Research Article

Positive Solutions for a Class of Fourth-Order p -Laplacian Boundary Value Problem Involving Integral Conditions

Yan Sun

Department of Mathematics, Shanghai Normal University, Shanghai 200234, China

Correspondence should be addressed to Yan Sun; ysun@shnu.edu.cn

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Under some conditions concerning the first eigenvalues corresponding to the relevant linear operator, we obtain sharp optimal criteria for the existence of positive solutions for p -Laplacian problems with integral boundary conditions. The main methods in the paper are constructing an available integral operator and combining fixed point index theory. The interesting point of the results is that the nonlinear term contains all lower-order derivatives explicitly. Finally, we give some examples to demonstrate the main results.

1. Introduction

In this paper, we discuss the existence of positive solutions for the following p -Laplacian integral boundary value problems:

$$\begin{aligned} & (\phi_p(y''(t)))'' \\ & = a(t) f(t, y(t), y'(t), y''(t), y'''(t)), \\ & \quad 0 < t < 1, \\ & y(0) = y(1) = \int_0^1 b(s) y(s) ds, \\ & y''(0) = y''(1) = \phi_q \left(\int_0^1 c(s) \phi_p(y''(s)) ds \right), \end{aligned} \quad (1)$$

where $\phi_p(t) = |t|^{p-2}t$, $p > 1$, $\phi_q = \phi_p^{-1}$, $1/p + 1/q = 1$, $a \in L^1[0, 1]$ is symmetric on the interval $[0, 1]$ (i.e., $a(1-t) = a(t)$) and $a(t) \geq 0$ for $t \in [0, 1]$, $f: [0, 1] \times [0, +\infty) \times (-\infty, +\infty) \times (-\infty, +\infty) \times (-\infty, +\infty) \rightarrow [0, +\infty)$ is continuous, and $b, c \in L^1[0, 1]$ are nonnegative symmetric on $[0, 1]$.

Boundary value problems of ordinary differential equations have become an important research field in recent years. Fourth-order p -Laplacian boundary value problems arise in applied mathematics, physics, gas diffusion through

porous media, engineering, elastic mechanics, electromagnetic waves of gravity driven flows, and the various areas of adiabatic tubular reactor processes, as well as biological problems; see [1–5] and the references therein.

In [6], by using the fixed point theorem for strict set contraction operator, Zhang et al. considered the existence of positive solutions for the following n th-order impulsive boundary value problems with integral boundary conditions in Banach spaces:

$$\begin{aligned} & z^{(n)}(t) + g(t, z(t), z'(t), \dots, z^{(n-2)}(t)) = \theta, \\ & \quad t \in J, \quad t \neq t_k, \quad k = 1, \dots, m, \\ & \Delta z^{(n-1)} \Big|_{t=t_k} = -I_k(z^{(n-2)}(t_k)), \quad k = 1, \dots, m, \\ & z^{(i)}(0) = \theta, \quad i = 0, 1, \dots, n-3, \\ & z^{(n-2)}(0) = z^{(n-2)}(1) = \int_0^1 h(t) z^{(n-2)}(t) dt, \end{aligned} \quad (2)$$

where $J = [0, 1]$, $0 < t_1 < t_2 < \dots < t_k < \dots < t_m < 1$, $g \in C[J \times P^{n-1}, P]$, $I_k \in C[P, P]$, $k = 1, \dots, m$, $h \in L^1[0, 1]$, and $\Delta z^{(n-1)}|_{t=t_k} = z^{(n-1)}(t_k^+) - z^{(n-1)}(t_k^-)$ denotes the jump of $z^{(n-1)}(t)$ at $t = t_k$. They also gave the upper and lower bounds for these positive solutions.

In [7], by means of the fixed point index theory, Sun and Zhang studied the following singular nonlinear Sturm-Liouville problems:

$$\begin{aligned}
 -(L\phi)(x) &= a(x)g(\phi(x)), \quad 0 < x < 1, \\
 R_1(\phi) &= \alpha_1\phi(0) + \beta_1\phi'(0) = 0, \\
 R_2(\phi) &= \alpha_2\phi(1) + \beta_2\phi'(1) = 0, \\
 -(L\phi)(x) &= f(x, \phi(x)), \quad 0 < x < 1, \\
 R_1(\phi) &= \alpha_1\phi(0) + \beta_1p(x)\phi'(0) = 0, \\
 R_2(\phi) &= \alpha_2\phi(1) + \beta_2p(x)\phi'(1) = 0,
 \end{aligned} \tag{3}$$

where $(L\phi)(x) = (p(x)\phi'(x))' + q(x)\phi(x)$ and $a(x)$ is allowed to be singular at $x = 0$ and $x = 1$.

In [2], by making use of the theory of the spectrum, Sergejeva studied the regions of solvability for three-point boundary value problem:

$$\begin{aligned}
 -y''(t) &= \mu y^+ - \lambda y^- + h(t, y, y''), \quad t \in (0, 1), \\
 y(0) &= 0, \\
 y(1) &= cy\left(\frac{1}{2}\right),
 \end{aligned} \tag{4}$$

where $y^+ = \max\{y, 0\}$, $y^- = \max\{-y, 0\}$, and h is a bounded function. The solvability results are established for the problem with $h \equiv 0$.

