $\Psi_1'(0) > 0$, that is, $(\gamma/c)(\partial\Phi(T_0, 0)/\partial V) > 1$, then there exists $V_1 \in (0, V^*)$ such that $\Psi_1(V_1) = 0$. From (62), we define a function Ψ_2 as

$$\Psi_2(T) = \pi - dT - \Phi(T, V_1) = 0. \quad (74)$$

Using Condition C1, we have $\Psi_2(0) = \pi > 0$ and $\Psi_2(T_0) = -\Phi(T_0, V_1) < 0$. Since Ψ_2 is a strictly decreasing function, then there exists a unique $T_1 \in (0, T_0)$ such that $\Psi_2(T_1) = 0$. It follows that $T_1^* = e^{-m_1\tau_1}(1-\xi)\Phi(T_1, V_1)/b > 0$ and $C_1^* = e^{-m_1\tau_1}\xi\Phi(T_1, V_1)/a > 0$. It means that an infected steady state without humoral immune response $S_1(T_1, T_1^*, C_1^*, V_1, 0)$ exists when $(\gamma/c)(\partial\Phi(T_0, 0)/\partial V) > 1$. Now, we define the parameter R_0 as

$$R_0 = \frac{\gamma}{c}\frac{\partial\Phi(T_0, 0)}{\partial V}. \quad (75)$$

The other possibility of (66) is $V_2 = \mu/g$. Insert V_2 in (62) and define Ψ_3 as

$$\Psi_3(T) = \pi - dT - \Phi(T, V_2) = 0. \quad (76)$$

Applying Condition C1, then Ψ_3 is a strictly decreasing function. Moreover, $\Psi_3(0) = \pi > 0$ and $\Psi_3(T_0) = -\Phi(T_0, V_2) < 0$. Thus, there exists a unique $T_2 \in (0, T_0)$ such that $\Psi_3(T_2) = 0$. It follows from (63)–(65) that

$$\begin{aligned} T_2^* &= \frac{e^{-m_1\tau_1}(1-\xi)\Phi(T_2, V_2)}{b}, \\ C_2^* &= \frac{e^{-m_1\tau_1}\xi\Phi(T_2, V_2)}{a}, \\ Z_2 &= \frac{c}{r}\left[\frac{\gamma\Phi(T_2, V_2)}{cV_2} - 1\right]. \end{aligned} \quad (77)$$

Thus, $T_2^* > 0$ and $C_2^* > 0$; moreover, $Z_2 > 0$ when $\gamma\Phi(T_2, V_2)/cV_2 > 1$. Now, we define parameter R_1 as

$$R_1 = \frac{\gamma\Phi(T_2, V_2)}{cV_2}. \quad (78)$$

Hence, Z_2 can be rewritten as $Z_2 = (c/r)(R_1 - 1)$. It follows that $S_2(T_2, T_2^*, C_2^*, V_2, Z_2)$ exists when $R_1 > 1$.

Conditions C1 and C2 imply that

$$\begin{aligned} R_1 &= \frac{\gamma\Phi(T_2, V_2)}{cV_2} \leq \frac{\gamma}{c}\frac{\partial\Phi(T_2, 0)}{\partial V} < \frac{\gamma}{c}\frac{\partial\Phi(T_0, 0)}{\partial V} \\ &= R_0. \end{aligned} \quad (79)$$

□

4.2. Global Stability Analysis

Theorem 11. Let Conditions C1 and C2 be satisfied and $R_0 \leq 1$; then, S_0 for system (57)–(61) is GAS.

Proof. Define

$$\begin{aligned} U_0 &= T - T_0 - \int_{T_0}^T \lim_{V \rightarrow 0^+} \frac{\Phi(T_0, V)}{\Phi(\rho, V)} d\rho + \eta_1 T^* + \eta_2 C^* \\ &\quad + \eta_3 V + \eta_4 Z + \int_0^{\tau_1} \Phi(T(t-\theta), V(t-\theta)) d\theta \\ &\quad + \eta_6 \int_0^{\tau_2} T^*(t-\theta) d\theta + \eta_7 \int_0^{\tau_3} C^*(t-\theta) d\theta. \end{aligned} \quad (80)$$

Then,

$$\begin{aligned} \frac{dU_0}{dt} &= \left(1 - \lim_{V \rightarrow 0^+} \frac{\Phi(T_0, V)}{\Phi(T, V)}\right)(\pi - dT - \Phi(T, V)) \\ &\quad + \eta_1(e^{-m_1\tau_1}(1-\xi)\Phi(T(t-\tau_1), V(t-\tau_1)) \\ &\quad - bT^*) + \eta_2(e^{-m_1\tau_1}\xi\Phi(T(t-\tau_1), V(t-\tau_1)) \\ &\quad - aC^*) + \eta_3(e^{-m_2\tau_2}N_T bT^*(t-\tau_2) \\ &\quad + e^{-m_3\tau_3}N_C aC^*(t-\tau_3) - rVZ - cV) + \eta_4(gVZ \\ &\quad - \mu Z) + \Phi(T, V) - \Phi(T(t-\tau_1), V(t-\tau_1)) \\ &\quad + \eta_6 T^* - \eta_6 T^*(t-\tau_2) + \eta_7 C^* - \eta_7 C^*(t-\tau_3). \end{aligned} \quad (81)$$

Equation (81) can be simplified as

$$\begin{aligned} \frac{dU_0}{dt} &= \left(1 - \lim_{V \rightarrow 0^+} \frac{\Phi(T_0, V)}{\Phi(T, V)}\right)(\pi - dT) \\ &\quad + \Phi(T, V) \lim_{V \rightarrow 0^+} \frac{\Phi(T_0, V)}{\Phi(T, V)} - c\eta_3 V - \eta_4 \mu Z \end{aligned}$$

