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Research Article

On Constructing Two-Point Optimal Fourth-Order Multiple-Root Finders with a Generic Error Corrector and Illustrating Their Dynamics

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With an error corrector via principal branch of the mth root of a function-to-function ratio, we propose optimal quartic-order multiple-root finders for nonlinear equations. The relevant optimal order satisfies Kung-Traub conjecture made in 1974. Numerical experiments performed for various test equations demonstrate convergence behavior agreeing with theory and the basins of attractions for several examples are presented.

1. Introduction

Iterative root-finding methods have been constantly developed by numerous researchers [1–3] to solve more accurately the root α of a nonlinear equation $f(x) = 0$ that arises frequently in a scientific world. Classical Newtons method below,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots, \tag{1}$$

is widely used for a simple root under normal circumstances, provided that an initial guess x_0 is chosen close enough to α. Likewise, modified Newtons method [4] of the form

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots, \tag{2}$$

is most popular for a multiple root α with its integer multiplicity $m \geq 1$.

Definition 1. Let $\{x_0, x_1, x_2, \ldots, x_n, \ldots\}$ be a sequence converging to α and $e_n = x_n - \alpha$ be the nth iterate error. If there exist real numbers $p \in \mathbb{R}$ and $b \in \mathbb{R} - \{0\}$ such that the following error equation holds:

$$e_{n+1} = be_n^p + O\left(e_n^{p+1}\right), \tag{3}$$

then b or $|b|$ is called the asymptotic error constant and p is called the order of convergence [5].

Note that both methods (1) and (2) are quadratically convergent one-point optimal [3] methods. Two-point higher-order methods for multiple roots can be found in papers [6–11]. Among these, we introduce, here, respectively, in (4), (5), and (6), some interesting works of Soleymani and Babajee [12], Kanwar et al. [13], and Zhou et al. [14] who have recently developed fourth-order multiple-root finders:

$$y_n = x_n - \frac{2m}{m+2}\frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = x_n + \frac{4md}{d(m^2+2m-4)-m^2\nu}\left[1-\frac{m^3(m-2)}{16d^2}\left(\frac{f'(y_n)}{f'(x_n)}-\frac{m+2}{m}d\right)^2\right]\cdot\frac{f(x_n)}{f'(x_n)},\quad d=\left(\frac{m}{m+2}\right)^m, \tag{4}$$

$$y_n = x_n - \frac{2m}{m+2}\frac{f(x_n)}{f'(x_n)-pf(x_n)},$$
$$x_{n+1} = x_n - \frac{mf(x_n)\left(m^2(3m-2)\{f'(y_n)\}^2 - C_1 f'(x_n)f'(y_n) + C_2\{f'(y_n)\}^2\right)}{2f'(x_n)\left((m-1)d^m f'(x_n) - mf'(x_n)\right)\left(mf'(y_n)\right)\left(mf'(y_n) - d^m(m+8)f'(x_n)\right)},\quad d=\left(\frac{m}{m+2}\right)^m, \tag{5}$$
$$\text{where } p=0,\ C_1 = md^m\left(6m^2+17m-14\right),\ C_2 = d^{2m}\left(3m^3+19m^2+16m+16\right),$$

$$y_n = x_n - m\frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = y_n - mG(w)\cdot\frac{f(x_n)}{f'(x_n)} = x_n - m\{1+G(w)\}\frac{f(x_n)}{f'(x_n)},\quad w = \sqrt[m-1]{\frac{f(y_n)}{f(x_n)}}, \tag{6}$$
$$\text{where } G \text{ is sufficiently differentiable at 0 with } G(0)=0,\ G'(0)=1,\ G''(0)=4.$$

By a close inspection of iterative methods (1), (2), and (4)–(6), we find that an iterative method can be constructed in the following form:

$$e_{n+1} = e_n - Q_f, \tag{7}$$

where Q_f is implicitly dependent upon x_n, for example, with $Q_f = f(x_n)/f'(x_n)$ in (1) and $Q_f = m\{1 + G(w)\}(f(x_n)/f'(x_n))$ in (6). We may regard Q_f as an error-correcting function. Consequently, it would be natural to call the function Q_f as the error corrector. Usually Q_f takes the form of $Q_f = W_f \cdot h$, with W_f as a weighting function that is widely known among many researchers. A more generic form of the error corrector Q_f will be investigated during the course of developing new quartic-order multiple-root finders.

The main aim of this paper is to design new two-step two-point optimal quartic-order multiple-root finders with multiplicity of $m \geq 1$. The first step is to compute $y_n(x_n)$ using $f(x_n)$, $f'(x_n)$ usually with a Newton-like method or other. The second step is to update x_n in the first step by introducing an error corrector Q_f formed by $f(x_n)/f'(x_n)$ and a principal branch of $[f(y_n)/f(x_n)]^{1/m}$. We will check the optimality based on the Kung-Traub conjecture [3] that a multipoint method [15] without memory can achieve its convergence order at most of 2^{k-1} for k functional evaluations.

This paper is comprised of four sections as follows. Following this introductory section, Section 2 describes main results with convergence analysis for newly proposed two-point optimal fourth-order multiple-root finders. Principal branch of a logarithmic function plays a crucial role in developing new methods in view of the relation $[f(y_n)/f(x_n)]^{1/m} = \exp[(1/m)\log(f(y_n)/f(x_n))]$. The convergence analysis includes the derivation of the error equation for the proposed methods. In Section 3, special cases of error-correcting functions are treated with tabulated results and labeled case numbers. Two types of error-correcting functions are constructed based on bivariate polynomials and rational functions. In the first part of Section 4, with error-correcting functions properly chosen from Section 3, a variety of numerical examples are presented for a wide selection of test functions. A comparison for the convergence behavior is made among the proposed methods and the listed existing methods (4)–(6). The second part of Section 4 discusses related dynamics of maps (8) behind the basins of attraction. Dynamical properties of the proposed methods along with their illustrative basins of attraction are displayed with detailed analyses and comments. Section 5 describes overall conclusion as well as possible future work.

2. Main Results

We first assume that a function $f : \mathbb{C} \to \mathbb{C}$ has a multiple root α with integer multiplicity $m \geq 1$ and is analytic [16] in a small neighborhood of α. Then with the concept of error corrector Q_f introduced in (7) we propose new two-step iterative multiple-root finders below, given an initial guess x_0 sufficiently close to α: for $n = 0, 1, 2, \ldots$,

$$y_n = x_n - \gamma\cdot h,\quad h = \frac{f(x_n)}{f'(x_n)},\ \gamma\in\mathbb{R},$$
$$x_{n+1} = x_n - Q_f(s,h),\quad s = \left[\frac{f(y_n)}{f(x_n)}\right]^{1/m}, \tag{8}$$

where $\gamma \in \mathbb{N}$ is a parameter and $Q_f : \mathbb{C}^2 \to \mathbb{C}$ is holomorphic [17] in a neighborhood of $(\lambda, 0)$, where $\lambda \in \mathbb{R}$ is to be determined later for optimal quartic-order convergence. Since s is a one-to-m multiple-valued function, we choose s as a principal analytic branch given by $s = \exp[(1/m)\log(f(y_n)/f(x_n))]$, with $\mathrm{Log}(f(y_n)/f(x_n)) = \log|f(y_n)/f(x_n)| + i\arg(f(y_n)/f(x_n))$ for $-\pi < \arg(f'(y_n)/f'(x_n)) \leq \pi$; this convention of $\arg(z)$ for $z \in \mathbb{C}$ agrees with that of $\log[z]$ command of Mathematica to be adopted in numerical experiments of Section 4. By means of further inspection of s, we find that $\lambda \in \mathbb{R}$ is characterized in such a way that $s = |f(y_n)/f(x_n)|^{1/m} \cdot \exp[(i/m)\arg(f(y_n)/f(x_n))] = \lambda + O(e_n)$, as shown by (15).