Motivated and inspired greatly by the above mentioned works, the aim of the paper is to improve, generalize, and supplement the previous results. We obtain the existence results on nonlinear fourth-order integral boundary value problems with one-dimensional p -Laplacian integral boundary value problem (1). The novelty of our results is in exploring some optimal criteria for the existence of positive solutions of problem (1). The methods used in our work will depend on an application of fixed point index theory together with the first eigenvalues corresponding to the relevant linear operator. It can also be seen that the nonlinear term involves all lower-order derivatives explicitly.

The paper is organized as follows. In Section 2, we state some lemmas and several preliminary results. The main results are formulated and proved in Section 3. Examples are given in Section 4.

2. Preliminaries

Throughout the paper, we make the following assumptions:

(H₁) $f \in C([0, 1] \times [0, +\infty) \times (-\infty, +\infty) \times (-\infty, +\infty) \times (-\infty, +\infty), [0, +\infty))$ and $h(t, y) \leq f(t, y, y', y'', y''') \leq g(t, y)$, $h, g : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$, are continuous and also $h(\cdot, y)$ and $g(\cdot, y)$ are symmetric on $[0, 1]$ for all $y \geq 0$ (i.e., $h(1-t, y) = h(t, y)$, $g(1-t, y) = g(t, y)$).

(H₂) $a, b, c \in L^1[0, 1]$ are all nonnegative and symmetric on $[0, 1]$, and $a(t) \not\equiv 0$ is continuous on $(0, 1)$, and $a(t)$ may

be singular at $t = 0$ and/or $t = 1$, and $\mu \in (0, 1)$ and $\nu \in (0, 1)$, where

$$\begin{aligned}
 \mu &= \int_0^1 b(s) ds, \\
 \nu &= \int_0^1 c(s) ds.
 \end{aligned} \tag{5}$$

In what follows, we will consider the Banach space $E = C[0, 1]$, $C^+[0, 1] = \{x \in C[0, 1] \mid x(t) \geq 0\}$, equipped with the maximum norm $\|z\| = \max_{0 \leq t \leq 1} |z(t)|$. Denote cone P by

$$P = \{z \in C^+[0, 1] \mid z(t) \text{ is concave for } t \in [0, 1]\}; \tag{6}$$

then, P is a positive cone in $C[0, 1]$.

Let $B_r = \{z \in C^+[0, 1] \mid \|z\| < r\}$ for $r (r > 0)$, and

$$\begin{aligned}
 H(t, s) &= G(t, s) + \frac{1}{1-\mu} \int_0^1 G(s, \tau) b(\tau) d\tau, \\
 H_1(t, s) &= G(t, s) + \frac{1}{1-\nu} \int_0^1 G(s, v) c(v) dv,
 \end{aligned} \tag{7}$$

$$G(t, s) = \begin{cases} s(1-t), & 0 \leq s \leq t \leq 1, \\ t(1-s), & 0 \leq t \leq s \leq 1. \end{cases}$$

Here $G(t, s)$, $H(t, s)$, and $H_1(t, s)$ are Green's functions of problem (1).

Lemma 1 (see [1]). *Suppose that (H₁) and (H₂) hold; then, for $t, s \in [0, 1]$, we have*

- (I) $G(t, s) \geq 0$, $H(t, s) \geq 0$, $H_1(t, s) \geq 0$;
- (II) $e(t)e(s) \leq G(t, s) \leq e(t) \leq 1/4$, where $e(t) = t(1-t)$;
- (III) $G(1-t, 1-s) = G(t, s)$, $H(1-t, 1-s) = H(t, s)$, and $H_1(1-t, 1-s) = H_1(t, s)$;
- (IV) $\rho e(s) \leq H(t, s) \leq \gamma s(1-s) = \gamma e(s) \leq (1/4)\gamma$; $\rho_1 e(s) \leq H_1(t, s) \leq \gamma_1 s(1-s) = \gamma_1 e(s) \leq (1/4)\gamma_1$, where $\gamma = 1/(1-\mu)$, $\rho = \int_0^1 e(\tau)b(\tau)d\tau/(1-\mu)$, $\gamma_1 = 1/(1-\nu)$, $\rho_1 = \int_0^1 e(\tau)c(\tau)d\tau/(1-\nu)$.

Define an integral operator $A : C^+[0, 1] \rightarrow C^+[0, 1]$ by

$$\begin{aligned}
 (Ay)(t) &= \int_0^1 H(t, s) \phi_q \left[\int_0^1 H_1(s, \tau) a(\tau) \right. \\
 &\quad \left. \cdot f(\tau, y(\tau), y'(\tau), y''(\tau), y'''(\tau)) d\tau \right] ds.
 \end{aligned} \tag{8}$$

It is well known that $y \in C^+[0, 1]$ is a solution of problem (1) if and only if y is a fixed point of operator equation $y = Ay$.

Lemma 2. *Suppose that (H₁) and (H₂) hold, if $y \in E$ is a solution of the following integral equation:*

$$\begin{aligned}
 y(t) &= \int_0^1 H(t, s) \phi_q \left[\int_0^1 H_1(s, \tau) a(\tau) \right. \\
 &\quad \left. \cdot f(\tau, y(\tau), y'(\tau), y''(\tau), y'''(\tau)) d\tau \right] ds;
 \end{aligned} \tag{9}$$

then, y is a solution of problem (1).

Lemma 3. Suppose that (H_1) and (H_2) hold; then, $A : P \rightarrow P$ is a completely continuous operator.

