Research Article

On Constructing Two-Point Optimal Fourth-Order Multiple-Root Finders with a Generic Error Corrector and Illustrating Their Dynamics

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With an error corrector via principal branch of the $m$th root of a function-to-function ratio, we propose optimal quartic-order multiple-root finders for nonlinear equations. The relevant optimal order satisfies Kung-Traub conjecture made in 1974. Numerical experiments performed for various test equations demonstrate convergence behavior agreeing with theory and the basins of attractions for several examples are presented.

1. Introduction

Iterative root-finding methods have been constantly developed by numerous researchers [1–3] to solve more accurately the root $\alpha$ of a nonlinear equation $f(x) = 0$ that arises frequently in a scientific world. Classical Newton’s method below,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots, \quad (1)$$

is widely used for a simple root under normal circumstances, provided that an initial guess $x_0$ is chosen close enough to $\alpha$. Likewise, modified Newton’s method [4] of the form

$$x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots, \quad (2)$$

is most popular for a multiple root $\alpha$ with its integer multiplicity $m \geq 1$.

Definition 1. Let $\{x_0, x_1, x_2, \ldots, x_n, \ldots\}$ be a sequence converging to $\alpha$ and $e_n = x_n - \alpha$ be the $n$th iterate error. If there exist real numbers $p \in \mathbb{R}$ and $b \in \mathbb{R} \setminus \{0\}$ such that the following error equation holds:

$$e_{n+1} = be_n^p + O(e_n^{p+1}), \quad (3)$$

then $b$ or $|b|$ is called the asymptotic error constant and $p$ is called the order of convergence [5].

Note that both methods (1) and (2) are quadratically convergent one-point optimal [3] methods. Two-point higher-order methods for multiple roots can be found in papers [6–11]. Among these, we introduce, here, respectively, in (4), (5), and (6), some interesting works of Soleymani and Babajee [12], Kanwar et al. [13], and Zhou et al. [14] who have recently developed fourth-order multiple-root finders:
$$y_n = x_n - \frac{2m}{m + 2} \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = x_n + \frac{4md}{d(m^2 + 2m - 4) - m^2v} \left[ 1 - \frac{m^3(m - 2)}{16d^2} \left( \frac{f'(y_n)}{f'(x_n)} - \frac{m + 2}{m}d \right)^2 \right] \frac{f(x_n)}{f'(x_n)}, \quad d = \left( \frac{m}{m + 2} \right)^m, \quad (4)$$

$$y_n = x_n - \frac{2m}{m + 2} \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = x_n - \frac{mf(x_n)}{2f'(x_n)((m - 1)d^m f'(x_n) - mf'(x_n))(mf'(y_n))(mf'(y_n) - d^m(m + 8)f'(x_n))}, \quad d = \left( \frac{m}{m + 2} \right)^m, \quad (5)$$

where $p = 0, \ C_1 = md^m(6m^2 + 17m - 14), \ C_2 = d^2m(3m^3 + 19m^2 + 16m + 16),$ 

$$y_n = x_n - m \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = y_n - m G(w) \cdot \frac{f(x_n)}{f'(x_n)} = x_n - m \left[ 1 + G(w) \right] \frac{f(x_n)}{f'(x_n)}, \quad w = \sqrt[3]{\frac{f(y_n)}{f(x_n)}}, \quad (6)$$

where $G$ is sufficiently differentiable at 0 with $G(0) = 0, \ G'(0) = 1, \ G''(0) = 4.$

By a close inspection of iterative methods (1), (2), and (4)–(6), we find that an iterative method can be constructed in the following form:

$$e_{m+1} = e_n - Q_f, \quad (7)$$

where $Q_f$ is implicitly dependent upon $x_n,$ for example, with $Q_f = f(x_n)/f'(x_n)$ in (1) and $Q_f = m[1 + G(w)]/(f(x_n)/f'(x_n))$ in (6). We may regard $Q_f$ as an error-correcting function. Consequently, it would be natural to call the function $Q_f$ as the error corrector. Usually $Q_f$ takes the form of $Q_f = W_f \cdot h,$ with $W_f$ as a weighting function that is widely known among many researchers. A more generic form of the error corrector $Q_f$ will be investigated during the course of developing new quartic-order multiple-root finders.

The main aim of this paper is to design new two-step two-point optimal quartic-order multiple-root finders with multiplicity of $m \geq 1.$ The first step is to compute $y_n(x_n)$ using $f(x_n), f'(x_n)$ usually with a Newton-like method or other. The second step is to update $x_n$ in the first step by introducing an error corrector $Q_f$ formed by $f(x_n)/f'(x_n)$ and a principal branch of $[f(y_n)/f(x_n)]^{1/m}.$ We will check the optimality based on the Kung-Traub conjecture [3] that a multipoint method [15] without memory can achieve its convergence order at most of $2^{k-1}$ for $k$ functional evaluations.

This paper is comprised of four sections as follows. Following this introductory section, Section 2 describes main results with convergence analysis for newly proposed two-point optimal fourth-order multiple-root finders. Principal branch of a logarithmic function plays a crucial role in developing new methods in view of the relation $[f(y_n)/f(x_n)]^{1/m} = \exp[(1/m)\log(f(y_n)/f(x_n))].$ The convergence analysis includes the derivation of the error equation for the proposed methods. In Section 3, special cases of error-correcting functions are treated with tabulated results and labeled case numbers. Two types of error-correcting functions are constructed based on bivariate polynomials and rational functions. In the first part of Section 4, with error-correcting functions properly chosen from Section 3, a variety of numerical examples are presented for a wide selection of test functions. A comparison for the convergence behavior is made among the proposed methods and the listed existing methods (4)–(6). The second part of Section 4 discusses related dynamics of maps (8) behind the basins of attraction. Dynamical properties of the proposed methods along with their illustrative basins of attraction are displayed with detailed analyses and comments. Section 5 describes overall conclusion as well as possible future work.