$$\begin{aligned}
&= \pi \left(1 - \frac{\partial \Phi(T_0, 0)/\partial V}{\partial \Phi(T, 0)/\partial V} \right) \left(1 - \frac{T}{T_0} \right) \\
&\quad + \Phi(T, V) \frac{\partial \Phi(T_0, 0)/\partial V}{\partial \Phi(T, 0)/\partial V} - c\eta_3 V - \eta_4 \mu Z \\
&\leq \pi \left(1 - \frac{\partial \Phi(T_0, 0)/\partial V}{\partial \Phi(T, 0)/\partial V} \right) \left(1 - \frac{T}{T_0} \right) \\
&\quad + V \frac{\partial \Phi(T_0, 0)}{\partial V} - c\eta_3 V - \eta_4 \mu Z \\
&= \pi \left(1 - \frac{\partial \Phi(T_0, 0)/\partial V}{\partial \Phi(T, 0)/\partial V} \right) \left(1 - \frac{T}{T_0} \right) \\
&\quad + c\eta_3 \left(\frac{1}{c\eta_3} \frac{\partial \Phi(T_0, 0)}{\partial V} - 1 \right) V - \eta_4 \mu Z \\
&= \pi \left(1 - \frac{\partial \Phi(T_0, 0)/\partial V}{\partial \Phi(T, 0)/\partial V} \right) \left(1 - \frac{T}{T_0} \right) \\
&\quad + c\eta_3 (R_0 - 1) V - \eta_4 \mu Z.
\end{aligned} \tag{82}$$

Based on Condition C2, the first term of (82) is less than or equal to zero. Therefore, if $R_0 \leq 1$, then $dU_0/dt \leq 0$, for all $T, V, Z > 0$. Similar to the previous sections, one can show that S_0 is GAS. $\square$

Condition C3. Consider

$$\left(\frac{\Phi(T, V)}{\Phi(T, V_i)} - \frac{V}{V_i} \right) \left(1 - \frac{\Phi(T, V_i)}{\Phi(T, V)} \right) \leq 0, \tag{83}$$

$\forall T, V \in (0, \infty), i = 1, 2.$

Lemma 12. Suppose that $R_0 > 1$ and Conditions C1–C3 hold. Then, $T_1, T_2, T_1^*, T_2^*, C_1^*, C_2^*, V_1, V_2$ exist satisfying

$$\operatorname{sgn}(R_1 - 1) = \operatorname{sgn}(T_2 - T_1) = \operatorname{sgn}(V_1 - V_2). \tag{84}$$

Proof. From Condition C1, for $T_1, T_2, V_1, V_2 > 0$, we have

$$(\Phi(T_2, V_2) - \Phi(T_1, V_2))(T_2 - T_1) > 0. \tag{85}$$

$$(\Phi(T_1, V_2) - \Phi(T_1, V_1))(V_2 - V_1) > 0, \tag{86}$$

Applying Condition C3 when $i = 1$, $T = T_1$, and $V = V_2$, we get

$$\begin{aligned}
&(\Phi(T_1, V_2) V_1 - \Phi(T_1, V_1) V_2) \\
&\quad \cdot (\Phi(T_1, V_2) - \Phi(T_1, V_1)) \leq 0.
\end{aligned} \tag{87}$$

It follows from inequality (86) that

$$(\Phi(T_1, V_2) V_1 - \Phi(T_1, V_1) V_2)(V_1 - V_2) > 0. \tag{88}$$

We assume that $\operatorname{sgn}(V_2 - V_1) = \operatorname{sgn}(T_2 - T_1)$. Then, we have

$$\begin{aligned}
&(\pi - dT_2) - (\pi - dT_1) = \Phi(T_2, V_2) - \Phi(T_1, V_1) \\
&= \Phi(T_2, V_2) - \Phi(T_1, V_2) + \Phi(T_1, V_2) \\
&\quad - \Phi(T_1, V_1),
\end{aligned} \tag{89}$$

and using inequalities (85) and (86), we obtain $\operatorname{sgn}(T_1 - T_2) = \operatorname{sgn}(T_2 - T_1)$, and this leads to contradiction. Therefore, $\operatorname{sgn}(V_1 - V_2) = \operatorname{sgn}(T_2 - T_1)$. From the conditions of S_1 , we have $\gamma\Phi(T_1, V_1)/cV_1 = 1$; then,

$$\begin{aligned}
R_1 - 1 &= \frac{\gamma}{c} \left(\frac{1}{V_2} (\Phi(T_2, V_2) - \Phi(T_1, V_2)) \right. \\
&\quad \left. + \frac{1}{V_1 V_2} (\Phi(T_1, V_2) V_1 - \Phi(T_1, V_1) V_2) \right).
\end{aligned} \tag{90}$$

Using inequalities (85) and (88), we obtain $\operatorname{sgn}(R_1 - 1) = \operatorname{sgn}(V_1 - V_2)$. $\square$

Theorem 13. Assume that $R_1 \leq 1 < R_0$ and Conditions C1–C3 are satisfied; then, S_1 for system (57)–(61) is GAS.

Proof. Let us define

$$\begin{aligned}
U_1 &= T - T_1 - \int_{T_1}^T \frac{\Phi(T_1, V_1)}{\Phi(\rho, V_1)} d\rho + \eta_1 T_1^* H\left(\frac{T^*}{T_1^*}\right) \\
&\quad + \eta_2 C_1^* H\left(\frac{C^*}{C_1^*}\right) + \eta_3 V_1 H\left(\frac{V}{V_1}\right) + \eta_4 Z \\
&\quad + \Phi(T_1, V_1) \int_0^{\tau_1} H\left(\frac{\Phi(T(t-\theta), V(t-\theta))}{\Phi(T_1, V_1)}\right) d\theta \\
&\quad + \eta_6 T_1^* \int_0^{\tau_2} H\left(\frac{T^*(t-\theta)}{T_1^*}\right) d\theta \\
&\quad + \eta_7 C_1^* \int_0^{\tau_3} H\left(\frac{C^*(t-\theta)}{C_1^*}\right) d\theta.
\end{aligned} \tag{91}$$