Proof. First, we prove that $A : P \rightarrow P$. For every $y \in P$, we have

$$\begin{aligned} (Ay)''(t) &= -\phi_q \left[\int_0^1 H_1(t, s) a(s) \right. \\ &\quad \cdot f(s, y(s), y'(s), y''(s), y'''(s)) ds \Big] \leq 0, \\ (Ay)(0) &= \int_0^1 H(0, s) \phi_q \left[\int_0^1 H_1(s, \tau) a(\tau) \right. \\ &\quad \cdot f(\tau, y(\tau), y'(\tau), y''(\tau), y'''(\tau)) d\tau \Big] ds \\ &\geq 0, \\ (Ay)(1) &= \int_0^1 H(1, s) \phi_q \left[\int_0^1 H_1(s, \tau) a(\tau) \right. \\ &\quad \cdot f(\tau, y(\tau), y'(\tau), y''(\tau), y'''(\tau)) d\tau \Big] ds \\ &\geq 0. \end{aligned} \quad (10)$$

Thus, $(Ay)(t) \geq 0$, for all $t \in [0, 1]$. Hence, $(Ay)(t)$ is nonnegative concave functional on $[0, 1]$.

By Lemma 1, for $t \in [0, 1]$, we obtain

$$\begin{aligned} (Ay)(t) &= \int_0^1 H(t, s) \phi_q \left[\int_0^1 H_1(s, \tau) a(\tau) \right. \\ &\quad \cdot f(\tau, y(\tau), y'(\tau), y''(\tau), y'''(\tau)) d\tau \Big] ds \\ &\leq \int_0^1 H(t, s) \phi_q \left[\int_0^1 H_1(s, \tau) a(\tau) \right. \\ &\quad \cdot g(\tau, y(\tau)) d\tau \Big] ds \leq \gamma \gamma_1^{q-1} \int_0^1 e(s) \\ &\quad \cdot \phi_q \left[\int_0^1 e(\tau) a(\tau) g(\tau, y(\tau)) d\tau \right] ds < +\infty. \end{aligned} \quad (11)$$

Hence, $Ay \in P$, and $A(P) \subset P$, which implies $A : P \rightarrow P$.

Next, we show that $A : P \rightarrow P$ is completely continuous.

For any natural number $n \geq 2$, set

$$a_n(t) = \begin{cases} \inf_{t \leq s \leq 1/n} a(s), & 0 \leq t \leq \frac{1}{n}, \\ a(t), & \frac{1}{n} \leq t \leq \frac{n-1}{n}, \\ \inf_{(n-1)/n \leq s \leq t} a(s), & \frac{n-1}{n} \leq t \leq 1. \end{cases} \quad (12)$$

Then, $a_n : [0, 1] \rightarrow [0, +\infty)$ is continuous, and $a_n(t) \leq a(t)$, $t \in [0, 1]$. Let

$$\begin{aligned} (A_n y)(t) &= \int_0^1 H(t, s) \phi_q \left[\int_0^1 H_1(s, \tau) a_n(\tau) \right. \\ &\quad \cdot f(\tau, y(\tau), y'(\tau), y''(\tau), y'''(\tau)) d\tau \Big] ds. \end{aligned} \quad (13)$$

Obviously, $A_n : P \rightarrow P$ is a completely continuous operator. From Lemma 1, we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \|A_n y - Ay\| &\leq \lim_{n \rightarrow \infty} \max_{0 \leq t \leq 1} \int_0^1 H(t, s) \\ &\quad \cdot \phi_q \left[\int_0^1 H_1(s, \tau) (a(\tau) - a_n(\tau)) g(\tau, y(\tau)) d\tau \right] ds \\ &\leq \gamma \gamma_1^{q-1} \lim_{n \rightarrow \infty} \int_0^1 e(s) \\ &\quad \cdot \phi_q \left[\int_0^1 e(\tau) a(\tau) g(\tau, y(\tau)) d\tau \right] ds = \gamma \gamma_1^{q-1} \\ &\quad \cdot \lim_{n \rightarrow \infty} \int_{e(n)} e(s) \phi_q \left[\int_0^1 e(\tau) a(\tau) g(\tau, y(\tau)) d\tau \right] ds \\ &= 0, \end{aligned} \quad (14)$$

where $e(n) = [0, 1/n] \cup [(n-1)/n, 1]$. Therefore, $A : P \rightarrow P$ is a completely continuous operator. The proof is completed. $\square$

Now we define linear integral operator as follows:

$$\begin{aligned} (Ty)(t) &= \int_0^1 H(t, s) \phi_q \left[\int_0^1 H_1(s, \tau) a(\tau) y(\tau) d\tau \right] ds, \\ L &= \max_{t \in [0, 1]} \int_0^1 H(t, s) \phi_q \left[\int_0^1 H_1(s, \tau) a(\tau) d\tau \right] ds. \end{aligned} \quad (15)$$

It is easy to see that $T : C^+[0, 1] \rightarrow C^+[0, 1]$ is a completely continuous linear operator, and $T(P) \subset P$.

Lemma 4. Assume that (H_1) and (H_2) hold. Then, the operator $T : P \rightarrow P$ defined by (15) is a completely continuous linear operator and $T(P) \subset P$, the spectral radius $r(T) \neq 0$, and T has a positive eigenfunction corresponding to its first eigenvalue $\lambda_1 = r^{-1}(T)$.