These suggested methods require one derivative and two functions in order to achieve optimal order of four. In this section, we establish a main theorem describing the convergence analysis regarding proposed methods (8) and find out how to select the parameter γ and the error-correcting function Q_f for optimal fourth-order convergence.

Theorem 2. *Let $f : \mathbb{C} \to \mathbb{C}$ have a zero α with integer multiplicity $m \geq 1$ and be analytic in a small neighborhood of α. Let $\gamma = m$ and $\theta_j = f^{(m+j)}(\alpha)/f^{(m)}(\alpha)$ for $j \in \mathbb{N}$. Let x_0 be an initial guess chosen in a sufficiently small neighborhood of α. Let $Q_f : \mathbb{C}^2 \to \mathbb{C}$ be holomorphic in a neighborhood of $(0,0)$. Let $Q_{ij} = (1/i!\,j!)(\partial^{i+j}/\partial s^i \partial h^j)Q_f(s,h)|_{(s=0,h=0)}$ for $1 \leq i, j \leq 4$. Suppose that relations $Q_{00} = Q_{02} = Q_{03} = Q_{10} = Q_{12} = Q_{20} = Q_{30} = 0$, $Q_{01} = Q_{11} = m$, and $Q_{21} = 2m$ hold. Then iterative methods (8) are optimal and of order four and possess the following error equation:*

$$\begin{aligned} e_{n+1} = {} & \left[-\frac{Q_{04}}{m^4} - \frac{Q_{13}}{m^4(m+1)}\theta_1 \right. \\ & - \frac{1}{m^2(m+1)^2(m+2)}\theta_1\theta_2 - \frac{Q_{22}}{m^4(m+1)^2}\theta_1^2 \\ & \left. + \frac{m(m+9) - 2Q_{31}}{2m^4(m+1)^3}\theta_1^3 - \frac{Q_{40}}{m^4(m+1)^4}\theta_1^4 \right] e_n^4 \\ & + O\left(e_n^5\right). \end{aligned} \tag{9}$$

Proof. Three functional evaluations evidently are eligible for optimal convergence order in the sense of Kung-Traub. Hence, it suffices to determine the constant parameter γ and some properties of the error-correcting function Q_f for fourth-order convergence. Applying the Taylors series expansion of f about α, we get the following relations:

$$\begin{aligned} f(x_n) = {} & \frac{f^{(m)}(\alpha)}{m!} \\ & \cdot e_n^m \left[1 + A_1 e_n + A_2 e_n^2 + A_3 e_n^3 + A_4 e_n^4 + O\left(e_n^5\right)\right], \end{aligned} \tag{10}$$

$$\begin{aligned} f'(x_n) = {} & \frac{f^{(m)}(\alpha)}{(m-1)!} \\ & \cdot e_n^{m-1} \left[1 + B_1 e_n + B_2 e_n^2 + B_3 e_n^3 + B_4 e_n^4 + O\left(e_n^5\right)\right], \end{aligned} \tag{11}$$

where $A_k = (m!/(m+k)!)\theta_k$, $B_k = ((m-1)!/(m+k-1)!)\theta_k$, and $\theta_k = f^{(m+k)}(\alpha)/f^{(m)}(\alpha)$ for $k = 1, 2, 3, 4$.

Dividing (10) by (11), we have

$$\frac{f(x_n)}{f'(x_n)} = \frac{1}{m}\left[e_n - K_1 e_n^2 - K_2 e_n^3 + K_3 e_n^4 + O\left(e_n^5\right)\right], \tag{12}$$

where $K_1 = -A_1 + B_1$, $K_2 = -A_2 + A_1 B_1 - B_1^2 + B_2$, and $K_3 = -A_3 + A_2 B_1 - A_1 B_1^2 + B_1^3 + A_1 B_2 - 2B_1 B_2 + B_3$.

Letting $t = 1 - \gamma/m$ for convenience with the above relation (12), we obtain

$$\begin{aligned} y_n = {} & \alpha + t e_n + K_1(1-t)e_n^2 + K_2(1-t)e_n^3 \\ & + K_3(1-t)e_n^4 + O\left(e_n^5\right). \end{aligned} \tag{13}$$

Expanding $f(y_n)$ about the multiple root α leads us to relation

$$\begin{aligned} f(y_n) = {} & \frac{f^{(m)}(\alpha)e_n^m}{m!}\left\{ t^{m-1}\left[t \right.\right. \\ & \left. + \left(A_1 t^2 + K_1 m(1-t)\right)\right] e_n + \frac{1}{2}t^{m-2}\left[K_1^2(m-1)\right. \\ & \cdot m(t-1)^2 - 2A_1K_1(m+1)(t-1)t^2 \\ & \left. + 2t\left(A_2 t^3 + K_2 m(1-t)\right)\right] e_n^2 + \frac{1}{6} \\ & \cdot t^{m-3}\left[-K_1^3(m-2)(m-1)m(t-1)^3\right] + 3A_1(m \\ & + 1)(t-1)t^2\left(K_1^2 m(t-1) - 2tK_2\right) + 6t^2\left(A_3 t^4\right. \\ & + K_3 m(1-t) - 6K_1(t-1) \\ & \left.\left. \cdot t\left(A_2(m+2)t^3 + K_2 m(m-1)(1-t)\right)e_n^3\right)\right\} \\ & + O\left(e_n^4\right). \end{aligned} \tag{14}$$

Hence, we get

$$\begin{aligned} s = \left[\frac{f(y_n)}{f(x_n)}\right]^{1/m} = {} & t + \frac{(t-1)(K_1 m + A_1 t)}{k}e_n \\ & - \frac{1}{m^2}(t-1)\left[\left(A_1 K_1 m(2t-1)\right.\right. \\ & + A_1^2 t(1 + m + (m-1)t) \\ & \left.\left. + 2m\left(K_2 m - A_2 t(t+a)\right)\right)\right] e_n^2 + \frac{1}{6m^3}(t-1) \\ & \cdot \left[3A_1^2 K_1 m\left(-1 - m + 4t + 3(m-1)t^2\right)\right. \end{aligned}$$

$$
\begin{aligned}
&+ A_1^3 t\left((t-1)^2 - 3m\left(t^2-1\right) + 2m^2\left(1+t+t^2\right)\right) \\
&+ 6m^2\left(-K_2 m + A_3 t\left(1+t+t^2\right)\right. \\
&+ A_2 K_1\left(1-3t^2\right)\Big) - 6A_1 m\Big(K_2 m(2t-1) \\
&+ K_1^2 m(1-t) + A_2 t\left(1+m+mt+(m-1)t^2\right)\Big)\Big] \\
&\cdot e_n^3 + O\left(e_n^4\right).
\end{aligned} \tag{15}
$$

Hence, λ mentioned in the description of (8) is selected to be t as desired. It must be emphasized that s denotes a principal analytic branch as mentioned earlier from (8). Noting that $O(s-t) = O(h) = O(e_n)$, Taylor expansion of $Q_f(s,h)$ about $(t,0)$ up to fourth-order terms in both variables yields after retaining up to fourth-order terms in e_n with $Q_{14} = Q_{23} = Q_{24} = Q_{32} = Q_{33} = Q_{34} = Q_{41} = Q_{42} = Q_{43} = Q_{44} = 0$:

$$
\begin{aligned}
Q_f(s,h) &= Q_{00} \\
&+ h\left[Q_{01} + Q_{11}(s-t) + Q_{21}(s-t)^2 + Q_{31}(s-t)^3\right] \\
&+ h^2\left[Q_{02} + Q_{12}(s-t) + Q_{22}(s-t)^2\right] \\
&+ h^3\left[Q_{03} + Q_{13}(s-t)\right] + h^3 Q_{04} + Q_{10}(s-t) \\
&+ Q_{20}(s-t)^2 + Q_{30}(s-t)^3 + Q_{40}(s-t)^4 \\
&+ O\left(e_n^4\right).
\end{aligned} \tag{16}
$$

Hence by substituting (10)–(16) into the proposed method (8), we obtain the error equation as

$$
\begin{aligned}
e_{n+1} &= e_n - Q_f(s,h) \\
&= \psi_1 e_n + \psi_2 e_n^2 + \psi_3 e_n^3 + \psi_4 e_n^4 + O\left(e_n^5\right),
\end{aligned} \tag{17}
$$

where $\psi_0 = -Q_{00}$ and the coefficients ψ_i, $i = 1,2,3,4$, generally depend on the parameters t, Q_{ij} $(0 \le i,j \le 4)$, and θ_i $(i = 1,2,\ldots)$ and the function $f(x)$.