Then,

$$\begin{aligned}
\frac{dU_1}{dt} &= \left(1 - \frac{\Phi(T_1, V_1)}{\Phi(T, V_1)} \right) (\pi - dT - \Phi(T, V)) \\
&\quad + \eta_1 \left(1 - \frac{T_1^*}{T^*} \right) \\
&\quad \cdot (e^{-m_1 \tau_1} (1 - \xi) \Phi(T(t - \tau_1), V(t - \tau_1)) \\
&\quad - bT^*) + \eta_2 \left(1 - \frac{C_1^*}{C^*} \right) \\
&\quad \cdot (e^{-m_1 \tau_1} \xi \Phi(T(t - \tau_1), V(t - \tau_1)) \\
&\quad - aC^*) + \eta_3 \left(1 - \frac{V_1}{V} \right) (e^{-m_2 \tau_2} N_T bT^*(t - \tau_2) \\
&\quad + e^{-m_3 \tau_3} N_C aC^*(t - \tau_3) - rVZ - cV) + \eta_4 (gVZ \\
&\quad - \mu Z) + \Phi(T, V) - \Phi(T(t - \tau_1), V(t - \tau_1))
\end{aligned}$$

$$\begin{aligned}
& + \Phi(T_1, V_1) \ln \frac{\Phi(T(t-\tau_1), V(t-\tau_1))}{\Phi(T, V)} + \eta_6 \left[T^* \right. \\
& - T^*(t-\tau_2) + T_1^* \ln \left(\frac{T^*(t-\tau_2)}{T^*} \right) \left. \right] + \eta_7 \left[C^* \right. \\
& - C^*(t-\tau_3) + C_1^* \ln \left(\frac{C^*(t-\tau_3)}{C^*} \right) \left. \right].
\end{aligned} \tag{92}$$

Collecting terms of (92), we obtain

$$\begin{aligned}
& \frac{dU_1}{dt} \\
& = \left(1 - \frac{\Phi(T_1, V_1)}{\Phi(T, V_1)} \right) (\pi - dT) \\
& + \Phi(T, V) \frac{\Phi(T_1, V_1)}{\Phi(T, V_1)} \\
& - e^{-m_1 \tau_1} \eta_1 (1 - \xi) \Phi(T(t-\tau_1), V(t-\tau_1)) \frac{T_1^*}{T^*} \\
& + \eta_1 b T_1^* \\
& - e^{-m_1 \tau_1} \eta_2 \xi \Phi(T(t-\tau_1), V(t-\tau_1)) \frac{C_1^*}{C^*} \\
& + \eta_2 a C_1^* - b \eta_1 \frac{T^*(t-\tau_2) V_1}{V} \\
& - a \eta_2 \frac{C^*(t-\tau_3) V_1}{V} - c \eta_3 V + c \eta_3 V_1 + \eta_3 r V_1 Z \\
& - \eta_4 \mu Z \\
& + \Phi(T_1, V_1) \ln \left(\frac{\Phi(T(t-\tau_1), V(t-\tau_1))}{\Phi(T, V)} \right) \\
& + \eta_1 b T_1^* \ln \left(\frac{T^*(t-\tau_2)}{T^*} \right) \\
& + \eta_2 a C_1^* \ln \left(\frac{C^*(t-\tau_3)}{C^*} \right).
\end{aligned} \tag{93}$$

Applying $\pi = dT_1 + \Phi(T_1, V_1)$, we get

$$\begin{aligned}
& \frac{dU_1}{dt} = \left(1 - \frac{\Phi(T_1, V_1)}{\Phi(T, V_1)} \right) (dT_1 - dT) + \Phi(T_1, V_1) \\
& \cdot \left(1 - \frac{\Phi(T_1, V_1)}{\Phi(T, V_1)} \right) + \Phi(T, V) \frac{\Phi(T_1, V_1)}{\Phi(T, V_1)} \\
& - e^{-m_1 \tau_1} \eta_1 (1 - \xi) \Phi(T(t-\tau_1), V(t-\tau_1)) \\
& \cdot \frac{T_1^*}{T^*} + \eta_1 b T_1^* \\
& - e^{-m_1 \tau_1} \eta_2 \xi \Phi(T(t-\tau_1), V(t-\tau_1)) \frac{C_1^*}{C^*}
\end{aligned}$$

$$\begin{aligned}
& + \eta_2 a C_1^* - \eta_1 b \frac{T^*(t-\tau_2) V_1}{V} - \eta_2 a \\
& \cdot \frac{C^*(t-\tau_3) V_1}{V} - c \eta_3 V + c \eta_3 V_1 + \eta_3 r V_1 Z \\
& - \eta_4 \mu Z + \Phi(T_1, V_1) e^{-m_1 \tau_1} \eta_1 (1 - \xi) \\
& \cdot \ln \left(\frac{\Phi(T(t-\tau_1), V(t-\tau_1))}{\Phi(T, V)} \right) \\
& + \Phi(T_1, V_1) e^{-m_1 \tau_1} \eta_2 \xi \\
& \cdot \ln \left(\frac{\Phi(T(t-\tau_1), V(t-\tau_1))}{\Phi(T, V)} \right) + \eta_1 b T_1^* \\
& \cdot \ln \left(\frac{T^*(t-\tau_2)}{T^*} \right) + \eta_2 a C_1^* \\
& \cdot \ln \left(\frac{C^*(t-\tau_3)}{C^*} \right).
\end{aligned} \tag{94}$$

Using the steady state conditions for S_1 ,

$$\begin{aligned}
(1 - \xi) \Phi(T_1, V_1) &= e^{m_1 \tau_1} b T_1^*, \\
\xi \Phi(T_1, V_1) &= e^{m_1 \tau_1} a C_1^*, \\
c V_1 &= e^{-m_2 \tau_2} N_T b T_1^* + e^{-m_3 \tau_3} N_C a C_1^*,
\end{aligned} \tag{95}$$

then we have $\Phi(T_1, V_1) = \eta_1 b T_1^* + \eta_2 a C_1^*$ and

$$\begin{aligned}
& \frac{dU_1}{dt} = dT_1 \left(1 - \frac{\Phi(T_1, V_1)}{\Phi(T, V_1)} \right) \left(1 - \frac{T}{T_1} \right) \\
& + \Phi(T_1, V_1) \frac{\Phi(T, V)}{\Phi(T, V_1)} \\
& + \eta_1 b T_1^* \left(1 - \frac{\Phi(T_1, V_1)}{\Phi(T, V_1)} \right) \\
& + \eta_2 a C_1^* \left(1 - \frac{\Phi(T_1, V_1)}{\Phi(T, V_1)} \right) \\
& - \eta_1 b T_1^* \frac{\Phi(T(t-\tau_1), V(t-\tau_1)) T_1^*}{\Phi(T_1, V_1) T^*} \\
& + \eta_1 b T_1^* \\
& - \eta_2 a C_1^* \frac{\Phi(T(t-\tau_1), V(t-\tau_1)) C_1^*}{\Phi(T_1, V_1) C^*} \\
& + \eta_2 a C_1^* - \eta_1 b T_1^* \frac{T^*(t-\tau_2) V_1}{V T_1^*} \\
& - \eta_2 a C_1^* \frac{C^*(t-\tau_3) V_1}{V C_1^*} + \eta_1 b T_1^* + \eta_2 a C_1^*
\end{aligned}$$