Proof. A simple modification of the argument in Lemma 3 yields that $T : P \rightarrow P$ is a completely continuous linear operator and $T(P) \subset P$. By (H_1) and (H_2) , there is $s_0 \in (0, 1)$ such that $H_1(s_0, s_0)a(s_0) > 0$. Choose constants u and v such that $s_0 \in (u, v) \subset [u, v] \subset (0, 1)$ and $H_1(s, s)a(s) > 0$, for all $t, s \in [u, v]$. Choose a nonnegative continuous functional

$g \in C[0, 1]$ such that $g(t) > 0$, for all $t \in (u, v)$. Then, for any $t \in (u, v)$, we have

$$\begin{aligned} (Tg)(t) &= \int_0^1 H(t, s) \phi_q \left[\int_0^1 H_1(s, \tau) a(\tau) g(\tau) d\tau \right] ds \\ &\geq \int_u^v H(t, s) \phi_q \left[\int_u^v H_1(s, \tau) a(\tau) g(\tau) d\tau \right] ds \\ &\geq 0. \end{aligned} \quad (16)$$

Thus, there exists a constant $w > 0$ such that $w(Tg)(t) \geq g(t)$, for all $t \in [0, 1]$. By famous Krein-Rutman theorems, we know that the spectral radius $r(T) \neq 0$ and T has a positive eigenfunction ψ corresponding to its first eigenvalue $\lambda_1 = r^{-1}(T)$; that is, $\psi = \lambda_1 T\psi$. This completes the proof. $\square$

Lemma 5 (see [8]). *Let P be a cone in Banach space E , and let $\Omega(P)$ be a bounded open set in P . Suppose that $A : \overline{\Omega}(P) \rightarrow P$ is completely continuous. If there exists $y_0 \in P \setminus \{\theta\}$, such that $y - Ay \neq \mu y_0$, for all $y \in \partial\Omega(P)$, $\mu \geq 0$. Then, the fixed point index $i(A, \Omega(P), P) = 0$.*

Lemma 6 (see [8]). *Let P be a cone in Banach space E , and let $\Omega(P)$ be a bounded open set in P with $\theta \in \Omega(P)$. Suppose that $A : \overline{\Omega}(P) \rightarrow P$ is a completely continuous operator. If $Ay \neq \mu y$, for all $y \in \partial\Omega(P)$, $\mu \geq 1$. Then, the fixed point index $i(A, \Omega(P), P) = 1$.*

Lemma 7 (see [9]). *Let P and P_1 be cones in Banach space E , and let $\Omega(P)$ be a bounded open set in P . Suppose that $A : \overline{\Omega}(P) \rightarrow P$ is a completely continuous operator, and A has no fixed point on $\partial\Omega(P)$. If there exist linear operators $L_1, L_2, L_3 : E \rightarrow E$ which satisfy $L_1(P) \subset P$, $L_2(P) \subset P_1$, $L_3(P_1) \subset P_1$, and $y_0 \in P \setminus \{\theta\}$, such that*

- (1) $L_2 L_1^n y_0 \geq L_2 y_0$, for certain natural number n ,
- (2) $L_2 L_1 y = L_3 L_2 y$, for all $y \in P$,
- (3) $L_2 A y \geq L_2 L_1 y$, for all $y \in \partial\Omega(P)$,

then the fixed point index $i(A, \Omega(P), P) = 0$.

3. Main Results

In this section, we establish sharp optimal criteria for the existence of positive solutions to problem (1) under superlinear cases and sublinear cases, respectively.

Theorem 8. *Suppose that (H_1) and (H_2) hold. In addition, assume that*

$$\liminf_{y \rightarrow +\infty} \frac{h(t, y(t))}{y} > \lambda_1, \quad \text{uniformly on } t \in [0, 1], \quad (17)$$

$$\limsup_{y \rightarrow 0+} \frac{g(t, y(t))}{y} < \lambda_1, \quad \text{uniformly on } t \in [0, 1], \quad (18)$$

where λ_1 is the first eigenvalue of T defined by (15); then, problem (1) has at least one positive solution.

To prove our main result, we need some preliminary results.

Lemma 9. *Suppose that (H_1) and (H_2) hold. Define*

$$P_1 = \left\{ y \in P \mid \int_0^1 \tilde{y}^*(t) y(t) dt \geq \lambda_1^{-1} \delta \|y\| \right\}, \quad (19)$$

where $\tilde{y}^*(t) = y^*(t)a(t)$, $y^* \in P$, is the positive eigenfunction corresponding to its first eigenvalue $\lambda_1 = r^{-1}(T)$; that is, $y^* = \lambda_1 T y^*$, and

$$\delta \leq \frac{\int_0^1 y^*(t) a(t) y(t) dt}{\int_0^1 H(t, s) \phi_q \left[\int_0^1 H_1(s, \tau) a(\tau) y(\tau) d\tau \right] ds}; \quad (20)$$

then, P_1 is a cone in $C^+[0, 1]$, and $T(P) \subset P_1$.

Proof. It is obvious that P_1 is a cone in $C^+[0, 1]$. For any $y \in P$, we obtain

$$\begin{aligned} \int_0^1 \tilde{y}^*(t) (Ty)(t) dt &= \int_0^1 \tilde{y}^*(t) dt \\ &\cdot \int_0^1 H(t, s) \phi_q \left[\int_0^1 H_1(s, \tau) a(\tau) y(\tau) d\tau \right] ds \\ &= \lambda_1^{-1} \int_0^1 \tilde{y}^*(t) y(t) dt \\ &= \lambda_1^{-1} \int_0^1 y^*(t) a(t) y(t) dt \geq \lambda_1^{-1} \delta \|Ty\|. \end{aligned} \quad (21)$$

Thus, $T(P) \subset P_1$. The proof is completed. $\square$

Proof of Theorem 8. It follows from (17) that there exists $\varepsilon > 0$, such that $h(t, y(t)) \geq (\lambda_1 + \varepsilon)y$, for y is sufficiently large. In view of (H_1) , we see that there exists $b \geq 0$ such that

$$h(t, y(t)) \geq (\lambda_1 + \varepsilon)y - b, \quad 0 \leq y < +\infty. \quad (22)$$

Let $R > Lb\lambda_1^2(\varepsilon\delta)^{-1} \int_0^1 \tilde{y}^*(t) dt$, where $\tilde{y}^*$ and δ are defined by Lemma 9.