2. Main Results

We first assume that a function $f : C \rightarrow C$ has a multiple root $\alpha$ with integer multiplicity $m \geq 1$ and is analytic [16] in a small neighborhood of $\alpha.$ Then with the concept of error corrector $Q_f$ introduced in (7) we propose new two-step iterative multiple-root finders below, given an initial guess $x_0$ sufficiently close to $\alpha$: for $n = 0, 1, 2, \ldots$,

$$y_n = x_n - \gamma \cdot h, \quad h = \frac{f(x_n)}{f'(x_n)}, \quad \gamma \in \mathbb{R},$$

$$x_{m+1} = x_n - Q_f(s, h), \quad s = \left[ \frac{f(y_n)}{f(x_n)} \right]^{1/m}, \quad (8)$$
where \( y \in \mathbb{N} \) is a parameter and \( Q_f : \mathbb{C}^2 \rightarrow \mathbb{C} \) is holomorphic [17] in a neighborhood of \((\lambda, 0)\), where \( \lambda \in \mathbb{R} \) is to be determined later for optimal quartic-order convergence. Since \( s \) is a one-to-\( m \) multiple-valued function, we choose \( s \) as a principal analytic branch given by \( s = \exp[(1/m)\log(f(y_n)/f(x_n))] \), with \( \log(f(y_n)/f(x_n)) = \log(f(y_n)/f(x_n)) + i \arg(f(y_n)/f(x_n)) \) for \( -\pi < \arg(f(y_n)/f(x_n)) \leq \pi \); this convention of \( \arg(z) \) for \( z \in \mathbb{C} \) agrees with that of \( \log[z] \) command of Mathematica to be adopted in numerical experiments of Section 4. By means of further inspection of \( s \), we find that \( \lambda \in \mathbb{R} \) is characterized in such a way that \( s = |f(y_n)/f(x_n)^{1/m}| \cdot \exp[(i/m)\arg(f(y_n)/f(x_n))] = \lambda + O(e_n) \), as shown by (15).

These suggested methods require one derivative and two functions in order to achieve optimal order of four. In this section, we establish a main theorem describing the convergence analysis regarding proposed methods (8) and find out how to select the parameter \( \gamma \) and the error-correcting function \( Q_f \) for optimal fourth-order convergence.

**Theorem 2.** Let \( f : \mathbb{C} \rightarrow \mathbb{C} \) have a zero \( \alpha \) with integer multiplicity \( m \geq 1 \) and be analytic in a sufficiently small neighborhood of \( \alpha \). Let \( y = m \) and \( \theta_j = f^{(m-j)}(\alpha)/f^{(m)}(\alpha) \) for \( j \in \mathbb{N} \). Let \( x_n \) be an initial guess chosen in a sufficiently small neighborhood of \( \alpha \). Let \( Q_f : \mathbb{C}^2 \rightarrow \mathbb{C} \) be holomorphic in a neighborhood of \((0, 0)\). Let \( Q_{f,i} = (1/2)(\partial f/\partial y)(\partial f/\partial y)Q_f(s, h)|_{(s = 0 \text{ or } h = 0)} \) for \( 1 \leq i \leq j \leq 4 \). Suppose that relations \( Q_{00} = Q_{02} = Q_{30} = Q_{32} = Q_{20} = Q_{30} = 0 \), \( Q_{01} = Q_{11} = m \), and \( 2Q_{21} = 2m \) hold. Then iterative methods (8) are optimal and of order four and possess the following error equation:

\[
e_{n+1} = \frac{Q_{04}}{m^4} - \frac{Q_{13}}{m^{4}(m + 1)}\theta_1
- \frac{1}{m^2(m + 1)^2} \cdot \theta_1 \theta_2
- \frac{Q_{32}}{m^4(m + 1)^2} \theta_1^2
+ \frac{m(m + 9) - 2Q_{31}}{2m^3(m + 1)^3} \theta_1^3
+ \frac{Q_{40}}{m^4(m + 1)^4} \theta_1^4
+ O(e_n^5).
\]  

(9)

**Proof.** Three functional evaluations evidently are eligible for optimal convergence order in the sense of Kung-Traub. Hence, it suffices to determine the constant parameter \( \gamma \) and some properties of the error-correcting function \( Q_f \) for fourth-order convergence. Applying the Taylor's series expansion of \( f \) about \( \alpha \), we get the following relations:

\[
f(x_n) = \frac{f^{(m)}(\alpha)}{m!} \cdot e_n^m + O(e_n^{m+1}),
\]

(10)

\[
f'(x_n) = \frac{f^{(m)}(\alpha)}{(m-1)!} \cdot e_n^{m-1} + O(e_n^m),
\]

(11)

where \( A_k = (m!/(m+k)!)\theta_k, B_k = ((m-1)!/(m+k-1)!)\theta_k \), and \( \theta_j = f^{(m+j)}(\alpha)/f^{(m)}(\alpha) \) for \( k = 1, 2, 3, 4 \). Dividing (10) by (11), we have

\[
\frac{f(x_n)}{f'(x_n)} = \frac{1}{m} \left[ e_n - K_1 e_n^2 - K_2 e_n^3 + K_3 e_n^4 + O(e_n^5) \right],
\]

(12)

where \( K_1 = -A_1 + B_1, K_2 = -A_2 + A_1 B_1 - B_1^2 + B_2, \) and \( K_3 = -K_1 A_1 B_1 - A_2 B_1^3 + B_1^4 - A_1, B_2 - 2B_1 B_2 + B_3 \).