$$\begin{aligned}
& -\Phi(T_1, V_1) \frac{V}{V_1} + \eta_3 r \left(V_1 - \frac{\mu}{g} \right) Z \\
& + \eta_1 b T_1^* \ln \left(\frac{\Phi(T(t-\tau_1), V(t-\tau_1))}{\Phi(T, V)} \right) \\
& + \eta_2 a C_1^* \ln \left(\frac{\Phi(T(t-\tau_1), V(t-\tau_1))}{\Phi(T, V)} \right) \\
& + \eta_1 b T_1^* \ln \left(\frac{T^*(t-\tau_2)}{T^*} \right) \\
& + \eta_2 a C_1^* \ln \left(\frac{C^*(t-\tau_3)}{C^*} \right).
\end{aligned} \tag{96}$$

Using the following equalities,

$$\begin{aligned}
& \ln \left(\frac{\Phi(T(t-\tau_1), V(t-\tau_1))}{\Phi(T, V)} \right) \\
& = \ln \left(\frac{\Phi(T(t-\tau_1), V(t-\tau_1)) T_j^*}{\Phi(T_j, V_j) T^*} \right) \\
& + \ln \left(\frac{\Phi(T_j, V_j)}{\Phi(T, V_j)} \right) + \ln \left(\frac{\Phi(T, V_j) V}{\Phi(T, V) V_j} \right) \\
& + \ln \left(\frac{V_j T^*}{V T_j^*} \right), \\
& \ln \left(\frac{\Phi(T(t-\tau_1), V(t-\tau_1))}{\Phi(T, V)} \right) \\
& = \ln \left(\frac{\Phi(T(t-\tau_1), V(t-\tau_1)) C_j^*}{\Phi(T_j, V_j) C^*} \right) \\
& + \ln \left(\frac{\Phi(T_j, V_j)}{\Phi(T, V_j)} \right) \ln \left(\frac{\Phi(T, V_j) V}{\Phi(T, V) V_j} \right) \\
& + \ln \left(\frac{V_j C^*}{V C_j^*} \right), \\
& \ln \left(\frac{T^*(t-\tau_2)}{T^*} \right) \\
& = \ln \left(\frac{T^*(t-\tau_2) V_j}{V T_j^*} \right) + \ln \left(\frac{V T_j^*}{V_j T^*} \right), \\
& \ln \left(\frac{C^*(t-\tau_3)}{C^*} \right) \\
& = \ln \left(\frac{C^*(t-\tau_3) V_j}{V C_j^*} \right) + \ln \left(\frac{V C_j^*}{V_j C^*} \right),
\end{aligned} \tag{97}$$

$j = 1, 2,$

in case of $j = 1$, we obtain

$$\begin{aligned}
\frac{dU_1}{dt} & = dT_1 \left(1 - \frac{\Phi(T_1, V_1)}{\Phi(T, V_1)} \right) \left(1 - \frac{T}{T_1} \right) + \Phi(T_1, V_1) \\
& \cdot \left[\frac{\Phi(T, V)}{\Phi(T, V_1)} - \frac{V}{V_1} - 1 \right. \\
& + \frac{\Phi(T, V_1) V}{\Phi(T, V) V_1} \left. \right] - \eta_1 b T_1^* \left[\frac{\Phi(T_1, V_1)}{\Phi(T, V_1)} - 1 \right. \\
& - \ln \left(\frac{\Phi(T_1, V_1)}{\Phi(T, V_1)} \right) \left. \right] - \eta_2 a C_1^* \left[\frac{\Phi(T_1, V_1)}{\Phi(T, V_1)} - 1 \right. \\
& - \ln \left(\frac{\Phi(T_1, V_1)}{\Phi(T, V_1)} \right) \left. \right] \\
& - \eta_1 b T_1^* \left[\frac{\Phi(T(t-\tau_1), V(t-\tau_1)) T_1^*}{\Phi(T_1, V_1) T^*} - 1 \right. \\
& - \ln \left(\frac{\Phi(T(t-\tau_1), V(t-\tau_1)) T_1^*}{\Phi(T_1, V_1) T^*} \right) \left. \right] \\
& - \eta_2 a C_1^* \left[\frac{\Phi(T(t-\tau_1), V(t-\tau_1)) C_1^*}{\Phi(T_1, V_1) C^*} - 1 \right. \\
& - \ln \left(\frac{\Phi(T(t-\tau_1), V(t-\tau_1)) C_1^*}{\Phi(T_1, V_1) C^*} \right) \left. \right] \\
& - \eta_1 b T_1^* \left[\frac{T^*(t-\tau_2) V_1}{V T_1^*} - 1 \right. \\
& - \ln \left(\frac{T^*(t-\tau_2) V_1}{V T_1^*} \right) \left. \right] - \eta_2 a C_1^* \left[\frac{C^*(t-\tau_3) V_1}{V C_1^*} \right. \\
& - 1 - \ln \left(\frac{C^*(t-\tau_3) V_1}{V C_1^*} \right) \left. \right] \\
& - \eta_1 b T_1^* \left[\frac{\Phi(T, V_1) V}{\Phi(T, V) V_1} - 1 - \ln \left(\frac{\Phi(T, V_1) V}{\Phi(T, V) V_1} \right) \right] \\
& - \eta_2 a C_1^* \left[\frac{\Phi(T, V_1) V}{\Phi(T, V) V_1} - 1 - \ln \left(\frac{\Phi(T, V_1) V}{\Phi(T, V) V_1} \right) \right] \\
& + r \eta_3 \left(V_1 - \frac{\mu}{g} \right) Z = dT_1 \left(1 - \frac{\Phi(T_1, V_1)}{\Phi(T, V_1)} \right) \left(1 \right. \\
& - \frac{T}{T_1} \left. \right) + \Phi(T_1, V_1) \left(\frac{\Phi(T, V)}{\Phi(T, V_1)} - \frac{V}{V_1} \right) \left(1 \right. \\
& - \frac{\Phi(T, V_1)}{\Phi(T, V)} \left. \right) - \eta_1 b T_1^* \left[H \left(\frac{\Phi(T_1, V_1)}{\Phi(T, V_1)} \right) \right. \\
& + H \left(\frac{\Phi(T(t-\tau_1), V(t-\tau_1)) T_1^*}{\Phi(T_1, V_1) T^*} \right) \\
& + H \left(\frac{T^*(t-\tau_2) V_1}{V T_1^*} \right) + H \left(\frac{\Phi(T, V_1) V}{\Phi(T, V) V_1} \right) \left. \right]
\end{aligned}$$