Solving $\psi_1 = 0$ for Q_{00}, we obtain

$$
Q_{00} = 0. \tag{18}
$$

Substituting (18) into $\psi_1 = 0$ and simplifying, we have

$$
\psi_1 = \frac{m - Q_{01}}{m} - \frac{Q_{10}(t-1)^2\theta_1}{m(m+1)} = 0 \tag{19}
$$

from which $m - Q_{01} = 0$ and $Q_{10}(t-1)^2 = 0$ must be satisfied independently of θ_1. Hence,

$$
\begin{aligned}
Q_{01} &= m, \\
Q_{10} &= 0.
\end{aligned} \tag{20}
$$

Substituting (18) and (20) into $\theta_2 = 0$, we get

$$
\begin{aligned}
\psi_2 &= -\frac{Q_{02}}{m^2} - \frac{m - Q_{11}(t-1)^2\theta_1}{m^2(m+1)} - \frac{Q_{20}(t-1)^4\theta_1^2}{m^2(m+1)^2} \\
&= 0.
\end{aligned} \tag{21}
$$

Solving the above equation independently of θ_1 for Q_{02}, Q_{11} and Q_{20}, we obtain

$$
\begin{aligned}
Q_{11} &= \frac{m}{(t-1)^2}, \\
Q_{02} &= Q_{20} = 0.
\end{aligned} \tag{22}
$$

Substituting (18), (20), and (22) into $\psi_3 = 0$ and simplifying, we have

$$
\begin{aligned}
\psi_3 = &-\frac{Q_{03}}{m^3} - \frac{Q_{12}(t-1)^2}{m^3(m+1)}\theta_1 - \frac{Q_{30}(t-1)^6}{m^3(m+1)^3}\theta_1^3 \\
&+ \frac{\left(-m(t-4) - 2Q_{21}(t-1)^4 + m^2 t\right)}{2m^3(m+1)^2}\theta_1^2 \\
&- \frac{t}{m(m+1)(m+2)}\theta_2 = 0.
\end{aligned} \tag{23}
$$

To make $\psi_3 = 0$ independently of θ_1 and θ_2, we obtain

$$
\begin{aligned}
t &= Q_{00} = Q_{10} = Q_{02} = Q_{20} = Q_{12} = Q_{03} = Q_{30} \\
&= 0, \\
Q_{01} &= Q_{11} = m, \\
Q_{21} &= 2m.
\end{aligned} \tag{24}
$$

With the aid of symbolic computation of Mathematica [18], we substitute (18)–(24) into ψ_4 to arrive at the following relation:

$$
\begin{aligned}
e_{n+1} = &\left[-\frac{Q_{04}}{m^4} - \frac{Q_{13}}{m^4(m+1)}\theta_1\right. \\
&- \frac{1}{m^2(m+1)^2(m+2)}\theta_1\theta_2 - \frac{Q_{22}}{m^4(m+1)^2}\theta_1^2 \\
&\left.+ \frac{m(m+9) - 2Q_{31}}{2m^4(m+1)^3}\theta_1^3 - \frac{Q_{40}}{m^4(m+1)^4}\theta_1^4\right] e_n^4 \\
&+ O\left(e_n^5\right).
\end{aligned} \tag{25}
$$

This completes the proof. □

3. Special Cases of Error-Correcting Functions

Using relations (18)–(24), the bivariate Taylor polynomial of $Q_f(s,h)$ is easily given by

$$
\begin{aligned}
Q_f(s,h) = &\left[m\left(1+s+2s^2\right) + Q_{31}s^3\right]h + h^2 Q_{22}s^2 \\
&+ h^3 Q_{13}s + h^4 Q_{04} + Q_{40}s^4.
\end{aligned} \tag{26}
$$

Here notaions $s = [f(y_n)/f(x_n)]^{1/m}$ and $h = f(x_n)/f'(x_n)$ are introduced for simplicity. Special cases of $Q_f(s,h)$ are considered here. In each case, relevant coefficients are determined based on relations (18)–(24). If $Q_{22} = Q_{13} = Q_{04} = Q_{40} = 0$, then (26) yields

$$
Q_f(s,h) = \left[m\left(1+s+2s^2\right) + Q_{31}s^3\right]h. \tag{27}
$$

Table 1: Typical subcases of Case 1 with selected parameters.

SN	Q_{31}	Q_{22}	Q_{13}	Q_{04}	Q_{40}	$Q_f(s,h)$
1A	0	0	0	0	0	$m(1+s+s^2)\cdot h$
1B	1	0	0	0	0	$[m(1+s+s^2)+s^3]\cdot h$
1C	$2m$	0	0	0	0	$m(1+s)(1+2s^2)\cdot h$
1D	0	1	0	0	0	$[h^3+m(1+s+2s^2)]\cdot h$
1E	$2m$	0	1	0	0	$[h^2s+m(1+s)(1+2s^2)]\cdot h$
1F	0	0	0	1	0	$[(m(1+s)+(h+2m)s^2)]\cdot h$
1G	$2m$	0	0	0	1	$s^4+m(1+s)(1+s^2)\cdot h$

Remark 3. If we take the Taylor polynomial of $G(s)$ being in the form of $G(s) = (1+s+2s^2)+(Q_{31}/m)s^3$, then $G(0) = 1$, $G'(0) = 1$ and $G''(0) = 4$ hold. Hence, method (6) is a subcase of this case in which $Q_f(s,h) = mG(s)\cdot h$ is satisfied under the restriction of f to $\mathbb{R}$.

Although a variety of forms of error-correcting functions $Q_f(s,h)$ are available in view of (26), we will limit ourselves to considering two cases of error correctors comprising low-order bivariate polynomials or simple rational functions.

Case 1 (Q_f with a bivariate polynomial). In this case, $Q_{31}, Q_{22}, Q_{13}, Q_{04}$, and Q_{40} can be regarded as parameters to be chosen to satisfy

$$Q_f(s,h) = \left[m\left(1+s+2s^2\right)+Q_{31}s^3\right]h + h^2Q_{22}s^2 + h^3Q_{13}s + h^4Q_{04} + Q_{40}s^4. \tag{28}$$

We list typical subcases with selected parameters and $Q_f(s,h)$ in Table 1, where SN stands for the corresponding subcase identification number.

Case 2 (Q_f with a bivariate polynomial and rational functions). Consider

$$Q_f(s,h) = \frac{a_1+a_2s+a_3s^2}{1+b_2s+b_3s^2}\cdot h + ds^4 + qh^4 + h^2s\frac{q_1h+q_2s}{1+rsh}, \tag{29}$$

where a_1, a_2, and a_3 and b_2, b_3 are determined using (18)–(24) with two of them as free parameters. We find that $a_1 = m, a_2 = a_3 - m(1+b_3)$, and $b_2 = -2-b_3+a_3/m$ and the remaining parameters d, q, r, q_1, and q_2 are free to be chosen.

In Table 2, we list typical subcases with interesting choices of parameters and $Q_f(s,h)$.

4. Numerical Experiments and Dynamics

In this section, we will first describe the computational experiments of proposed methods (8) and then illustrate the complex dynamics [19–26] related to the basins of attraction [27–29] of iterative maps (26) along with comparisons among existing methods.