Letting \( t = 1 - \gamma/m \) for convenience with the above relation (12), we obtain

\[
y_n = \alpha + te_n + K_1 (1 - t) e_n^2 + K_2 (1 - t) e_n^3 + K_3 (1 - t) e_n^4 + O(e_n^5).\]

(13)

Expanding \( f(y_n) \) about the multiple root \( \alpha \) leads us to relation

\[
f(y_n) = \frac{f^{(m)}(\alpha)}{m!} e_n^m \left\{ m^{-1} \left[ t + \frac{(A_1 t^2 + K_1 m (1 - t)) e_n + \frac{1}{2} t m^{-2} \left[ K_2^2 (m - 1) - m(t - 1)^2 - 2A_1 K_1 (m + 1)(1 - t)^2 + 2t (A_2 t^3 + K_2 m (1 - t)) e_n^2 + \frac{1}{6} t m^{-3} \left[ -K_3^3 (m - 2) (m - 1) (m - 1)^2 \right] + 3A_1 (m + 1)(1 - t)^2 + 2K_2 m (1 - t)) e_n^3 + K_3 m (1 - t) - 6K_1 (1 - t) \right] + t (A_2 (m + 2) t^3 + K_2 m (m - 1)(1 - t) e_n^3) \right\} + O(e_n^5).\]

(14)

Hence, we get

\[
s = \left[ \frac{f(y_n)}{f'(x_n)} \right]^{1/m} = t + \frac{(t - 1)(K_1 m + A_1 t) e_n^5}{k} - \frac{1}{m^2} (t - 1) \left[ (A_1 K_1 m (2t - 1) + A_2^2 t (1 + m + (m - 1) t)
+ 2m (K_2 m - A_2 t (t + a))) e_n^3 + \frac{1}{6m^3} (t - 1) + 3A_2 K_1 m (-1 - m + 4t + 3 (m - 1) t^2) \right].
\]
\[+ A_3^3 t \left( (t-1)^2 - 3m(t^2 -1) + 2m^2 \left(1 + t + t^2\right)\right) + 6m^2 \left(-K_5 m + A_3 t \left(1 + t + t^2\right) \right) + A_2 K_1 \left(1 - 3t^2\right) - 6A_1 m \left(K_5 m(2t - 1) + K_4^2 m(t-1) + A_2 m(t - 1) + A_2 t \left(1 + m + mt + (m-1)t^2\right)\right) \]
\[\cdot e_n^t + O\left(e_n^t\right).\] (15)

Hence, \(\lambda\) mentioned in the description of (8) is selected to be \(t\) as desired. It must be emphasized that \(s\) denotes a principal analytic branch as mentioned earlier from (8). Noting that 
\(O(s-t) = O(h) = O(e_n^t)\), Taylor expansion of \(Q_f(s, h)\) about \((t, 0)\) up to fourth-order terms in both variables yields after retaining up to fourth-order terms in \(e_n\) with \(Q_{14} = Q_{23} = Q_{24} = Q_{32} = Q_{33} = Q_{41} = Q_{42} = Q_{43} = Q_{44} = 0:\)
\[Q_f(s,h) = Q_{00} + h^2 \left[Q_{01} + Q_{11} (s-t) + 6m^2 \left(-K_5 m + A_3 t \left(1 + t + t^2\right) \right) + A_2 K_1 \left(1 - 3t^2\right) - 6A_1 m \left(K_5 m(2t - 1) + K_4^2 m(t-1) + A_2 m(t - 1) + A_2 t \left(1 + m + mt + (m-1)t^2\right)\right) \right]\]
\[\cdot e_n + O\left(e_n^t\right).\] (16)

Solving the above equation independently of \(\theta_1\) for \(Q_{02}, Q_{11}\) and \(Q_{20}\), we obtain
\[Q_{11} = \frac{m}{(t-1)^2},\] (22)
\[Q_{02} = Q_{20} = 0.\]

Substituting (18), (20), and (22) into \(\psi_3 = 0\) and simplifying, we have
\[\psi_3 = -\frac{Q_{03} m^3}{m^3} + Q_{12} (t-1)^3 \theta_1 - \frac{Q_{20}}{m^3 (m+1)^3} \theta_1^3 + \frac{Q_{22}}{m^2 (m+1)^2} \theta_1^2 + \frac{Q_{22}}{m^4 (m+1)^4} \theta_1^2 + O\left(e_n^t\right).\] (23)

To make \(\psi_3 = 0\) independently of \(\theta_1\) and \(\theta_2\), we obtain
\[t = Q_{00} = Q_{10} = Q_{20} = Q_{12} = Q_{03} = Q_{30} = 0,\]
\[Q_{01} = Q_{11} = m,\]
\[Q_{21} = 2m.\]

With the aid of symbolic computation of Mathematica [18], we substitute (18)–(24) into \(\psi_4\) to arrive at the following relation:
\[e_{n+1} = -\frac{Q_{04}}{m^4} - \frac{Q_{13} m^3}{m^3 (m+1)} \theta_1 - \frac{Q_{22}}{m^2 (m+1)^2} \theta_1^2 + \frac{Q_{22}}{m^4 (m+1)^4} \theta_1^2 + \frac{m(m+9) - 2Q_{21}}{2m^4 (m+1)^3} \theta_1^3 + \frac{Q_{40}}{m^4 (m+1)^4} \theta_1^4 + O\left(e_n^t\right).\] (25)