In the following, we prove that

$$y - Ay \neq \mu y^*, \quad \forall y \in \partial B_R \cap P, \mu \geq 0, \quad (23)$$

where $y^* \in P$ is a positive eigenfunction corresponding to its first eigenvalue $\lambda_1 = r^{-1}(T)$.

Otherwise, there exist $y_1 \in \partial B_R \cap P$ and $\mu_0 \geq 0$, such that

$$y_1 - Ay_1 = \mu_0 y^*. \quad (24)$$

Since $A(P) \subset P_1$ and $T(P) \subset P_1$, we know from (24) that $y_1 \in P_1$. Therefore, it follows from (19) and (22) that

$$\begin{aligned}
& \int_0^1 \tilde{y}^*(t) (Ay_1)(t) dt - \int_0^1 \tilde{y}^*(t) y_1(t) dt \geq \int_0^1 dt \\
& \cdot \int_0^1 \tilde{y}^*(t) H(t, s) \\
& \cdot \phi_q \left[\int_0^1 H_1(s, \tau) a(\tau) h(\tau, y_1(\tau)) d\tau \right] ds \\
& - \int_0^1 \tilde{y}^*(t) y_1(t) dt \geq (\lambda_1 + \varepsilon) \int_0^1 dt \int_0^1 \tilde{y}^*(t) \\
& \cdot H(t, s) \phi_q \left[\int_0^1 H_1(s, \tau) a(\tau) y_1(\tau) d\tau \right] ds \\
& - b \int_0^1 dt \int_0^1 \tilde{y}^*(t) H(t, s) \\
& \cdot \phi_q \left[\int_0^1 H_1(s, \tau) a(\tau) d\tau \right] ds - \int_0^1 \tilde{y}^*(t) \\
& \cdot y_1(t) dt = (\lambda_1 + \varepsilon) \lambda_1^{-1} \int_0^1 \tilde{y}^*(t) y_1(t) dt \\
& - bL \int_0^1 \tilde{y}^*(t) dt - \int_0^1 \tilde{y}^*(t) y_1(t) dt \\
& = \varepsilon \lambda_1^{-1} \int_0^1 \tilde{y}^*(t) y_1(t) dt - bL \int_0^1 \tilde{y}^*(t) dt \\
& \geq \varepsilon \lambda_1^{-1} (\lambda_1^{-1} \delta \|y_1\|) - bL \int_0^1 \tilde{y}^*(t) dt = \varepsilon \lambda_1^{-2} \delta R \\
& - bL \int_0^1 \tilde{y}^*(t) dt > 0.
\end{aligned} \tag{25}$$

On the other hand, it follows from (24) that

$$\begin{aligned}
& \int_0^1 \tilde{y}^*(t) y_1(t) dt - \int_0^1 \tilde{y}^*(t) (Ay_1)(t) dt \\
& = \mu_0 \int_0^1 \tilde{y}^*(t) y^*(t) dt \geq 0,
\end{aligned} \tag{26}$$

which is a contradiction; thus, by Lemma 5, we see

$$i(A, B_R \cap P, P) = 0. \tag{27}$$

It follows from (18) that there exists $0 \leq r \leq R$, such that

$$g(t, y(t)) < \lambda_1 y, \quad \forall 0 \leq y \leq r. \tag{28}$$

Suppose that there exist $y_2 \in \partial B_r \cap P$, $\mu_1 \geq 1$, such that $Ay_2 = \mu_1 y_2$. Without loss of generality, we may suppose that $\mu_1 > 1$ (otherwise, the proof is completed). Then,

$$\begin{aligned}
\mu_1 y_2(t) & \leq \int_0^1 H(t, s) \\
& \cdot \phi_q \left[\int_0^1 H_1(s, \tau) a(\tau) g(\tau, y_2(\tau)) d\tau \right] ds \\
& < \lambda_1 \int_0^1 H(t, s) \\
& \cdot \phi_q \left[\int_0^1 H_1(s, \tau) a(\tau) y_2(\tau) d\tau \right] ds.
\end{aligned} \tag{29}$$

Multiplying $\tilde{y}^*$ and then integrating by (29), we have

$$\begin{aligned}
\mu_1 \int_0^1 \tilde{y}^*(t) y_2(t) dt & < \lambda_1 \int_0^1 dt \int_0^1 \tilde{y}^*(t) H(t, s) \\
& \cdot \phi_q \left[\int_0^1 H_1(s, \tau) a(\tau) y_2(\tau) d\tau \right] ds = \lambda_1 \int_0^1 dt \\
& \cdot \int_0^1 H(t, s) \tilde{y}^*(t) \\
& \cdot \phi_q \left[\int_0^1 H_1(s, \tau) a(\tau) y_2(\tau) d\tau \right] ds = \int_0^1 \tilde{y}^*(t) \\
& \cdot y_2(t) dt.
\end{aligned} \tag{30}$$