Throughout the experiments, we have moderately assigned 112 significant digits as the minimum number of precision digits, via Mathematica [18] command MinPrecision = 112, to achieve the desired accuracy ensuring convergence of the proposed methods. It is necessary to compute $e_n = x_n - \alpha$ with high accuracy for desired numerical results. In case that α is not exact, it is replaced by a more accurate value which has a larger number of significant digits than the assigned \$MinPrecision = 112.

Definition 4 (computational convergence order). Assume that theoretical asymptotic error constant $\eta = \lim_{n\to\infty}(|e_n|/|e_{n-1}|^p)$ and convergence order $p \geq 1$ are known. Define $p_n = \log|e_n/\eta|/\log|e_{n-1}|$ as the computational convergence order. Note that $\lim_{n\to\infty}p_n = p$.

If α and x_n both mutually have the same accuracy of \$MinPrecision = 112, then $e_n = x_n - \alpha$ would be nearly zero as n becomes large and thus computing $|e_{n+1}/e_n^4|$ would unfavorably cause a numeric overflow. Computed values of x_n are accurate up to 112 significant digits. To observe the reliable convergence behavior, we desire α with enough accuracy of 16 digits higher than MinPrecision, which has 128 significant digits. To supply such α, a set of following Mathematica commands are used:

```
sol = FindRoot[f(x), {x, x0}, PrecisionGoal ⟶ 16
    + $MinPrecision, WorkingPrecision ⟶ 2
    * $MinPrecision];
α = sol[[1, 2]].
```
(30)

Although the number of significant digits of x_n and α is 112 and 128, respectively, we list the two values at most up to 15 significant digits due to the limited paper space. We set the error bound ε to $(1/2)\times 10^{-80}$ satisfying $|x_n - \alpha| < \varepsilon$.

Iterative methods (26) with all subcases of both Cases 1 and 2 were, respectively, identified by **Y1A**, **Y1B**, **Y1C**, **Y1D**, **Y1E**, **Y1F**, and **Y1G** and **Y2A**, **Y2B**, **Y2C**, **Y2D**, **Y2E**, **Y2F**, **Y2G**, and **Y2H**, being **Y**-prefixed. Among them, typical three methods have been successfully applied to three test functions shown below:

Method **Y1D**: $F_1(x) = \left[4\sin^2\frac{\pi}{x} + x^2 - 12\right]^4$,

$$m = 4,\ \alpha = -3.$$

Method **Y2B**:

$$F_2(x) = \left[2x + 4i - 2\pi + \sin(x+2i)\log\left(x^2+1\right)\right]\cdot\sin^2(x+2i),\quad m = 3,\ \alpha = -2i+\pi,\ i = \sqrt{-1}. \tag{31}$$

Method **Y2E**: $F_3(x) = \left[\left\{\sin^{-1}(x-1)\right\}^2 + e^x - 6\right]^2$,

$$m = 2,\ \alpha \approx 1.69008688293711,$$

where $\log z$ $(z \in \mathbb{C})$

represents a principal analytic branch such that

$$-\pi < \operatorname{Im}(\log z) \leq \pi.$$

Table 2: Typical subcases of Case 2 with selected parameters.

SN	a_1	a_2	a_3	b_2	b_3	d	q	r	q_1	q_2	$Q_f(s)$
2A	m	$-m$	0	-2	0	0	0	0	0	0	$\frac{m(1-s)}{1-2s}\cdot h$
2B	m	0	0	-1	-1	0	0	0	0	0	$\frac{m}{1-s-s^2}\cdot h$
2C	m	0	m	-1	0	0	0	0	0	0	$\frac{m(1+s^2)}{1-s}\cdot h$
2D	m	$-m$	0	-2	0	1	0	0	0	0	$s^4+\frac{m(1-s)}{1-2s}\cdot h$
2E	m	0	0	-1	-1	0	1	0	0	0	$\left[h^3+\frac{m}{1-s-s^2}\right]\cdot h$
2F	m	0	m	-1	0	0	0	1	0	1	$\left[\frac{s^2h}{1+sh}+\frac{m(1+s^2)}{1-s}\right]\cdot h$
2G	m	$-m$	0	-2	0	0	0	1	1	0	$\left[\frac{sh^2}{1+sh}+\frac{m(1-s)}{1-2s}\right]\cdot h$
2H	m	0	0	-1	-1	0	0	0	0	1	$\left[s^2h+\frac{m}{1-s-s^2}\right]\cdot h$

Methods **Y1D**, **Y2B**, and **Y2E** in Table 3 have clearly confirmed quartic-order convergence. Table 3 lists iteration indexes n, approximate zeros x_n, residual errors $|F(x_n)|$, errors $|e_n|$, and computational asymptotic error constants η_n as well as the theoretical asymptotic error constant η and computational asymptotic convergence order p_n. The values of η_n agree up to 10 significant digits with η. Undoubtedly, the computed asymptotic order of convergence well approaches 4. The computational asymptotic error constant reveals a good agreement with the theory developed in Section 3.

Additional functions below are tested to further confirm the convergence of methods (8):

$$f_1(x) = x^2\left\{x+\log\left(1+x^2\right)\right\}, \quad \alpha = 0,\ m = 3,\ x_0 = 0.1,$$

$$f_2(x) = \left[5-6x^2+\cos(2x)-\sin(3x)\right]^2 \cdot\left[5-6x^2+\cos(2x)(1-2\sin x)-\sin x\right]^2, \quad \alpha \approx 0.84777738787,\ m = 4,\ x_0 = 1.0,$$

$$f_3(x) = \left[1+\cos\left(x^2-1\right)-x\log\left(x^2-\pi\right)\right]^4 \cdot\left(x^2-1-\pi\right)^2, \quad \alpha = \sqrt{1+\pi},\ m = 6,\ x_0 = 2.0,$$

$$f_4(x) = \left(x^2+e^{-x^2}+\sin\left(\frac{1}{x}-3\right)^7\right), \quad \alpha = 1.51115415815562,\ m = 7,\ x_0 = 1.0,$$

$$f_5(x) = \left[x-\sqrt{3}x^3\cos\left(\frac{\pi x}{6}\right)+\frac{1}{1+x^2}-\frac{11}{5}+4\sqrt{3}\right]^2 \cdot(x-2)^3, \quad \alpha = 2,\ m = 5,\ x_0 = 1.7,$$

$$f_6(x) = \left[(x-2)^2+\frac{3}{16}+\frac{1}{x^2}\log\left\{(x-2)^2+\frac{19}{16}\right\}\right] \cdot\left[(x-2)^2+\frac{3}{16}\right], \quad \alpha = 2-\frac{i\sqrt{3}}{4},\ m = 2,\ x_0 = 1.93-0.41i,$$

$$f_7(x) = \left[x^2(1-\log x)-\frac{1}{\sqrt{x}}\right]^8, \quad \alpha = 1,\ m = 8,\ x_0 = 0.9;$$

here $\log z$ $(z \in \mathbb{C})$ represents a principal analytic branch such that $-\pi < \operatorname{Im}(\log z) \le \pi$. (32)

In Table 8, we directly compare numerical errors $|x_n-\alpha|$ of proposed methods **Y1B**, **Y1C**, **Y1E**, and **Y1G** in Case 1 with those of existing optimal fourth-order multiple-root finders. Abbreviations **Kan**, **Zhou**, and **Sol** denote existing optimal fourth-order multiple root finders obtained by Kanwar et al. (5), Zhou et al. (6) with $G(w) = (1+w+2w^2)$, and Soleymani and Babajee (4), respectively. The least errors within the prescribed error bound are highlighted in boldface. Method **Y1E** shows best convergence for f_1, f_5, and f_7 and method **Y1G** for f_2, f_4 and method **Sol** for f_3, f_6.