This completes the proof. \(\square\)

3. Special Cases of Error-Correcting Functions

Using relations (18)–(24), the bivariate Taylor polynomial of \(Q_f(s, h)\) is easily given by
\[Q_f(s, h) = \left[m \left(1 + s + 2s^2\right) + Q_{31} s^3\right] h + h^2 Q_{22} s^2 + h^3 Q_{13} s + h^4 Q_{04} + O(s^4).\] (26)

Here notations \(s = \left[f(y_n)/f(x_n)\right]^{1/m}\) and \(h = f(x_n)/f'(x_n)\) are introduced for simplicity. Special cases of \(Q_f(s, h)\) are considered here. In each case, relevant coefficients are determined based on relations (18)–(24). If \(Q_{22} = Q_{13} = Q_{04} = Q_{40} = 0\), then (26) yields
\[Q_f(s, h) = \left[m \left(1 + s + 2s^2\right) + Q_{31} s^3\right] h.\] (27)
Table 1: Typical subcases of Case 1 with selected parameters.

<table>
<thead>
<tr>
<th>SN</th>
<th>Q_{31}</th>
<th>Q_{22}</th>
<th>Q_{13}</th>
<th>Q_{04}</th>
<th>Q_{40}</th>
<th>Q_{f}(s,h)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1A</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>m(1 + s + s^2) \cdot h</td>
</tr>
<tr>
<td>1B</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>[m(1 + s + s^2) + s^3] \cdot h</td>
</tr>
<tr>
<td>1C</td>
<td>2m</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>m(1 + s + s^2) \cdot h</td>
</tr>
<tr>
<td>1D</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>[h^3 + m(1 + s^2)] \cdot h</td>
</tr>
<tr>
<td>1E</td>
<td>2m</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>[h^2 + m(1 + s^2)] \cdot h</td>
</tr>
<tr>
<td>1F</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>[m(1 + s) + (h + 2m)s^2] \cdot h</td>
</tr>
<tr>
<td>1G</td>
<td>2m</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>s^3 + m(1 + s)(1 + s^2) \cdot h</td>
</tr>
</tbody>
</table>

Remark 3. If we take the Taylor polynomial of $G(s)$ being in the form of $G(s) = (1 + s + 2s^2) + (Q_{31}/m)s^3$, then $G(0) = 1$, $G'(0) = 1$ and $G''(0) = 4$ hold. Hence, method (6) is a subcase of this case in which $Q_f(s,h) = mG(s) \cdot h$ is satisfied under the restriction of $f$ to $R$.

Although a variety of forms of error-correcting functions $Q_f(s,h)$ are available in view of (26), we will limit ourselves to considering two cases of error correctors comprising low-order bivariate polynomials or simple rational functions.

Case 1 ($Q_f$ with a bivariate polynomial). In this case, $Q_{31}, Q_{22}, Q_{13}, Q_{04}$, and $Q_{40}$ can be regarded as parameters to be chosen to satisfy

$$Q_f(s,h) = \left[m \left(1 + s + 2s^2\right) + Q_{31}s^3\right]h + h^2Q_{22}s^2 + h^4Q_{13}s + h^6Q_{04} + Q_{40}s^4.$$  \hspace{1cm} (28)

We list typical subcases with selected parameters and $Q_f(s,h)$ in Table 1, where SN stands for the corresponding subcase identification number.

Case 2 ($Q_f$ with a bivariate polynomial and rational functions). Consider

$$Q_f(s,h) = \frac{a_1 + a_2s + a_3s^2}{b_1s + b_2s^2} \cdot h + ds^4 + qh^4 + h^2q_1h + q_2s + 1 + rhs,$$  \hspace{1cm} (29)

where $a_1, a_2, a_3$ and $b_1, b_2, b_3$ are determined using (18)–(24) with two of them as free parameters. We find that $a_1 = m, a_2 = a_3 - m(1 + b_2)$, and $b_2 = -2 - b_1 + a_3/m$ and the remaining parameters $d, q, r, q_1, q_2$ are free to be chosen.

In Table 2, we list typical subcases with interesting choices of parameters and $Q_f(s,h)$.

4. Numerical Experiments and Dynamics

In this section, we will first describe the computational experiments of proposed methods (8) and then illustrate the complex dynamics [19–26] related to the basins of attraction [27–29] of iterative maps (26) along with comparisons among existing methods.

Throughout the experiments, we have moderately assigned 112 significant digits as the minimum number of precision digits, via Mathematica [18] command $\text{MinPrecision} = 112$, to achieve the desired accuracy ensuring convergence of the proposed methods. It is necessary to compute $e_n = x_n - \alpha$ with high accuracy for desired numerical results. In case that $\alpha$ is not exact, it is replaced by a more accurate value which has a larger number of significant digits than the assigned $\text{MinPrecision} = 112$.

Definition 4 (computational convergence order). Assume that theoretical asymptotic error constant $\eta = \lim_{n \to \infty} (|e_n|/|e_{n-1}|)^p$ and convergence order $p \geq 1$ are known. Define $p_n = \log|e_n|/\log|e_{n-1}|$ as the computational convergence order. Note that $\lim_{n \to \infty} p_n = p$.