$$\begin{aligned}
& -\eta_2 a C_1^* \left[H \left(\frac{\Phi(T_1, V_1)}{\Phi(T, V_1)} \right) \right. \\
& + H \left(\frac{\Phi(T(t-\tau_1), V(t-\tau_1)) C_1^*}{\Phi(T_1, V_1) C^*} \right) \\
& + H \left(\frac{C^*(t-\tau_3) V_1}{V C_1^*} \right) + H \left(\frac{\Phi(T, V_1) V}{\Phi(T, V) V_1} \right) \Big] \\
& + r \eta_3 \left(V_1 - \frac{\mu}{g} \right) Z.
\end{aligned} \tag{98}$$

Since $R_1 \leq 1$, then Conditions C1 and C3 and Lemma 12 imply that $dU_1/dt \leq 0$, for all $T, T^*, C^*, V, Z > 0$, where the equality occurs at S_1 . LIP implies that S_1 is GAS. $\square$

Theorem 14. Let $R_1 > 1$ and Conditions C1–C3 be satisfied; then, S_2 for system (57)–(61) is GAS.

Proof. We consider a Lyapunov functional:

$$\begin{aligned}
U_2 = & T - T_2 - \int_{T_2}^T \frac{\Phi(T_2, V_2)}{\Phi(\rho, V_2)} d\rho + \eta_1 T_2^* H \left(\frac{T^*}{T_2^*} \right) \\
& + \eta_2 C_2^* H \left(\frac{C^*}{C_2^*} \right) + \eta_3 V_2 H \left(\frac{V}{V_2} \right) \\
& + \eta_4 Z_2 H \left(\frac{Z}{Z_2} \right) + \Phi(T_2, V_2) \\
& \cdot \int_0^{\tau_1} H \left(\frac{\Phi(T(t-\theta), V(t-\theta))}{\Phi(T_2, V_2)} \right) d\theta \\
& + \eta_6 T_2^* \int_0^{\tau_2} H \left(\frac{T^*(t-\theta)}{T_2^*} \right) d\theta \\
& + \eta_7 C_2^* \int_0^{\tau_3} H \left(\frac{C^*(t-\theta)}{C_2^*} \right) d\theta.
\end{aligned} \tag{99}$$

Then,

$$\begin{aligned}
\frac{dU_2}{dt} = & \left(1 - \frac{\Phi(T_2, V_2)}{\Phi(T, V_2)} \right) (\pi - dT - \Phi(T, V)) \\
& + \eta_1 \left(1 - \frac{T_2^*}{T^*} \right) \\
& \cdot (e^{-m_1 \tau_1} (1 - \xi) \Phi(T(t-\tau_1), V(t-\tau_1)) \\
& - bT^*) + \eta_2 \left(1 - \frac{C_2^*}{C^*} \right) \\
& \cdot (e^{-m_1 \tau_1} \xi \Phi(T(t-\tau_1), V(t-\tau_1)) \\
& - aC^*) + \eta_3 \left(1 - \frac{V_2}{V} \right) (e^{-m_2 \tau_2} N_T bT^*(t-\tau_2) \\
& + e^{-m_3 \tau_3} N_C aC^*(t-\tau_3) - cV - rVZ) + \eta_4 \left(1 \right.
\end{aligned}$$

$$\begin{aligned}
& - \frac{Z_2}{Z} \Big) (gVZ - \mu Z) + \Phi(T, V) \\
& - \Phi(T(t-\tau_1), V(t-\tau_1)) + \Phi(T_2, V_2) \\
& \cdot \ln \left(\frac{\Phi(T(t-\tau_1), V(t-\tau_1))}{\Phi(T, V)} \right) + \eta_6 \left[T^* \right. \\
& - T^*(t-\tau_2) + T_2^* \ln \left(\frac{T^*(t-\tau_2)}{T^*} \right) \Big] + \eta_7 \left[C^* \right. \\
& - C^*(t-\tau_3) + C_2^* \ln \left(\frac{C^*(t-\tau_3)}{C^*} \right) \Big].
\end{aligned} \tag{100}$$

Using the steady state conditions for S_2 ,

$$\begin{aligned}
\pi &= dT_2 + \Phi(T_2, V_2), \\
(1 - \xi) \Phi(T_2, V_2) &= e^{m_1 \tau_1} bT_2^*, \\
\xi \Phi(T_2, V_2) &= e^{m_1 \tau_1} aC_2^*, \\
cV_2 &= e^{-m_2 \tau_2} N_T bT_2^* + e^{-m_3 \tau_3} N_C aC_2^* \\
&\quad - rV_2 Z_2,
\end{aligned} \tag{101}$$

we get $\Phi(T_2, V_2) = \eta_1 bT_2^* + \eta_2 aC_2^*$, and

$$\begin{aligned}
\frac{dU_2}{dt} = & dT_2 \left(1 - \frac{\Phi(T_2, V_2)}{\Phi(T, V_2)} \right) \left(1 - \frac{T}{T_2} \right) \\
& + \Phi(T_2, V_2) \frac{\Phi(T, V)}{\Phi(T, V_2)} \\
& + \eta_1 bT_2^* \left(1 - \frac{\Phi(T_2, V_2)}{\Phi(T, V_2)} \right) \\
& + \eta_2 aC_2^* \left(1 - \frac{\Phi(T_2, V_2)}{\Phi(T, V_2)} \right) \\
& - \eta_1 bT_2^* \frac{\Phi(T(t-\tau_1), V(t-\tau_1)) T_2^*}{\Phi(T_2, V_2) T^*} \\
& + \eta_1 bT_2^* \\
& - \eta_2 aC_2^* \frac{\Phi(T(t-\tau_1), V(t-\tau_1)) C_2^*}{\Phi(T_2, V_2) C^*} \\
& + \eta_2 aC_2^* - \eta_1 bT_2^* \frac{T^*(t-\tau_2) V_2}{V T_2^*} \\
& - \eta_2 aC_2^* \frac{C^*(t-\tau_3) V_2}{V C_2^*} + \eta_1 bT_2^* + \eta_2 aC_2^* \\
& - \Phi(T_2, V_2) \frac{V}{V_2} \\
& + \eta_1 bT_2^* \ln \left(\frac{\Phi(T(t-\tau_1), V(t-\tau_1))}{\Phi(T, V)} \right)
\end{aligned}$$