By maximum principle, $y^*(t) > 0$ and $y_2(t) > 0$, $t \in [0, 1]$; then $\int_0^1 \tilde{y}^*(t) y_2(t) dt > 0$. Hence, (30) implies $\mu_1 < 1$ which is a contradiction. So by Lemma 6, we have

$$i(A, B_r \cap P, P) = 1. \tag{31}$$

It follows from (27) and (31) that

$$\begin{aligned}
& i(A, (B_R \cap P) \setminus (\overline{B_r} \cap P), P) \\
& = i(A, B_R \cap P, P) - i(A, B_r \cap P, P) = 0 - 1 = -1.
\end{aligned} \tag{32}$$

Then, A has at least one fixed point on $(B_R \cap P) \setminus (\overline{B_r} \cap P)$. This means problem (1) has at least one positive solution. The proof is completed. $\square$

Corollary 10. Suppose that (H_1) and (H_2) hold. In addition, assume that

$$\begin{aligned}
0 & \leq \limsup_{y \rightarrow 0^+} \frac{g(t, y(t))}{y} = g^0 < h_\infty \\
& = \liminf_{y \rightarrow +\infty} \frac{h(t, y(t))}{y} \leq +\infty.
\end{aligned} \tag{33}$$

Then, for any

$$\xi \in \left(\frac{\lambda_1}{h_\infty}, \frac{\lambda_1}{g^0} \right), \quad (34)$$

where λ_1 is the first eigenvalue of T defined by (15), the following p -Laplacian integral boundary value problem

$$\begin{aligned} & (\phi_p(y''(t)))'' \\ &= \xi a(t) f(t, y(t), y'(t), y''(t), y'''(t)), \\ & 0 < t < 1, \end{aligned} \quad (35)$$

$$y(0) = y(1) = \int_0^1 b(s) y(s) ds,$$

$$\phi_p(y''(0)) = \phi_p(y''(1)) = \int_0^1 c(s) \phi_p(y''(s)) ds$$

has at least one positive solution.

Proof. By (34), we obtain

$$\begin{aligned} \liminf_{y \rightarrow +\infty} \frac{\xi h(t, y(t))}{y} &> \lambda_1, \\ \limsup_{y \rightarrow 0^+} \frac{\xi g(t, y(t))}{y} &< \lambda_1. \end{aligned} \quad (36)$$

Therefore, it follows from Theorem 8 that Corollary 10 holds. $\square$

Theorem 11. Suppose that (H_1) and (H_2) hold. In addition, assume that

$$\liminf_{y \rightarrow 0^+} \frac{h(t, y(t))}{y} > \lambda_1, \quad \text{uniformly on } t \in [0, 1]; \quad (37)$$

$$\limsup_{y \rightarrow +\infty} \frac{g(t, y(t))}{y} < \lambda_1, \quad \text{uniformly on } t \in [0, 1], \quad (38)$$

where λ_1 is the first eigenvalue of T defined by (15); then, problem (1) has at least one positive solution.

Proof. It follows from (37) that there exists $r_1 > 0$ such that

$$g(t, y(t)) \geq \lambda_1 y, \quad \forall 0 \leq y \leq r_1. \quad (39)$$

Let $y^* \in P$ be the positive eigenfunction corresponding to its first eigenvalue $\lambda_1 = r^{-1}(T)$. Then, $y^* = \lambda_1 T y^*$; that is,

$$\begin{aligned} y^*(t) &= \lambda_1 (T y^*)(t) \\ &= \lambda_1 \int_0^1 H(t, s) \phi_q \left[\int_0^1 H_1(s, \tau) a(\tau) y^*(\tau) d\tau \right] ds, \\ (y^*(t))'' &= \lambda_1 ((T y^*)(t))'' \\ &= -\lambda_1 \phi_q \left[\int_0^1 H_1(t, s) a(s) y(s) ds \right] \leq 0, \\ & t \in [0, 1], \end{aligned} \quad (40)$$

$$\begin{aligned} y^*(0) &= \lambda_1 \int_0^1 H(0, s) \phi_q \left[\int_0^1 H_1(s, \tau) a(\tau) y^*(\tau) d\tau \right] ds \\ &\geq 0, \\ y^*(1) &= \lambda_1 \int_0^1 H(1, s) \phi_q \left[\int_0^1 H_1(s, \tau) a(\tau) y^*(\tau) d\tau \right] ds \\ &\geq 0. \end{aligned}$$

Hence, $y^*(t) \geq 0$, for all $t \in [0, 1]$. Then, $y^*(t)$ is nonnegative concave on $[0, 1]$. Thus, $y^* \in P \setminus \{\theta\}$.

Let $(T_2 y)(t) = \lambda_1 (T y)(t)$, $y \in C^+[0, 1]$. Then, $T_2 : C^+[0, 1] \rightarrow C^+[0, 1]$ is a completely continuous linear operator, and $T_2(P_1) \subset P$, $T_2 y^* = \lambda_1 T y^* = y^*$.