If $\alpha$ and $x_n$ both mutually have the same accuracy of $\text{MinPrecision} = 112$, then $e_n = x_n - \alpha$ would be nearly zero as $n$ becomes large and thus computing $|e_{n+1}/e_n|$ would unfavorably cause a numeric overflow. Computed values of $x_n$ are accurate up to 112 significant digits. To observe the reliable convergence behavior, we desire $\alpha$ with enough accuracy of 16 digits higher than $\text{MinPrecision}$, which has 128 significant digits. To supply such $\alpha$, a set of following Mathematica commands are used:

$$\text{sol} = \text{FindRoot} \left\{ f(x), \{x, x_0\} \right\}, \text{PrecisionGoal} \to 16$$
$$+ \text{MinPrecision}, \text{WorkingPrecision} \to 2$$
$$\text{\ast MinPrecision} \};$$  \hspace{1cm} (30)

Although the number of significant digits of $x_n$ and $\alpha$ is 112 and 128, respectively, we list the two values at most up to 15 significant digits due to the limited paper space. We set the error bound $\epsilon$ to $(1/2) 	imes 10^{-80}$ satisfying $|x_n - \alpha| < \epsilon$.

Iterative methods (26) with all subcases of both Cases 1 and 2 were, respectively, identified by Y1A, Y1B, Y1C, Y1D, Y1E, Y1F, and Y1G and Y2A, Y2B, Y2C, Y2D, Y2E, Y2F, Y2G, and Y2H, being Y-prefixed. Among them, typical three methods have been successfully applied to three test functions shown below:

Method Y1D: $F_1(x) = \left(4 \sin^2 \frac{\pi}{x} + \frac{x^2}{2} - 12 \right)^4$, $m = 4, \alpha = -3$.

Method Y2B:

$$F_2(x) = \left[2x + 4i - 2i + \sin(x + 2i)\log\left(x^2 + 1\right)\right]$$
$$\cdot \sin^2(x + 2i), \quad m = 3, \alpha = -2i + \pi, \quad i = \sqrt{-1}.$$  \hspace{1cm} (31)

Method Y2E: $F_3(x) = \left[\left(\sin^2(x - 1)\right)^2 + e^x - 6\right]^2$, $m = 2, \alpha = 1.69008688293711$,

where $\log z$ ($z \in C$) represents a principal analytic branch such that $-\pi < \text{Im} (\log z) \leq \pi$. 


Table 2: Typical subcases of Case 2 with selected parameters.

<table>
<thead>
<tr>
<th>SN</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$a_3$</th>
<th>$b_2$</th>
<th>$b_3$</th>
<th>$d$</th>
<th>$q$</th>
<th>$r$</th>
<th>$q_1$</th>
<th>$q_2$</th>
<th>$Q_f(\delta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2A</td>
<td>$m$</td>
<td>$-m$</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\frac{m(1-s)}{1-2s}$ \cdot h</td>
</tr>
<tr>
<td>2B</td>
<td>$m$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\frac{m}{1-s}$ \cdot h</td>
</tr>
<tr>
<td>2C</td>
<td>$m$</td>
<td>0</td>
<td>$m$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\frac{m(1+s^2)}{1-s}$ \cdot h</td>
</tr>
<tr>
<td>2D</td>
<td>$m$</td>
<td>$-m$</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$s^4 + \frac{m(1-s)}{1-s} \cdot h$</td>
</tr>
<tr>
<td>2E</td>
<td>$m$</td>
<td>0</td>
<td>0</td>
<td>$-1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$h_i + \frac{m(1+s^2)}{1-s}$ \cdot h</td>
</tr>
<tr>
<td>2F</td>
<td>$m$</td>
<td>0</td>
<td>0</td>
<td>$m$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$s^2h_i + \frac{m}{1-s} \cdot h$</td>
</tr>
<tr>
<td>2G</td>
<td>$m$</td>
<td>$-m$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$s^2h_i + \frac{m(1-s)}{1-s} \cdot h$</td>
</tr>
<tr>
<td>2H</td>
<td>$m$</td>
<td>0</td>
<td>0</td>
<td>$-1$</td>
<td>$-1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$s^2h_i + \frac{m(1-s^2)}{1-s} \cdot h$</td>
</tr>
</tbody>
</table>

Methods Y1D, Y2B, and Y2E in Table 3 have clearly confirmed quartic-order convergence. Table 3 lists iteration indexes $n$, approximate zeros $x_n$, residual errors $|F(x_n)|$, errors $|e_n|$, and computational asymptotic error constants $\eta_n$ as well as the theoretical asymptotic error constant $\eta$ and computational asymptotic convergence order $p_n$. The values of $\eta_n$ agree up to 10 significant digits with $\eta$. Undoubtedly, the computational asymptotic order of convergence well approaches 4. The computational asymptotic error constant reveals a good agreement with the theory developed in Section 3.

Additional functions below are tested to further confirm the convergence of methods (8):

\[ f_1(x) = x^2 \left\{ x + \log \left( 1 + x^3 \right) \right\}, \quad \alpha = 0, \ m = 3, \ x_0 = 0.1, \]

\[ f_2(x) = \left[ 5 - 6x^2 + \cos(2x) - \sin(3x) \right]^2, \]

\[ \cdot \left[ 5 - 6x^2 + \cos(2x) \left( 1 - 2\sin x \right) - \sin x \right]^2, \]

\[ \alpha = 0.84777738787, \ m = 4, \ x_0 = 1.0, \]

\[ f_3(x) = \left[ 1 + \cos(x^2 - 1) - x\log(x^2 - \pi) \right]^4, \]

\[ \cdot \left( x^2 - 1 - \pi \right)^4, \quad \alpha = \frac{1}{\sqrt{1 + \pi}}, \ m = 6, \ x_0 = 2.0, \]