$$\begin{aligned}
& + \eta_2 a C_2^* \ln \left(\frac{\Phi(T(t-\tau_1), V(t-\tau_1))}{\Phi(T, V)} \right) \\
& + \eta_1 b T_2^* \ln \left(\frac{T^*(t-\tau_2)}{T^*} \right) \\
& + \eta_2 a C_2^* \ln \left(\frac{C^*(t-\tau_3)}{C^*} \right).
\end{aligned} \tag{102}$$

Using (97), when $j = 2$, we get

$$\begin{aligned}
\frac{dU_2}{dt} = & dT_2 \left(1 - \frac{\Phi(T_2, V_2)}{\Phi(T, V_2)} \right) \left(1 - \frac{T}{T_2} \right) \\
& + \Phi(T_2, V_2) \left[\frac{\Phi(T, V)}{\Phi(T, V_2)} - \frac{V}{V_2} - 1 + \frac{\Phi(T, V_2)V}{\Phi(T, V)V_2} \right] \\
& - \eta_1 b T_2^* \left[\frac{\Phi(T_2, V_2)}{\Phi(T, V_2)} - 1 - \ln \left(\frac{\Phi(T_2, V_2)}{\Phi(T, V_2)} \right) \right] \\
& - \eta_2 a C_2^* \left[\frac{\Phi(T_2, V_2)}{\Phi(T, V_2)} - 1 - \ln \left(\frac{\Phi(T_2, V_2)}{\Phi(T, V_2)} \right) \right] \\
& - \eta_1 b T_2^* \left[\frac{\Phi(T(t-\tau_1), V(t-\tau_1)) T_2^*}{\Phi(T_2, V_2) T^*} - 1 \right. \\
& \left. - \ln \left(\frac{\Phi(T(t-\tau_1), V(t-\tau_1)) T_2^*}{\Phi(T_2, V_2) T^*} \right) \right] \\
& - \eta_2 a C_2^* \left[\frac{\Phi(T(t-\tau_1), V(t-\tau_1)) C_2^*}{\Phi(T_2, V_2) C^*} - 1 \right. \\
& \left. - \ln \left(\frac{\Phi(T(t-\tau_1), V(t-\tau_1)) C_2^*}{\Phi(T_2, V_2) C^*} \right) \right] \\
& - \eta_1 b T_2^* \left[\frac{T^*(t-\tau_2) V_2}{V T_2^*} - 1 \right. \\
& \left. - \ln \left(\frac{T^*(t-\tau_2) V_2}{V T_2^*} \right) \right] - \eta_2 a C_2^* \left[\frac{C^*(t-\tau_3) V_2}{V C_2^*} \right. \\
& \left. - 1 - \ln \left(\frac{C^*(t-\tau_3) V_2}{V C_2^*} \right) \right] \\
& - \eta_1 b T_2^* \left[\frac{\Phi(T, V_2)V}{\Phi(T, V)V_2} - 1 - \ln \left(\frac{\Phi(T, V_2)V}{\Phi(T, V)V_2} \right) \right] \\
& - \eta_2 a C_2^* \left[\frac{\Phi(T, V_2)V}{\Phi(T, V)V_2} - 1 - \ln \left(\frac{\Phi(T, V_2)V}{\Phi(T, V)V_2} \right) \right] \\
= & dT_2 \left(1 - \frac{\Phi(T_2, V_2)}{\Phi(T, V_2)} \right) \left(1 - \frac{T}{T_2} \right) + \Phi(T_2, V_2) \\
& \cdot \left(\frac{\Phi(T, V)}{\Phi(T, V_2)} - \frac{V}{V_2} \right) \left(1 - \frac{\Phi(T, V_2)}{\Phi(T, V)} \right)
\end{aligned}$$

$$\begin{aligned}
& - \eta_1 b T_2^* \left[H \left(\frac{\Phi(T_2, V_2)}{\Phi(T, V_2)} \right) \right. \\
& + H \left(\frac{\Phi(T(t-\tau_1), V(t-\tau_1)) T_2^*}{\Phi(T_2, V_2) T^*} \right) \\
& + H \left(\frac{T^*(t-\tau_2) V_2}{V T_2^*} \right) + H \left(\frac{\Phi(T, V_2)V}{\Phi(T, V)V_2} \right) \left. \right] \\
& - \eta_2 a C_2^* \left[H \left(\frac{\Phi(T_1, V_1)}{\Phi(T, V_1)} \right) \right. \\
& + H \left(\frac{\Phi(T(t-\tau_1), V(t-\tau_1)) C_2^*}{\Phi(T_2, V_2) C^*} \right) \\
& + H \left(\frac{C^*(t-\tau_3) V_2}{V C_2^*} \right) + H \left(\frac{\Phi(T, V_2)V}{\Phi(T, V)V_2} \right) \left. \right].
\end{aligned} \tag{103}$$