Likewise, in Table 9, we compare numerical errors $|x_n-\alpha|$ of proposed methods **Y2A**, **Y2D**, **Y2F**, and **Y2G** in Case 2 with those of existing optimal fourth-order multiple-root finders. Method **Y2G** shows best convergence for f_1, f_3, f_5, and f_7 and method **Y2D** for f_4, f_6, while method shows best convergence **Y2F** for f_2.

Recall that $\lim_{n\to\infty}((x_{n+1}-\alpha)/(x_n-\alpha)^p) = g_f'(\alpha)/p!$ for an iterative method $x_{n+1} = g_f(x_n)$ converging to α with order p; note that the iteration function $g_f(x)$ is dependent upon a given nonlinear equation $f(x)$. Consequently, with a given order of convergence p, the local convergence behavior is dependent on the function $f(x)$, an initial value x_0, and a multiple zero α itself. Hence, for given test functions, zeros, and initial values, no method is expected to always show better performance than the others.

Table 3: Convergence for test functions $F_1(x) - F_3(x)$ with methods **M1F**, **M2E**, and **M2F**.

$\binom{\mathbf{MT}}{f_i}$	n	x_n	$\lvert F(x_n)\rvert$	$\lvert e_n = x_n - \alpha\rvert$	$\lvert e_n/e_{n-1}^4\rvert$	η	p_n
	0	−3.3	5.68515	0.3			
	1	−3.00038943156575	1.212×10^{-11}	3.894×10^{-4}	0.04807797108	0.06814101693	4.28967
$\binom{\mathbf{Y1C}}{F_1}$	2	−3.00000000000000	3.173×10^{-57}	1.566×10^{-15}	0.06811440433		4.00005
	3	−3.00000000000000	1.495×10^{-239}	4.104×10^{-61}	0.06814101693		4.00000
	4	−3.00000000000000	0.0×10^{-443}	0.0×10^{-111}			
	0	$\binom{3.1}{-1.95} *$	0.000338602	0.0650381			
	1	$\binom{3.14216523070554}{2.00080488739465}$	1.213×10^{-9}	9.877×10^{-4}	55.20587710	0.02458677729	1.17628
$\binom{\mathbf{Y1D}}{F_2}$	2	$\binom{3.14159265358979}{2.00000000000000}$	1.614×10^{-41}	2.340×10^{-14}	0.02459092337		3.99998
	3	$\binom{3.14159265358979}{2.00000000000000}$	5.066×10^{-169}	4.794×10^{-57}	0.02458677729		4.00000
	4	$\binom{3.14159265358979}{2.00000000000000}$	0.0×10^{-557}	0.0×10^{-111}			
	0	1.6	0.400529	0.0900869			
	1	1.70503564060025	0.0129610	0.0149488	226.9649843	0.3269289253	1.28174
$\binom{\mathbf{Y1E}}{F_3}$	2	1.69008689870921	1.408×10^{-14}	1.577×10^{-8}	0.3158415015		4.00821
	3	1.69008688293711	2.317×10^{-62}	2.023×10^{-32}	0.3269289253		4.00000
	4	1.69008688293711	0.0×10^{-222}	0.0×10^{-111}			

MT means method, $p_n = (\log\lvert e_n/\eta\rvert)/(\log\lvert e_{n-1}\rvert)$, and $* \binom{3.1}{-1.95} = 3.1 - 1.95i$.

Table 4: Typical Example 1 with $P_1(z) = (9z^2 + z - 7)^2$, $m = 2$.

Methods	**CPU**	**TCON**	**AVG**	**TDIV**
Kan	104	360,000	5.3	0
Sol	74	352,462	5.6	7538
Zhou	54	360,000	5.9	0
Y1D	83	360,000	5.5	5758
Y1G	77	359,998	5.7	2
Y2D	100	360,000	5.3	0
Y2G	90	360,000	5.0	0

Table 5: Typical Example 2 with $P_2(z) = z^3(z + 1)$, $m = 3$.

Method	**CPU**	**TCON**	**AVG**	**TDIV**
Kan	108	360,000	5.2	0
Sol	96	359,194	5.1	806
Zhou	69	360,000	5.3	0
Y1D	333	360,000	4.8	0
Y1G	166	360,000	4.8	0
Y2D	526	360,000	4.6	0
Y2G	251	359,996	4.7	4

Table 6: Typical Example 3 with $P_3(z) = (z^3 - z)^4$, $m = 4$.

Method	**CPU**	**TCON**	**AVG**	**TDIV**
Kan	320	360,000	6.4	0
Sol	201	359,860	6.2	140
Zhou	80	359,988	7.4	2
Y1D	112	360,000	6.7	0
Y1G	138	360,000	6.4	0
Y2D	172	360,000	5.7	0
Y2G	187	360,000	5.5	0

Table 7: Typical Example 4 with $P_4(z) = (z^2 - 1)^7$, $m = 7$.

Method	**CPU**	**TCON**	**AVG**	**TDIV**
Kan	157	360,000	5.5	0
Sol	146	359,960	5.4	40
Zhou	146	360,000	5.9	0
Y1D	58	360,000	5.4	0
Y1G	61	359,994	5.7	0
Y2D	62	359,998	5.3	2
Y2G	74	360,000	4.7	0

The efficiency index [5] is defined by $\mathrm{EI} = p^{1/d}$, where p is the order of convergence and d is the number of distinct functional or derivative evaluations per iteration. The proposed methods (8) as well as all other listed methods have the same $\mathrm{EI} = 4^{1/3} \approx 1.587$ agreeing with Kung-Traub optimality conjecture.

At this point, we are now ready to discuss the dynamical behavior of iterative maps (8). It is important to properly select initial values close to zero α for guaranteed convergence of iterative methods. It is not, however, an easy task to determine how close the initial values are to zero α. A convenient way of employing stable initial values is to directly use visual basins of attraction. In view of inspection of the area of convergence on the basins of attraction, the larger area of convergence would imply a better method. Clearly measuring the size of area of convergence requires a quantitative analysis. To this end, we provide Tables 5–8 featuring statistical data for the average number of iterations per point and the number of divergent points including **CPU** time. In the following 4 examples, we select a 6 by 6 square region centered at the origin and containing all the zeros of the given test polynomial functions. We then take a 600 × 600 uniform grid in the square to display initial points for the iterative methods via basins of attraction. We color

Table 8: Comparison of $|x_n - \alpha|$ among existing and proposed (Case 1) multiple-root finders.