\[ f_4(x) = \left( x^2 + e^{-x^2} + \sin \left( \frac{1}{x} - 3 \right) \right), \]

\[ \alpha = 1.51115415815562, \ m = 7, \ x_0 = 1.0, \]

\[ f_5(x) = \left[ x - \sqrt{3x^3} \cos \left( \frac{\pi x}{6} \right) - \frac{1}{1 + x^2} - \frac{11}{5} + 4\sqrt{3} \right]^2 \]

\[ \cdot \left( x - 2 \right)^3, \quad \alpha = 2, \ m = 5, \ x_0 = 1.7, \]

\[ f_6(x) = \left[ (x - 2)^2 + \frac{3}{16} + \frac{1}{x^2} \log \left( (x - 2)^2 + \frac{19}{16} \right) \right] \]

\[ \cdot \left( x - 2 \right)^2 + \frac{3}{16}, \]

\[ \alpha = 2 - \frac{i\sqrt{3}}{4}, \ m = 2, \ x_0 = 1.93 - 0.41i, \]

\[ f_7(x) = \left[ x^2 \left( 1 - \log x \right) - \frac{1}{x} \right]^8, \]

\[ \quad \alpha = 1, \ m = 8, \ x_0 = 0.9; \]

here $\log z \ (z \in \mathbb{C})$ represents a principal analytic branch such that

\[ -\pi < \text{Im} \left( \log z \right) \leq \pi. \] (32)

In Table 8, we directly compare numerical errors $|x_n - \alpha|$ of proposed methods Y1B, Y1C, Y1E, and Y1G in Case 1 with those of existing optimal fourth-order multiple-root finders. Abbreviations Kan, Zhou, and Sol denote existing optimal fourth-order multiple root finders obtained by Kanwar et al. (5), Zhou et al. (6) with $G(w) = (1 + w + 2w^2)$, and Soleymani and Babajee (4), respectively. The least errors within the prescribed error bound are highlighted in boldface. Method Y1E shows best convergence for $f_1, f_3, f_5$, and method Y1G for $f_2, f_4$ and method Sol for $f_5, f_6$.

Likewise, in Table 9, we compare numerical errors $|x_n - \alpha|$ of proposed methods Y2A, Y2D, Y2F, and Y2G in Case 2 with those of existing optimal fourth-order multiple-root finders. Method Y2G shows best convergence for $f_1, f_3, f_5$, and $f_7$ and method Y2D for $f_4, f_6$, while method shows best convergence Y2F for $f_2$.

Recall that $\lim_{n \to \infty} ((x_{n+1} - \alpha)/(x_n - \alpha))^p = g_f(\alpha)/p!$ for an iterative method $x_{n+1} = g_f(x_n)$ converging to $\alpha$ with order $p$; note that the iteration function $g_f(x)$ is dependent upon a given nonlinear equation $f(x)$. Consequently, with a given order of convergence $p$, the local convergence behavior is dependent on the function $f(x)$, an initial value $x_0$, and a multiple zero $\alpha$ itself. Hence, for given test functions, zeros, and initial values, no method is expected to always show better performance than the others.
Table 3: Convergence for test functions $F_1(x) - F_3(x)$ with methods M1F, M2E, and M2F.

| Method | $n$ | $x_n$ | $|F(x_n)|$ | $|e_n = x_n - a|$ | $|e_{n+1}/|e_n|$ | $\eta$ | $p_n$ |
|---------|-----|-------|------------|-----------------|----------------|-------|-------|
| M1F     | 0   | -3.3  | 5.68515    | 0.3             | 3.894e-4       | 0.04807797108 | 0.06814101693 | 4.28967 |
|         | 1   | -3.00038943156575 | 1.212 × 10^{-1} | 3.984 × 10^{-4} | 0.0671440433 | 4.00005 |
|         | 2   | -3.0000000000000000 | 3.173 × 10^{-57} | 1.566 × 10^{-15} | 0.06814101693 | 4.00000 |
|         | 3   | -3.0000000000000000 | 1.495 × 10^{-239} | 4.104 × 10^{-61} | 0.06814101693 | 4.00000 |
|         | 4   | -3.0000000000000000 | 0.0 × 10^{-443} | 0.0 × 10^{-111} |               |       |

Table 4: Typical Example 1 with $P_1(z) = (9z^2 + z - 7)^4, m = 2$.  

<table>
<thead>
<tr>
<th>Methods</th>
<th>CPU</th>
<th>TCON</th>
<th>AVG</th>
<th>TDIV</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kan</td>
<td>104</td>
<td>360,000</td>
<td>5.3</td>
<td>0</td>
</tr>
<tr>
<td>Sol</td>
<td>74</td>
<td>352,462</td>
<td>5.6</td>
<td>7538</td>
</tr>
<tr>
<td>Zhou</td>
<td>54</td>
<td>360,000</td>
<td>5.9</td>
<td>0</td>
</tr>
<tr>
<td>Y1D</td>
<td>83</td>
<td>360,000</td>
<td>5.5</td>
<td>5758</td>
</tr>
<tr>
<td>Y1G</td>
<td>77</td>
<td>359,998</td>
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<td>2</td>
</tr>
<tr>
<td>Y2D</td>
<td>100</td>
<td>360,000</td>
<td>5.3</td>
<td>0</td>
</tr>
<tr>
<td>Y2G</td>
<td>90</td>
<td>360,000</td>
<td>5.0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 5: Typical Example 2 with $P_2(z) = z^2 + 1, m = 3$.  