It is easy to show that S_2 is GAS. $\square$

5. Numerical Simulations

In this section, we show an example of the general model (50)-(57) where Conditions C1-C3 can be satisfied:

$$\begin{aligned}
\dot{T}(t) &= \pi - dT(t) - \frac{(1-\varepsilon)kT(t)V(t)}{1 + \omega T(t) + \beta V(t)}, \\
\dot{T}^*(t) &= e^{-m_1 \tau_1} (1-\xi) \frac{(1-\varepsilon)kT(t-\tau_1)V(t-\tau_1)}{1 + \omega T(t-\tau_1) + \beta V(t-\tau_1)} \\
&\quad - bT^*(t), \\
\dot{C}^*(t) &= e^{-m_1 \tau_1} \xi \frac{(1-\varepsilon)kT(t-\tau_1)V(t-\tau_1)}{1 + \omega T(t-\tau_1) + \beta V(t-\tau_1)} \\
&\quad - aC^*(t), \\
\dot{V}(t) &= e^{-m_2 \tau_2} N_T b T^*(t-\tau_2) \\
&\quad + e^{-m_3 \tau_3} N_C a C^*(t-\tau_3) - rV(t)Z(t) \\
&\quad - cV(t), \\
\dot{Z}(t) &= gV(t)Z(t) - \mu Z(t),
\end{aligned} \tag{104}$$

where $\omega, \beta \geq 0$. In this example, the incidence rate is given by

$$\Phi(T, V) = \frac{(1-\varepsilon)kTV}{1 + \omega T + \beta V}. \tag{105}$$

We have the following:

$$\begin{aligned}
\Phi(T, V) &> 0, \\
\Phi(T, 0) &= \Phi(0, V) = 0,
\end{aligned}$$

$$\forall T > 0, V > 0,$$

$$\begin{aligned}
\frac{\partial \Phi(T, V)}{\partial T} &= \frac{(1 - \varepsilon)kV(1 + \beta V)}{(1 + \omega T + \beta V)^2} > 0, \\
&\forall T > 0, V > 0, \\
\frac{\partial \Phi(T, V)}{\partial V} &= \frac{(1 - \varepsilon)kT(1 + \omega T)}{(1 + \omega T + \beta V)^2} > 0, \\
&\forall T > 0, V > 0, \\
\frac{\partial \Phi(T, 0)}{\partial V} &= \frac{(1 - \varepsilon)kT}{1 + \omega T} > 0, \quad \forall T > 0, \\
\Phi(T, V) &= \frac{(1 - \varepsilon)kTV}{1 + \omega T + \beta V} \leq \frac{(1 - \varepsilon)kTV}{1 + \omega T} \\
&= V \frac{\partial \Phi(T, 0)}{\partial V}, \quad \forall T > 0, V > 0, \\
\left(\frac{\Phi(T, V)}{\Phi(T, V_i)} - \frac{V}{V_i} \right) &\left(1 - \frac{\Phi(T, V_i)}{\Phi(T, V)} \right) \\
&= \frac{-\beta(1 + \omega T)(V - V_i)^2}{V_i(1 + \omega T + \beta V_i)(1 + \omega T + \beta V)} \leq 0, \\
&\forall T, V > 0, i = 1, 2.
\end{aligned} \tag{106}$$

Then, Conditions C1–C3 are satisfied. The parameters R_0 and R_1 will be

$$\begin{aligned}
R_0 &= \frac{(1 - \varepsilon)kT_0 \left[e^{-(m_1\tau_1 + m_2\tau_2)}(1 - \xi)N_T + e^{-(m_1\tau_1 + m_3\tau_3)}\xi N_C \right]}{c(1 + \omega T_0)}, \\
R_1 &= \frac{(1 - \varepsilon)kT_2 \left[e^{-(m_1\tau_1 + m_2\tau_2)}(1 - \xi)N_T + e^{-(m_1\tau_1 + m_3\tau_3)}\xi N_C \right]}{c(1 + \omega T_2 + \beta V_2)},
\end{aligned} \tag{107}$$

where

$$\begin{aligned}
T_2 &= \frac{1}{2\omega} \left(\omega T_0 - (1 + \zeta V_2) \right. \\
&\quad \left. + \sqrt{[(1 + \zeta V_2) - \omega T_0]^2 + 4\omega T_0(1 + \beta V_2)} \right), \\
V_2 &= \frac{\mu}{g}, \quad \zeta = \beta + \frac{(1 - \varepsilon)k}{d}.
\end{aligned} \tag{108}$$

The values of some parameters of model (104) are listed in Table 1. The remaining parameters of the model will be chosen varied. Without loss of generality, we let $\tau_e = \tau_1 = \tau_2 = \tau_3$. We have used MATLAB for the numerical computations. Now, we study the following cases.

TABLE 1: The parameters of model (104).

Parameter	Value
π	10 cells mm ⁻³ day ⁻¹
d	0.01 day ⁻¹
b	0.3 day ⁻¹
ω	0.05 cells ⁻¹ mm ³
β	0.05 virus ⁻¹ mm ³
a	0.25 day ⁻¹
N_T	100 virus cells ⁻¹
ξ	0.5
N_C	50 virus cells ⁻¹
r	0.01 cells ⁻¹ mm ³ day ⁻¹
μ	0.05 day ⁻¹
m_1	1 day ⁻¹
m_2	1 day ⁻¹
m_3	1 day ⁻¹
c	3 day ⁻¹

Case 1 (effect of the parameters k and g on the stability of the system). Now, we confirm the results of Theorems 11–14. The evolution of the dynamics of model (104) was observed over a time interval $[0, 1000]$. We have chosen the initial conditions as follows: $T(0) = 600$, $T^*(0) = 10$, $C^*(0) = 5$, $V(0) = 10$, and $Z(0) = 80$. We use three sets for the parameters k and g to get the following three subcases. We fix the values $\tau_e = 0.5$ and $\varepsilon = 0$:

- (i) $R_0 \leq 1$. The values of k and g are chosen as $k = 0.002$ virus⁻¹ mm³ day⁻¹ and $g = 0.0005$ virus⁻¹ mm³ day⁻¹. The values of the two bifurcation parameters are computed as $R_0 = 0.3607 < 1$ and $R_1 = 0.3114 < 1$. This means that S_0 is GAS. One can see from Figures 1–5 that the numerical results confirm the results of Theorem 11. We observe that the states of the system eventually approach the steady state S_0 (1000, 0, 0, 0, 0). In this case, the HIV will be removed from the blood.
- (ii) $R_1 \leq 1 < R_0$. We take the values $k = 0.01$ virus⁻¹ mm³ day⁻¹ and $g = 0.0005$ virus⁻¹ mm³ day⁻¹. In this case, $R_1 = 0.8291 < 1 < R_0 = 1.8033$. According to Theorem 13, S_1 is GAS. From Figures 1–5, we can see that there is a consistency between the numerical and theoretical results of Theorem 13. Moreover, the states of the system converge to the steady S_1 (120.22, 8.89, 10.67, 80.91, 0).
- (iii) $R_1 > 1$. We choose $k = 0.01$ virus⁻¹ mm³ day⁻¹ and $g = 0.002$ virus⁻¹ mm³ day⁻¹. Then, we compute $R_0 = 1.8033 > 1$ and $R_1 = 1.6976 > 1$. From Figures 1–5, we can see that the states of the system approach the infected steady state with humoral immune response S_2 (538.56, 4.66, 5.60, 25.00, 209.27). This

FIGURE 1: The concentration of uninfected $CD4^+$ T cells for model (104).