$f, x_0; m$	$\lvert x_n - \alpha\rvert$	**Kan**	**Sol**	**Zhou**	**Y1B**	**Y1C**	**Y1E**	**Y1G**
f_1, 0.1; 3	$\lvert x_1 - \alpha\rvert$	$1.23e-5$	$1.12e-5$	$1.54e-5$	$1.47e-5$	$1.11e-5$	$1.02e-5$	$1.05e-5$
	$\lvert x_2 - \alpha\rvert$	$3.44e-21$	$2.15e-21$	$1.28e-20$	$1.00e-20$	$2.31e-21$	**$1.48e-21$**	$1.68e-21$
	$\lvert x_3 - \alpha\rvert$	$2.08e-83$	$2.89e-84$	$6.00e-81$	$2.10e-81$	$4.23e-84$	$6.58e-85$	$1.09e-84$
f_2, 1.0; 4	$\lvert x_1 - \alpha\rvert$	$3.47e-5$	$4.61e-5$	$7.32e-5$	$6.75e-5$	$6.49e-6$	$2.53e-5$	$1.92e-5$
	$\lvert x_2 - \alpha\rvert$	$2.33e-20$	$1.32e-19$	$2.92e-18$	$1.92e-18$	$2.23e-21$	$9.36e-21$	**$2.16e-21$**
	$\lvert x_3 - \alpha\rvert$	$4.75e-81$	$9.15e-78$	$7.36e-72$	$1.26e-72$	$3.14e-83$	$1.74e-82$	$3.46e-85$
f_3, 2.0; 6	$\lvert x_1 - \alpha\rvert$	$8.68e-6$	$7.39e-6$	$1.64e-5$	$1.58e-5$	$9.40e-6$	$9.39e-6$	$1.44e-5$
	$\lvert x_2 - \alpha\rvert$	$4.03e-20$	$1.87e-20$	$9.76e-19$	$8.12e-19$	$5.59e-20$	$5.56e-20$	$5.01e-19$
	$\lvert x_3 - \alpha\rvert$	$1.87e-77$	**$7.68e-79$**	$1.20e-71$	$5.53e-72$	$6.99e-77$	$6.86e-77$	$7.29e-73$
	$\lvert x_4 - \alpha\rvert$	$0.0e-111$	$0.0e-111$	$0.0e-111$	$0.0e-111$	$0.0e-111$	$0.0e-111$	$0.0e-111$
f_4, 1.0; 7	$\lvert x_1 - \alpha\rvert$	$4.93e-5$	$4.64e-5$	$5.65e-5$	$5.52e-5$	$3.81e-5$	$3.80e-5$	$3.29e-5$
	$\lvert x_2 - \alpha\rvert$	$6.98e-18$	$5.14e-18$	$1.44e-17$	$1.28e-17$	$1.85e-18$	$1.83e-18$	$8.25e-19$
	$\lvert x_3 - \alpha\rvert$	$2.79e-69$	$7.79e-70$	$6.19e-68$	$3.70e-68$	$1.03e-71$	$9.87e-72$	**$3.24e-73$**
	$\lvert x_4 - \alpha\rvert$	$0.0e-111$	$0.0e-111$	$0.0e-111$	$0.0e-111$	$0.0e-111$	$0.0e-111$	$0.0e-111$
f_5, 1.7; 5	$\lvert x_1 - \alpha\rvert$	$2.89e-3$	$2.87e-3$	$1.26e-3$	$1.25e-3$	$1.14e-3$	$1.13e-3$	$1.16e-3$
	$\lvert x_2 - \alpha\rvert$	$2.13e-10$	$2.02e-10$	$4.83e-12$	$4.53e-12$	$2.39e-12$	$2.30e-12$	$2.80e-12$
	$\lvert x_3 - \alpha\rvert$	$6.46e-39$	$5.09e-39$	$1.05e-45$	$7.91e-46$	$4.59e-47$	**$3.90e-47$**	$9.67e-47$
	$\lvert x_4 - \alpha\rvert$	$0.0e-111$	$0.0e-111$	$0.0e-111$	$0.0e-111$	$0.0e-111$	$0.0e-111$	$0.0e-111$
f_6, 1.93, $-0.41i$; 2	$\lvert x_1 - \alpha\rvert$	$1.22e-4$	$9.90e-5$	$2.51e-4$	$2.26e-4$	$1.53e-4$	$1.57e-4$	$1.98e-4$
	$\lvert x_2 - \alpha\rvert$	$8.28e-16$	$2.91e-16$	$3.03e-14$	$1.79e-14$	$2.48e-15$	$2.87e-15$	$7.78e-15$
	$\lvert x_3 - \alpha\rvert$	$1.74e-60$	**$2.20e-62$**	$6.38e-54$	$6.98e-55$	$1.41e-29$	$1.90e-29$	$1.39e-28$
	$\lvert x_4 - \alpha\rvert$	$0.0e-111$	$0.0e-111$	$0.0e-111$	$0.0e-111$	$8.31e-58$	$8.31e-58$	$0.0e-111$
f_7, 0.9; 8	$\lvert x_1 - \alpha\rvert$	$5.76e-5$	$5.42e-5$	$6.65e-5$	$6.51e-5$	$4.33e-5$	$4.32e-5$	$4.94e-5$
	$\lvert x_2 - \alpha\rvert$	$9.45e-18$	$7.03e-18$	$1.96e-17$	$1.75e-17$	$2.13e-18$	$2.11e-18$	$4.29e-18$
	$\lvert x_3 - \alpha\rvert$	$6.82e-69$	$1.99e-69$	$1.48e-67$	$9.20e-68$	$1.25e-71$	**$1.20e-71$**	$2.45e-70$
	$\lvert x_4 - \alpha\rvert$	$0.0e-111$	$0.0e-112$	$0.0e-112$	$0.0e-112$	$0.0e-112$	$0.0e-112$	$0.0e-112$

Here, $1.23e-5$ denotes 1.23×10^{-5}.

the grid point, according to the iteration number for convergence and the root it converged to. This way we can find out if the method converged within the maximum number of iteration allowed and if it converged to the root closer to the initial grid point. To continue our discussion, let us first identify four members of iterative map (8) associated with Cases **1D**, **1G**, **2D**, and **2G** by **Y1D**, **Y1G**, **Y2D**, and **Y2G**, respectively.

For illustrating the complex dynamics of (8) with the desired basins of attraction, we take applied to various polynomials having multiple roots with multiplicity $m = 2, 3, 4, 7$. Statistical data for the basins of attraction are tabulated in Tables 4–7. In this tables, abbreviations **CPU**, **TCON**, **AVG,** and **TDIV** denote the value of **CPU** time for convergence, the value of total convergent points, the value of average iteration number for convergence, and the value of divergent points.

In the first example, we have taken the following polynomial:

$$P_1(z) = \left(9z^2 + z - 7\right)^2, \tag{33}$$

whose roots $z = -0.939221, 0.82811$ are both real with multiplicity $m = 2$. Based on Table 4 and Figure 1, we can find out that **Y2G** is best in view of lower **AVG** and **TDIV**, followed by **Kan** and **Y2D**. As can be seen in Figure 1, **Sol** and **Y1D** have shown considerable amount of black points. These points causing divergence behavior were expected from the last column of Table 4. The best result for **CPU** is by **Zhou** and the worst one is by **Kan**.

Our next sample has triple roots. The polynomial has three identical roots at the origin:

$$P_2(z) = z^3(z+1). \tag{34}$$

The results are listed in Table 5 and Figure 2. The method **Y2D** performs best in view of lower **AVG** and **TDIV**. These proposed methods **Y1D**, **Y1G**, **Y2D,** and **Y2G** appear to perform better than **Kan**, **Sol**, and **Zhou**. As can be seen in Figure 2, **Sol** has shown considerable amount of black points, while **Y2G** has shown a few black ones. The best result for **CPU** is by **Zhou** and the worst one is by **Y2D**.

As a third example, we take the following polynomial whose roots are all of multiplicity four:

$$P_3(z) = \left(z^3 - z\right)^4. \tag{35}$$

As can be seen in Figure 3, Sol has shown considerable amount of black points, while **Zhou** has shown several black ones. The best result for **CPU** is by **Zhou** and the worst one is by **Kan**.

Table 9: Comparison of $|x_n - \alpha|$ among existing and proposed (Case 2) multiple-root finders.