<table>
<thead>
<tr>
<th>Methods</th>
<th>CPU</th>
<th>TCON</th>
<th>AVG</th>
<th>TDIV</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kan</td>
<td>108</td>
<td>360,000</td>
<td>5.2</td>
<td>0</td>
</tr>
<tr>
<td>Sol</td>
<td>96</td>
<td>359,194</td>
<td>5.1</td>
<td>806</td>
</tr>
<tr>
<td>Zhou</td>
<td>69</td>
<td>360,000</td>
<td>5.3</td>
<td>0</td>
</tr>
<tr>
<td>Y1D</td>
<td>333</td>
<td>360,000</td>
<td>4.8</td>
<td>0</td>
</tr>
<tr>
<td>Y1G</td>
<td>166</td>
<td>360,000</td>
<td>4.8</td>
<td>0</td>
</tr>
<tr>
<td>Y2D</td>
<td>526</td>
<td>360,000</td>
<td>4.6</td>
<td>0</td>
</tr>
<tr>
<td>Y2G</td>
<td>251</td>
<td>359,996</td>
<td>4.7</td>
<td>4</td>
</tr>
</tbody>
</table>

Table 6: Typical Example 3 with $P_3(z) = (z^2 - 1)^4, m = 4$.  

<table>
<thead>
<tr>
<th>Methods</th>
<th>CPU</th>
<th>TCON</th>
<th>AVG</th>
<th>TDIV</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kan</td>
<td>140</td>
<td>360,000</td>
<td>6.4</td>
<td>0</td>
</tr>
<tr>
<td>Sol</td>
<td>201</td>
<td>359,860</td>
<td>6.2</td>
<td>140</td>
</tr>
<tr>
<td>Zhou</td>
<td>80</td>
<td>359,988</td>
<td>7.4</td>
<td>2</td>
</tr>
<tr>
<td>Y1D</td>
<td>112</td>
<td>360,000</td>
<td>6.7</td>
<td>0</td>
</tr>
<tr>
<td>Y1G</td>
<td>138</td>
<td>360,000</td>
<td>6.4</td>
<td>0</td>
</tr>
<tr>
<td>Y2D</td>
<td>172</td>
<td>360,000</td>
<td>5.7</td>
<td>0</td>
</tr>
<tr>
<td>Y2G</td>
<td>187</td>
<td>360,000</td>
<td>5.5</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 7: Typical Example 4 with $P_4(z) = (z^2 - 1)^7, m = 7$.  

<table>
<thead>
<tr>
<th>Methods</th>
<th>CPU</th>
<th>TCON</th>
<th>AVG</th>
<th>TDIV</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kan</td>
<td>157</td>
<td>360,000</td>
<td>5.5</td>
<td>0</td>
</tr>
<tr>
<td>Sol</td>
<td>146</td>
<td>359,960</td>
<td>5.4</td>
<td>40</td>
</tr>
<tr>
<td>Zhou</td>
<td>146</td>
<td>360,000</td>
<td>5.9</td>
<td>0</td>
</tr>
<tr>
<td>Y1D</td>
<td>58</td>
<td>360,000</td>
<td>5.4</td>
<td>0</td>
</tr>
<tr>
<td>Y1G</td>
<td>61</td>
<td>359,994</td>
<td>5.7</td>
<td>0</td>
</tr>
<tr>
<td>Y2D</td>
<td>62</td>
<td>359,998</td>
<td>5.3</td>
<td>2</td>
</tr>
<tr>
<td>Y2G</td>
<td>74</td>
<td>360,000</td>
<td>4.7</td>
<td>0</td>
</tr>
</tbody>
</table>

The efficiency index [5] is defined by $EI = p^{1/d}$, where $p$ is the order of convergence and $d$ is the number of distinct functional or derivative evaluations per iteration. The proposed methods (8) as well as all other listed methods have the same $EI = 4^{1/3} = 1.587$ agreeing with Kung-Traub optimality conjecture.

At this point, we are now ready to discuss the dynamical behavior of iterative maps (8). It is important to properly select initial values close to zero $a$ for guaranteed convergence of iterative methods. It is not, however, an easy task to determine how close the initial values are to zero $a$. A convenient way of employing stable initial values is to directly use visual basins of attraction. In view of inspection of the area of convergence on the basins of attraction, the larger area of convergence would imply a better method. Clearly measuring the size of area of convergence requires a quantitative analysis. To this end, we provide Tables 5–8 featuring statistical data for the average number of iterations per point and the number of divergent points including CPU time. In the following 4 examples, we select a 6 by 6 square region centered at the origin and containing all the zeros of the given test polynomial functions. We then take a $600 \times 600$ uniform grid in the square to display initial points for the iterative methods via basins of attraction. We color...
The grid point, according to the iteration number for convergence and the root it converged to. This way we can find out if the method converged within the maximum number of iteration allowed and if it converged to the root closer to the initial grid point. To continue our discussion, let us first identify four members of iterative map (8) associated with Cases 1D, 1G, 2D, and 2G by Y1D, Y1G, Y2D, and Y2G, respectively.

For illustrating the complex dynamics of (8) with the desired basins of attraction, we take applied to various polynomials having multiple roots with multiplicity $m = 2, 3, 4, 7$. Statistical data for the basins of attraction are tabulated in Tables 4–7. In this tables, abbreviations CPU, TCON, AVG, and TDIV denote the value of CPU time for convergence, the value of total convergent points, the value of average iteration number for convergence, and the value of divergent points.