FIGURE 3: The concentration of long-lived chronically infected cells for model (104).

FIGURE 2: The concentration of short-lived infected cells for model (104).

FIGURE 4: The concentration of free virus particles for model (104).

supports the results of Theorem 14 that the infected steady state without humoral immune response S_2 is GAS.

Case 2 (effect of the parameter ε on the stability of the system). For this case, we take $\tau_e = 0.5 \text{ day}^{-1}$, $k = 0.01 \text{ virus}^{-1} \text{ mm}^3 \text{ day}^{-1}$, and $g = 0.001 \text{ virus}^{-1} \text{ mm}^3 \text{ day}^{-1}$. In Figures 6–10, we show the effect of the drug efficacy ε on the stability of the steady states of the system. We can see from Figures 6–10 that as the drug efficacy ε is increased, the concentration of uninfected $CD4^+$ T cells is increased,

while the concentrations of free virus particles, B cells, short-lived infected cells, and long-lived chronically infected cells are decreased. From these figures and Table 2, we can see that the values of R_0 and R_1 are decreased as ε is increased. Using the values of the parameters given in Table 1, we obtain the following:

- (i) If $0 \leq \varepsilon < 0.3730$, then S_2 exists and it is GAS.
- (ii) If $0.3730 \leq \varepsilon < 0.4455$, then S_1 exists and it is GAS.
- (iii) If $0.4455 \leq \varepsilon < 1$, then S_0 is GAS.

It means that the numerical results and the results of Theorems 11–14 are compatible. In this case, the treatment

FIGURE 5: The concentration of B cells for model (104).

FIGURE 6: The concentration of uninfected CD4⁺ T cells for model (104).TABLE 2: The values of equilibria, R_0 , and R_1 for model (104) with different values of ε .

	Equilibria	R_0	R_1
$\varepsilon = 0.0$	E_2 (231.88, 7.77, 9.32, 50, 123.86)	1.8033	1.4129
$\varepsilon = 0.2$	E_2 (337.44, 6.70, 8.04, 50, 65.61)	1.4427	1.2187
$\varepsilon = 0.3730$	E_1 (456.34, 5.50, 6.60, 50, 0)	1.1308	1
$\varepsilon = 0.4$	E_1 (572.42, 4.32, 5.19, 39.32, 0)	1.0820	0.9624
$\varepsilon = 0.4455$	E_0 (1000, 0, 0, 0, 0)	1	0.8974
$\varepsilon = 0.6$	E_0 (1000, 0, 0, 0, 0)	0.7213	0.6632
$\varepsilon = 0.8$	E_0 (1000, 0, 0, 0, 0)	0.3607	0.3388

with sufficient drug efficacy can succeed to clear the HIV from the plasma.

FIGURE 7: The concentration of short-lived infected cells for model (104).

FIGURE 8: The concentration of long-lived chronically infected cells for model (104).

Case 3 (effect of the parameter τ_e on the stability of the system). We fix the parameters $\varepsilon = 0.4$, $k = 0.01$ virus⁻¹ mm³ day⁻¹, and $g = 0.001$ virus⁻¹ mm³ day⁻¹. In Figures 11–15, we show the effect of the time delay parameter τ_e on the stability of the steady states of the system. We can see from Figures 11–15 that the time delay parameter τ_e plays a similar role as the drug efficacy parameter ε . From these figures and Table 3, we can see that the values of R_0 and R_1 are decreased when τ_e is increased. Using the values of the parameters given in Table 1, we obtain the following:

- (i) If $0 \leq \tau_e < 0.4808$, then S_2 exists and it is GAS.

FIGURE 9: The concentration of free virus particles for model (104).

FIGURE 10: The concentration of B cells for model (104).

TABLE 3: The values of equilibria, R_0 , and R_1 for model (104) with different values of τ_e .

Delay	Equilibria	R_0	R_1
$\tau_e = 0.0$	E_2 (476.81, 8.72, 10.46, 50., 484.77)	2.9412	2.6160
$\tau_e = 0.1$	E_2 (476.81, 7.89, 9.47, 50, 342.53)	2.4080	2.1418
$\tau_e = 0.3$	E_2 (476.81, 6.46, 7.75, 50, 130.70)	1.6142	1.4357
$\tau_e = 0.4808$	E_1 (476.81, 5.39, 6.47, 50, 0)	1.1243	1.0
$\tau_e = 0.5$	E_1 (572.42, 4.32, 5.18, 39.32, 0)	1.0820	0.9624
$\tau_e = 0.5394$	E_0 (1000, 0, 0, 0, 0)	1.0	0.8894
$\tau_e = 0.7$	E_0 (1000, 0, 0, 0, 0)	0.7253	0.6451
$\tau_e = 1.0$	E_0 (1000, 0, 0, 0, 0)	0.3981	0.3540
$\tau_e = 2.0$	E_0 (1000, 0, 0, 0, 0)	0.0539	0.0479

(ii) If $0.4808 \leq \tau_e < 0.5394$, then S_1 exists and it is GAS.(iii) If $0.5394 \leq \tau_e$, then S_0 is GAS. FIGURE 11: The concentration of uninfected $CD4^+$ T cells for model (104).

FIGURE 12: The concentration of short-lived infected cells for model (104).

Figures 11–15 show that the numerical results are also compatible with the results of Theorems 11–14.

6. Conclusions

In this paper, we have proposed three HIV infection models with humoral immune response and two types of infected cells. The models incorporate two types of discrete delays. The incidence rate of infection is given by bilinear, saturated functional response and general nonlinear function in the

FIGURE 13: The concentration of long-lived chronically infected cells for model (104).

FIGURE 15: The concentration of B cells for model (104).

FIGURE 14: The concentration of free virus particles for model (104).

first, second, and third model, respectively. We have derived two bifurcation parameters, R_0 and R_1 . The global stability of all steady states of the models has been established using Lyapunov method. We have presented an example for the general incidence rate and performed numerical simulations to support our theoretical results.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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