For any $y \in \partial B_{r_1} \cap P$, it follows from (39) that

$$\begin{aligned} (A y)(t) &\geq \int_0^1 H(t, s) \phi_q \left[\int_0^1 H_1(s, \tau) a(\tau) h(\tau, y(\tau)) d\tau \right] ds \\ &\geq \lambda_1 \int_0^1 H(t, s) \phi_q \left[\int_0^1 H_1(s, \tau) a(\tau) y(\tau) d\tau \right] ds \\ &= (T_2 y)(t), \quad t \in [0, 1]. \end{aligned} \quad (41)$$

Setting $\Omega(P) = B_{r_1} \cap P$, $L_1 = L_3 = T_2$, $L_2 = I$ denotes the identical operator, and $n = 1$, in Lemma 7. We have

$$i(A, B_{r_1} \cap P, P) = 0. \quad (42)$$

It follows from (38) that there exist $r_2 > r_1$ and $0 < \sigma < 1$, such that

$$g(t, y(t)) \leq \sigma \lambda_1 y, \quad \forall y \geq r_2. \quad (43)$$

Set $(T_1 y)(t) = \sigma \lambda_1 (T y)(t)$. Then, $T_1 : C^+[0, 1] \rightarrow C^+[0, 1]$ is a completely continuous linear operator, and $T_1(P_1) \subset P$. Let $r_4 > r_2 > r_1$.

Now we prove that $i(A, B_{r_4} \cap P, P) = 1$.

Suppose that there exist $y_2 \in \partial B_{r_4} \cap P$, $\mu_0 \geq 1$, such that $Ay_2 = \mu_0 y_2$. Let $e(y_2) = \{t \in [0, 1] \mid y_2(t) > r_2\}$; then, from (43) we have

$$\begin{aligned}
 (Ay_2)(t) &\leq \int_0^1 H(t, s) \\
 &\cdot \phi_q \left[\int_0^1 H_1(s, \tau) a(\tau) g(\tau, y_2(\tau)) d\tau \right] ds \\
 &= \int_{e(y_2)} H(t, s) \\
 &\cdot \phi_q \left[\int_0^1 H_1(s, \tau) a(\tau) g(\tau, y_2(\tau)) d\tau \right] ds \\
 &+ \int_{[0,1] \setminus e(y_2)} H(t, s) \\
 &\cdot \phi_q \left[\int_0^1 H_1(s, \tau) a(\tau) g(\tau, y_2(\tau)) d\tau \right] ds \quad (44) \\
 &\leq \sigma \lambda_1 \int_0^1 H(t, s) \\
 &\cdot \phi_q \left[\int_0^1 H_1(s, \tau) a(\tau) y_2(\tau) d\tau \right] ds \\
 &+ \int_0^1 H(t, s) \\
 &\cdot \phi_q \left[\int_0^1 H_1(s, \tau) a(\tau) g(\tau, y_2(\tau)) d\tau \right] ds \\
 &\leq (T_1 y_2)(t) + M,
 \end{aligned}$$

where $M = \sup_{y \in \partial B_{r_2} \cap P} \int_0^1 H(t, s) \phi_q \left[\int_0^1 H_1(s, \tau) a(\tau) g(\tau, y_2(\tau)) d\tau \right] ds$. Thus, $0 \leq \mu_0 y_2(t) = Ay_2(t) \leq (T_1 y_2)(t) + M$, $t \in [0, 1]$. Since $T_1(C^+[0, 1]) \subset C^+[0, 1]$, we have $0 \leq (T_1^j(Ay_2))(t) \leq (T_1^j(T_1 y_2 + M))(t)$, $t \in [0, 1]$, for $j = 1, \dots, N-1$. It means that $\|T_1^j(Ay_2)\| \leq \|T_1^j(T_1 y_2 + M)\|$, $j = 1, \dots, N-1$. Let

$$\varepsilon = \frac{1}{2} (1 - r(T_1)). \quad (45)$$

For any $y \in C^+[0, 1]$, denote

$$\|y\|^* = \sum_{i=1}^N [r(T_1) + \varepsilon]^{N-i} \|T_1^{i-1} y\|. \quad (46)$$

It is easy to see that

$$\mu_0 \|y_2\|^* < \left[r(T_1) + \frac{3\varepsilon}{2} \right] \|y_2\|^*. \quad (47)$$

Since $\mu_0 \geq 1$, we have from (47) that $1 \leq r(T_1) + 3\varepsilon/2$, which contradicts (45). Thus, for any $y \in \partial B_{r_4} \cap P$, $\mu \geq 1$, we have that $Ay \neq \mu y$. Thus, from Lemma 6, we know that

$$i(A, B_{r_4} \cap P, P) = 1. \quad (48)$$

From (42) and (48), we see that

$$\begin{aligned}
 i(A, (B_{r_4} \cap P) \setminus (\bar{B}_{r_1} \cap P), P) \\
 = i(A, B_{r_4} \cap P, P) - i(A, \bar{B}_{r_1} \cap P, P) = 1 - 0 = 1.
 \end{aligned} \quad (49)$$

Then, A has at least one fixed point on $(B_{r_4} \cap P) \setminus (\bar{B}_{r_1} \cap P)$. It means problem (1) has at least one positive solution. The proof is completed. $\square$

Corollary 12. Suppose that (H_1) and (H_2) hold. In addition, assume that

$$\begin{aligned}
 0 \leq g^\infty = \limsup_{y \rightarrow +\infty} \frac{g(t, y(t))}{y} &\leq \liminf_{y \rightarrow 0^+} \frac{g(t, y(t))}{y} \\
 &= g_0 \leq +\infty.
 \end{aligned} \quad (50)$$

Then, for any

$$\xi \in \left(\frac{\lambda_1}{h_0}, \frac{\lambda_1}{g^\infty} \right), \quad (51)$$

where λ_1 is the first eigenvalue of T defined by (15), problem (35) has at least one positive solution.