$f, x_0; m$	$\lvert x_n - \alpha\rvert$	**Kan**	**Sol**	**Zhou**	**Y2A**	**Y2D**	**Y2F**	**Y2G**
f_1, 0.1; 3	$\lvert x_1 - \alpha\rvert$	$1.23e-5$	$1.12e-5$	$1.54e-5$	$6.34e-6$	$5.71e-6$	$1.02e-5$	$5.38e-6$
	$\lvert x_2 - \alpha\rvert$	$3.44e-21$	$2.15e-21$	$1.28e-20$	$1.19e-22$	$6.59e-23$	$1.48e-21$	$\mathbf{5.20e-23}$
	$\lvert x_3 - \alpha\rvert$	$2.08e-83$	$2.89e-84$	$6.00e-81$	$1.52e-89$	$1.16e-90$	$6.58e-85$	$4.53e-91$
f_2, 1.0; 4	$\lvert x_1 - \alpha\rvert$	$3.47e-5$	$4.61e-5$	$7.32e-5$	$2.86e-5$	$3.71e-5$	$2.14e-5$	$3.11e-5$
	$\lvert x_2 - \alpha\rvert$	$2.33e-20$	$1.32e-19$	$2.92e-18$	$5.43e-10$	$9.18e-10$	$\mathbf{4.47e-21}$	$6.42e-10$
	$\lvert x_3 - \alpha\rvert$	$4.75e-81$	$9.15e-78$	$7.36e-72$	$3.98e-39$	$4.10e-38$	$8.37e-84$	$8.66e-39$
f_3, 2.0; 6	$\lvert x_1 - \alpha\rvert$	$8.68e-6$	$7.39e-6$	$1.64e-5$	$8.53e-7$	$5.87e-6$	$9.12e-6$	$8.45e-7$
	$\lvert x_2 - \alpha\rvert$	$4.03e-20$	$1.87e-20$	$9.76e-19$	$5.55e-25$	$6.51e-21$	$4.99e-20$	$\mathbf{5.31e-25}$
	$\lvert x_3 - \alpha\rvert$	$1.87e-77$	$7.68e-79$	$1.20e-71$	$9.94e-98$	$9.82e-81$	$4.47e-77$	$8.25e-98$
f_4, 1.0; 7	$\lvert x_1 - \alpha\rvert$	$4.93e-5$	$4.64e-5$	$5.65e-5$	$1.59e-5$	$1.07e-5$	$3.69e-5$	$1.58e-5$
	$\lvert x_2 - \alpha\rvert$	$6.98e-18$	$5.14e-18$	$1.44e-17$	$2.13e-20$	$\mathbf{2.06e-21}$	$1.60e-18$	$2.07e-20$
	$\lvert x_3 - \alpha\rvert$	$2.79e-69$	$7.79e-70$	$6.19e-68$	$6.85e-80$	$2.86e-84$	$5.77e-72$	$6.11e-80$
f_5, 1.7; 5	$\lvert x_1 - \alpha\rvert$	$2.89e-3$	$2.87e-3$	$1.26e-3$	$1.00e-3$	$1.01e-3$	$1.15e-3$	$9.93e-4$
	$\lvert x_2 - \alpha\rvert$	$2.13e-10$	$2.02e-10$	$4.83e-12$	$8.63e-13$	$1.08e-12$	$2.45e-12$	$8.23e-13$
	$\lvert x_3 - \alpha\rvert$	$6.46e-39$	$5.09e-39$	$1.05e-45$	$4.77e-49$	$1.43e-48$	$5.13e-47$	$\mathbf{3.92e-49}$
	$\lvert x_4 - \alpha\rvert$	$0.0e-111$	$0.0e-111$	$0.0e-111$	$0.0e-111$	$0.0e-111$	$0.0e-111$	$0.0e-111$
f_6, 1.93, $-0.41i$; 2	$\lvert x_1 - \alpha\rvert$	$1.22e-4$	$9.90e-5$	$2.51e-4$	$4.88e-5$	$1.04e-4$	$1.41e-4$	$5.34e-5$
	$\lvert x_2 - \alpha\rvert$	$8.28e-16$	$2.91e-16$	$3.03e-14$	$8.58e-18$	$2.95e-16$	$1.83e-15$	$1.34e-17$
	$\lvert x_3 - \alpha\rvert$	$1.74e-60$	$2.20e-62$	$6.38e-54$	$1.69e-34$	$\mathbf{1.89e-62}$	$7.75e-30$	$4.14e-34$
	$\lvert x_4 - \alpha\rvert$	$0.0e-111$	$0.0e-111$	$0.0e-111$	$0.0e-111$	$0.0e-111$	$1.38e-58$	$0.0e-111$
f_7, 0.9; 8	$\lvert x_1 - \alpha\rvert$	$5.76e-5$	$5.42e-5$	$6.65e-5$	$1.50e-5$	$2.11e-5$	$4.24e-5$	$1.49e-5$
	$\lvert x_2 - \alpha\rvert$	$9.45e-18$	$7.03e-18$	$1.96e-17$	$1.06e-20$	$6.43e-20$	$1.98e-18$	$\mathbf{1.03e-20}$
	$\lvert x_3 - \alpha\rvert$	$6.82e-69$	$1.99e-69$	$1.48e-67$	$2.64e-81$	$5.52e-78$	$9.47e-72$	$2.35e-81$

Here, $1.23e-5$ denotes 1.23×10^{-5}.

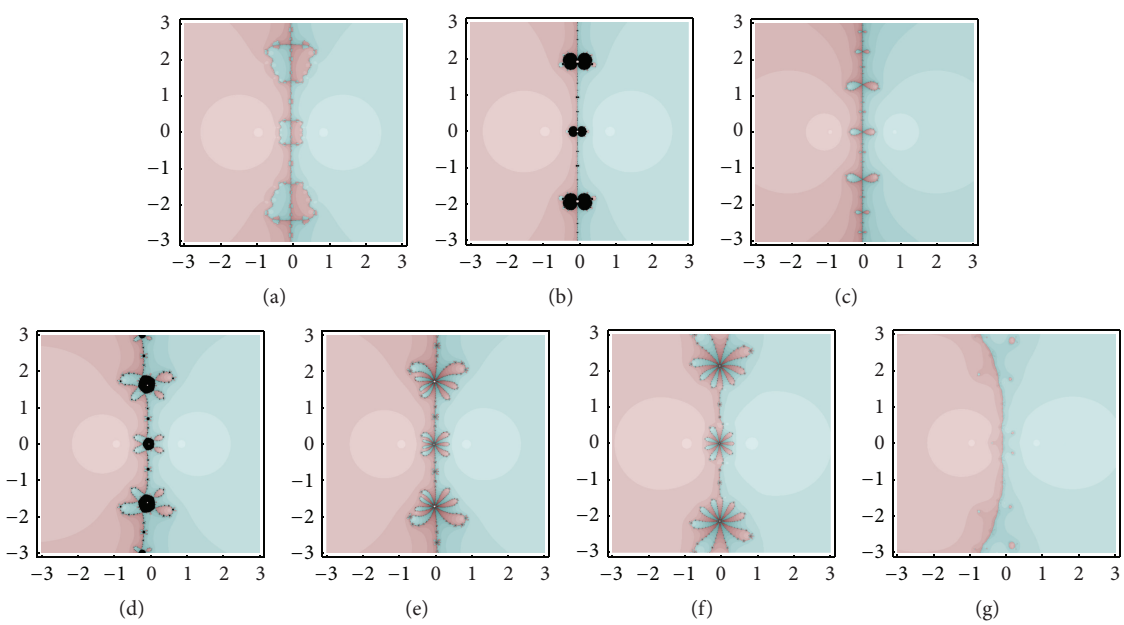

Figure 1: (a) for **Kan**, (b) for **Sol**, and (c) for **Zhou** and (d) for **1D**, (e) for **1G**, (f) for **2D**, and (g) for **2G**, for the roots of the polynomial $P_1(z) = (9z^2 + z - 7)^2$.