In the first example, we have taken the following polynomial:

$$P_1(z) = (9z^2 + z - 7)^2,$$  \hspace{1cm} (33)

whose roots $z = -0.939221, 0.82811$ are both real with multiplicity $m = 2$. Based on Table 4 and Figure 1, we can find out that Y2G is best in view of lower AVG and TDIV, followed by Kan and Y2D. As can be seen in Figure 1, Sol and Y1D have shown considerable amount of black points. These points causing divergence behavior were expected from the last column of Table 4. The best result for CPU is by Zhou and the worst one is by Kan.

Our next sample has triple roots. The polynomial has three identical roots at the origin:

$$P_2(z) = z^3(z + 1).$$  \hspace{1cm} (34)

The results are listed in Table 5 and Figure 2. The method Y2D performs best in view of lower AVG and TDIV. These proposed methods Y1D, Y1G, Y2D, and Y2G appear to perform better than Kan, Sol, and Zhou. As can be seen in Figure 2, Sol has shown considerable amount of black points, while Y2G has shown a few black ones. The best result for CPU is by Zhou and the worst one is by Y2D.

As a third example, we take the following polynomial whose roots are all of multiplicity four:

$$P_3(z) = (z^3 - z)^4.$$  \hspace{1cm} (35)

As can be seen in Figure 3, Sol has shown considerable amount of black points, while Zhou has shown several black ones. The best result for CPU is by Zhou and the worst one is by Kan.
Table 9: Comparison of $|x_n - \alpha|$ among existing and proposed (Case 2) multiple-root finders.

| $f(x_i; m)$ | $|x_n - \alpha|$ | Kan | Sol | Zhou | Y2A | Y2D | Y2F | Y2G |
|-------------|------------------|-----|-----|------|-----|-----|-----|-----|
| $f_1$, 0.1; 3 | $|x_1 - \alpha|$ | 1.23e-5 | 1.12e-5 | 1.54e-5 | 6.34e-6 | 5.71e-6 | 1.02e-5 | 5.38e-6 |
| $f_1$, 1.0; 3 | $|x_1 - \alpha|$ | 3.44e-21 | 2.15e-21 | 1.28e-20 | 1.19e-22 | 6.59e-23 | 1.48e-21 | 5.20e-23 |
| $f_1$, 2.0; 6 | $|x_1 - \alpha|$ | 2.08e-83 | 2.89e-84 | 6.00e-81 | 1.52e-89 | 1.16e-90 | 6.58e-85 | 4.53e-91 |
| $f_2$, 1.0; 4 | $|x_1 - \alpha|$ | 3.47e-5 | 4.61e-5 | 7.32e-5 | 2.86e-5 | 3.71e-5 | 2.14e-5 | 3.11e-5 |
| $f_2$, 1.0; 7 | $|x_1 - \alpha|$ | 3.43e-18 | 5.14e-18 | 1.44e-17 | 2.13e-20 | 2.06e-21 | 1.60e-18 | 2.07e-20 |
| $f_3$, 1.7; 5 | $|x_1 - \alpha|$ | 2.89e-3 | 2.87e-3 | 1.26e-3 | 1.00e-3 | 1.01e-3 | 1.15e-3 | 9.93e-4 |
| $f_4$, 1.93, -0.4I; 2 | $|x_1 - \alpha|$ | 1.22e-4 | 9.90e-5 | 2.51e-4 | 4.88e-5 | 1.04e-4 | 1.41e-4 | 5.34e-5 |
| $f_5$, 0.9; 8 | $|x_1 - \alpha|$ | 5.76e-5 | 5.42e-5 | 6.65e-5 | 1.50e-5 | 2.13e-5 | 4.24e-5 | 1.49e-5 |

Here, 1.23e – 5 denotes $1.23 \times 10^{-5}$.

**Figure 1:** (a) for Kan, (b) for Sol, and (c) for Zhou and (d) for 1D, (e) for 1G, (f) for 2D, and (g) for 2G, for the roots of the polynomial $P(z) = (9z^2 + z - 7)^2$. 
Figure 2: (a) for Kan, (b) for Sol, and (c) for Zhou and (d) for 1D, (e) for 1G, (f) for 2D, and (g) for 2G, for the roots of the polynomial $P_2(z) = z^3(z + 1)$.

Figure 3: (a) for Kan, (b) for Sol, and (c) for Zhou and (d) for 1D, (e) for 1G, (f) for 2D, and (g) for 2G, for the roots of the polynomial $P_3(z) = (z^3 - z)^4$.

The results are presented in Table 6 and Figure 3. The method 2G is best in view of lower AVG and TDIV. In the last example, we use the polynomial having two roots of unity

$$P_4(z) = (z^2 - 1)^7.$$  \hspace{1cm} (36)

The results are given in Table 7 and Figure 4. The method 2G is best in view of lower AVG and TDIV. As can be seen in Figure 4, Sol has shown many black points, while has shown Y2G a few black ones. The best result for CPU is by Y1D and the worst one is by Kan.

5. Conclusion

We have shown that optimal quartic-order multiple-root finders are constructed with an error-correcting function generated by $f(x_n)/f'(x_n)$ and a principal branch of $\lfloor f(y_n)/f(x_n) \rfloor^{1/m}$ along with the derivation of their relevant
error equation. To select initial values near zero $\alpha$ for ensured convergence, we should investigate the dynamics behind the basins of attraction of the corresponding iterative maps applied to various polynomials. Our future work will strengthen the current approach to pursue higher-order methods by controlling free parameters of the error-correcting functions that enhance relevant basins of attraction for a wide class of polynomials.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References