Proof. By (51), we obtain

$$\begin{aligned}
 \liminf_{y \rightarrow 0^+} \frac{\xi h(t, y(t))}{y} &> \lambda_1, \\
 \limsup_{y \rightarrow +\infty} \frac{\xi g(t, y(t))}{y} &< \lambda_1.
 \end{aligned} \quad (52)$$

It follows from Theorem 11 that Corollary 12 holds. The proof is completed. $\square$

4. Examples

Example 1. Consider the existence of positive solutions for the following p -Laplacian integral boundary value problems:

$$\begin{aligned}
 (\phi_p(y''(t)))'' &= (1+t)(\pi-1)y^2 \\
 &+ \frac{5}{2016} \ln(1+y), \quad 0 < t < 1, \\
 y(0) = y(1) &= \int_0^1 b(s)y(s)ds, \\
 y''(0) = y''(1) &= \phi_q \left(\int_0^1 c(s)\phi_p(y''(s))ds \right),
 \end{aligned} \quad (53)$$

where $f(t, y(t), y'(t), y''(t), y'''(t)) = (1+t)(\pi-1)y^2 + (5/2016) \ln(1+y)$. $\phi_p(t) = |t|^{p-2}t$, $p > 1$, $\phi_q = \phi_p^{-1}$, $1/p + 1/q = 1$, $a \in L^1[0, 1]$ is symmetric on the interval $[0, 1]$, $a(t) = 1$ for $t \in [0, 1]$ is continuous, and $b, c \in L^1[0, 1]$ are nonnegative symmetric on $[0, 1]$.

Let $h(t, y) = (5/2016) \ln(1+y) + (\pi-1)y^2$ and let $g(t, y) = (1+t)(\pi-1)y^2 + (5/2016) \ln(1+y)$. Obviously, $h(t, y) \leq f(t, y(t), y'(t), y''(t), y'''(t)) \leq g(t, y)$, and

$$\begin{aligned} \liminf_{y \rightarrow +\infty} \frac{h(t, y(t))}{y} &= +\infty, \quad \text{uniformly on } t \in [0, 1], \\ \limsup_{y \rightarrow 0^+} \frac{g(t, y(t))}{y} &= \frac{5}{2016}, \\ &\text{uniformly on } t \in [0, 1]. \end{aligned} \quad (54)$$

All conditions of Theorem 8 are satisfied. Consequently, following from Theorem 8, we know that the boundary value problem (53) has at least one positive solution.

Example 2. Consider the existence of positive solutions for the following p -Laplacian integral boundary value problems:

$$\begin{aligned} &(\phi_p(y''(t)))'' \\ &= y^3 + \exp\{y(t) - |y'(t)| - |y'''(t)| - |y''(t)|\} \\ &\quad - 1 - 6 \sin y \\ &\quad - |\sin^6(y(t) + |y'(t)| + |y''(t)| + |y'''(t)|)| \\ &\quad - \left| \frac{\sin \pi t (1-t)}{\sqrt{\pi t (1-t)}} \right|, \quad 0 < t < 1, \\ &y(0) = y(1) = \int_0^1 b(s) y(s) ds, \\ &y''(0) = y''(1) = \phi_q \left(\int_0^1 c(s) \phi_p(y''(s)) ds \right), \end{aligned} \quad (55)$$

where $\phi_p(t) = |t|^{p-2}t$, $p > 1$, $\phi_q = \phi_p^{-1}$, $1/p + 1/q = 1$, $a \in L^1[0, 1]$ is symmetric on the interval $[0, 1]$, $a(t) = 1$ for $t \in [0, 1]$ is continuous, and $b, c \in L^1[0, 1]$ are nonnegative symmetric on $[0, 1]$. For $(t, y, y', y'', y''') \in (0, 1) \times [0, +\infty) \times (-\infty, +\infty) \times (-\infty, +\infty) \times (-\infty, +\infty)$,

$$\begin{aligned} &f(t, y(t), y'(t), y''(t), y'''(t)) \\ &= y^3 + \exp\{y(t) - |y'(t)| - |y'''(t)| - |y''(t)|\} \\ &\quad - 1 - 6 \sin y \\ &\quad - |\sin^6(y(t) + |y'(t)| + |y''(t)| + |y'''(t)|)| \\ &\quad - \left| \frac{\sin \pi t (1-t)}{\sqrt{\pi t (1-t)}} \right|, \quad 0 < t < 1. \end{aligned} \quad (56)$$

Let

$$\begin{aligned} g(t, y) &= y^3 + e^y - 1 - 6 \sin y, \\ h(t, y) &= y^3 - 216 - 3 \left| \frac{\sin \pi t (1-t)}{\sqrt{\pi t (1-t)}} \right|, \\ &0 < t < 1. \end{aligned} \quad (57)$$

Obviously, $h(t, y) \leq f(t, y(t), y'(t), y''(t), y'''(t)) \leq g(t, y)$, and

$$\begin{aligned} \liminf_{y \rightarrow +\infty} \frac{h(t, y(t))}{y} &= +\infty, \quad \text{uniformly on } t \in [0, 1], \\ \limsup_{y \rightarrow 0^+} \frac{g(t, y(t))}{y} &= -5 \neq \lambda_1, \\ &(\lambda_1 \text{ is the first eigenvalue of } T), \quad \text{uniformly on } t \in [0, 1]. \end{aligned} \quad (58)$$

All conditions of Theorem 8 are satisfied. Consequently, following from Theorem 8, we know that the boundary value problem (55) has at least one positive solution.

Conflict of Interests

The author declares that she has no competing interests.

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