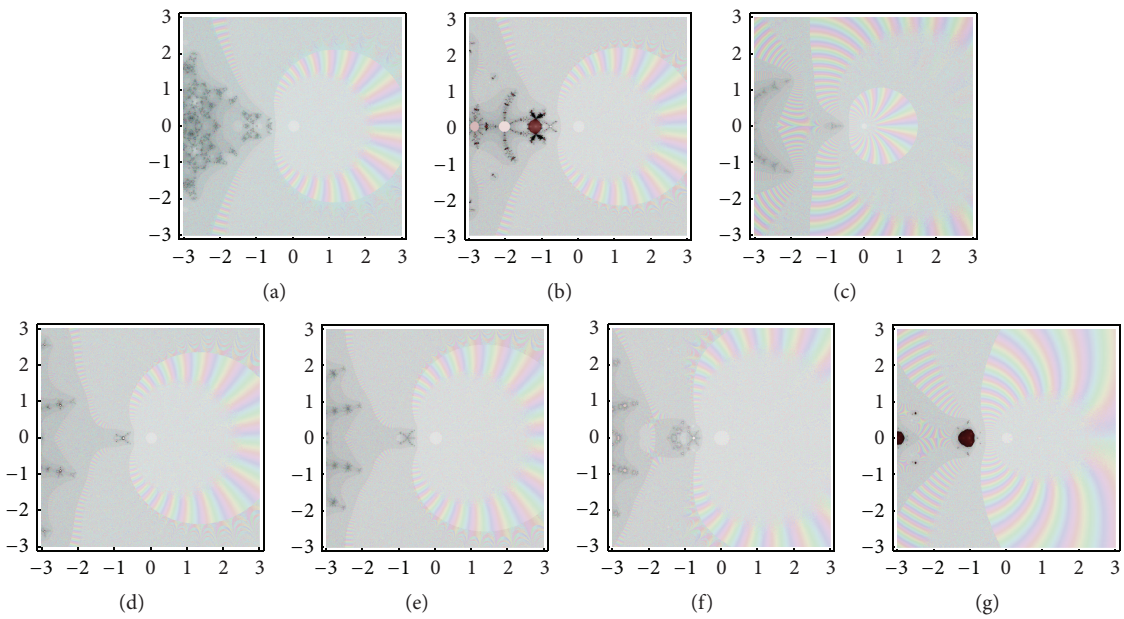

Figure 2: (a) for **Kan**, (b) for **Sol**, and (c) for **Zhou** and (d) for **1D**, (e) for **1G**, (f) for **2D**, and (g) for **2G**, for the roots of the polynomial $P_2(z) = z^3(z+1)$.

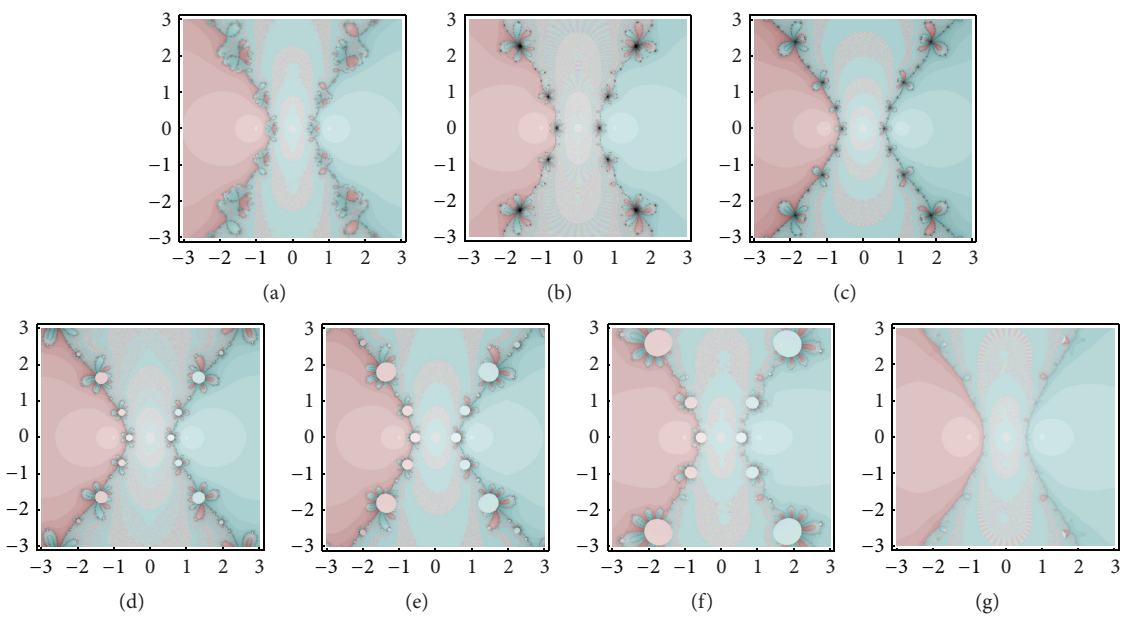

Figure 3: (a) for **Kan**, (b) for **Sol**, and (c) for **Zhou** and (d) for **1D**, (e) for **1G**, (f) for **2D**, and (g) for **2G**, for the roots of the polynomial $P_3(z) = (z^3 - z)^4$.

The results are presented in Table 6 and Figure 3. The method **2G** is best in view of lower **AVG** and **TDIV**. In the last example, we use the polynomial having two roots of unity

$$P_4(z) = \left(z^2 - 1\right)^7. \tag{36}$$

The results are given in Table 7 and Figure 4. The method **2G** is best in view of lower **AVG** and **TDIV**. As can be seen in Figure 4, **Sol** has shown many black points, while has shown **Y2G** a few black ones. The best result for **CPU** is by **Y1D** and the worst one is by **Kan**.

5. Conclusion

We have shown that optimal quartic-order multiple-root finders are constructed with an error-correcting function generated by $f(x_n)/f'(x_n)$ and a principal branch of $[f(y_n)/f(x_n)]^{1/m}$ along with the derivation of their relevant

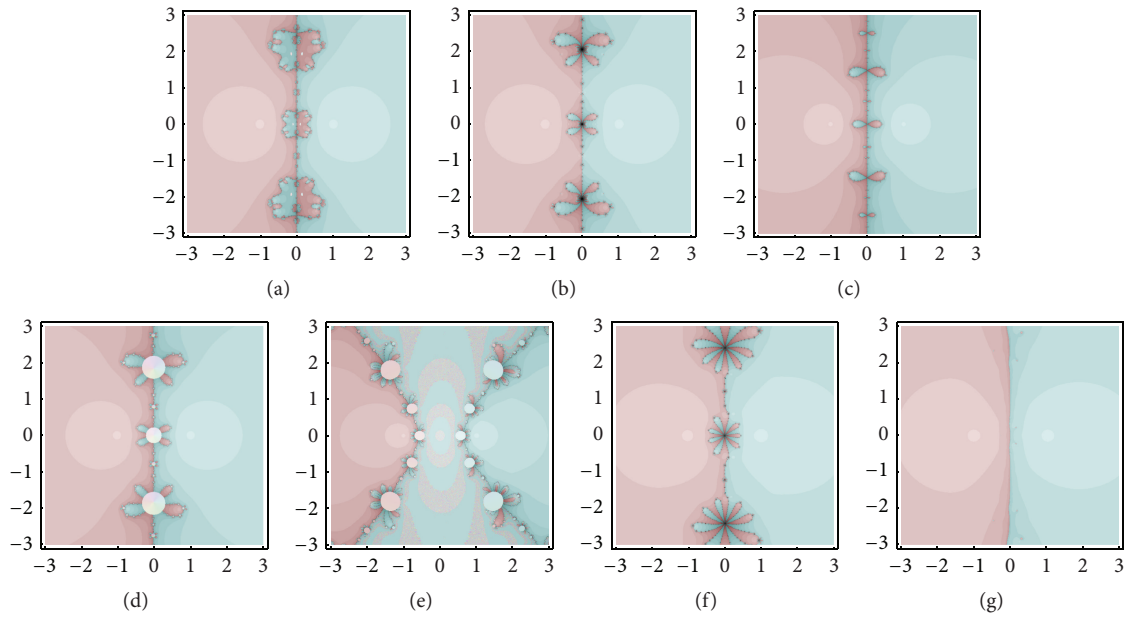

FIGURE 4: (a) for **Kan**, (b) for **Sol**, and (c) for **Zhou** and (d) for **1D**, (e) for **1G**, (f) for **2D**, and (g) for **2G**, for the roots of the polynomial $P_4(z) = (z^2 - 1)^7$.

error equation. To select initial values near zero α for ensured convergence, we should investigate the dynamics behind the basins of attraction of the corresponding iterative maps applied to various polynomials. Our future work will strengthen the current approach to pursue higher-order methods by controlling free parameters of the error-correcting functions that enhance relevant basins of attraction for a wide class of polynomials.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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