Research Article

On the Existence and Stability of Standing Waves for 2-Coupled Nonlinear Fractional Schrödinger System

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We study a system of 2-coupled nonlinear fractional Schrödinger equations. Firstly, we construct constrained minimization problem to the system. Next, we prove the existence of standing waves for the system by using the concentration-compactness and commutator estimates method. Lastly, we also consider the set of minimizers of the constrained minimization problem. We prove that it is a stable set for initial value of the problem; that is, a solution to the system with initial value which is near the set will remain near it for all time.

1. Introduction

In the recent years, more and more researchers study the application of fractional calculus and fractional integrodifferential equations in physics and other areas (see [1–4]). The concept of fractional calculus is firstly put forward by Leibniz as a generalization of standard calculus. Afterwards several kinds of definitions of them have been established such as Riemann-Liouville fractional derivative, Caputo fractional derivative, and Weyl fractional derivative (see [5, 6]). We all know that a variety of fractional calculus is brought in inspired by standard calculus, but the calculation laws of fractional calculus are much different (see [5, 7]). To avoid the complicated fractional calculus, in this paper we are only concerned with the fractional Laplacian operator.

Here we study the existence and stability of standing waves for the following system of 2-coupled nonlinear fractional Schrödinger equations:

$$\begin{aligned} i\psi_{1t} + (-\Delta)^\alpha \psi_1 &= \gamma \left(|\psi_1|^{p-2} + |\psi_1|^{p/2-2} |\psi_2|^{p/2} \right) \psi_1, \\ x &\in \mathbb{R}^N, \\ i\psi_{2t} + (-\Delta)^\alpha \psi_2 &= \gamma \left(|\psi_2|^{p-2} + |\psi_2|^{p/2-2} |\psi_1|^{p/2} \right) \psi_2, \\ x &\in \mathbb{R}^N, \end{aligned} \quad (1)$$

where $i^2 = -1$, $N \geq 2$, $2 < p < 2 + 4\alpha/N$, $\gamma > 0$, $0 < \alpha < 1$, and $(-\Delta)^\alpha$ is a pseudo-differential operator which is defined by $(-\Delta)^\alpha \psi_i(\xi) = |\xi|^{2\alpha} \widehat{\psi_i}(\xi)$, the Fourier transform $F\psi_i(\xi) = \widehat{\psi_i}(\xi) = (1/(2\pi)^{N/2}) \int_{\mathbb{R}^N} \psi_i(x) e^{-i\xi \cdot x} dx$.

It is well known that the coupled nonlinear Schrödinger equations play an important role in describing nonrelativistic quantum mechanical behavior. In particular, the fractional Schrödinger equations can describe better some real physical phenomenon. Schrödinger type equation(s) is first derived by Feynman and Hibbs, applying path integrals over Brownian paths in [8]. In [9–11], Laskin showed the path integral over Lévy-like quantum mechanical paths approaches to a generalization of quantum mechanics. As the path integral over Brownian trajectories, it leads to the standard Schrödinger equations. And as the path integral over Lévy-like quantum mechanical paths, it leads to the fractional Schrödinger equation. Recently, there are some researchers studying fractional Schrödinger equation(s) and its physical applications (see [12–14]). The research of the existence and stability of standing waves for the nonlinear Schrödinger equations arises in various fields of physics such as plasma physics, constructive field theory, and nonlinear optics (see [15–22]). There are many papers about this topic for the standard Schrödinger equations with different nonlinear terms such as [17, 19, 20]. There are some results concerned with the existence and stability of standing waves for the nonlinear

fractional Schrödinger equations in [21, 22]. The author [23] obtained the global solution for a class of systems of fractional nonlinear Schrödinger equations with boundary condition. However, there are few results about the standing waves of fractional nonlinear Schrödinger equations (1). In the present paper, we consider the existence and stability of standing waves for the system of 2-coupled nonlinear fractional Schrödinger equations.

The rest of this paper is arranged as follows. In Section 2, we recall some basic definitions and introduce preparation results. In Section 3, we give the main results and proof of them.

2. Preliminaries

In this section, we firstly introduce the constrained minimization problem in order to study the existence and stability of standing waves for (1). We know that the standing waves of (1) have the special form $(\psi_1(x, t), \psi_2(x, t)) = (e^{i\omega_1 t} u_1(x), e^{i\omega_2 t} u_2(x))$, where $\omega_i \in \mathbb{R}$ ($i = 1, 2$). Then we only need to find $\omega_1, \omega_2, u_1(x), u_2(x)$ satisfying the following equations to study the existence and stability of standing waves of (1):

$$\begin{aligned} (-\Delta)^\alpha u_1 - \omega_1 u_1 &= \gamma \left(|u_1|^{p-2} + |u_1|^{p/2-2} |u_2|^{p/2} \right) u_1, \\ x &\in \mathbb{R}^N, \\ (-\Delta)^\alpha u_2 - \omega_2 u_2 &= \gamma \left(|u_2|^{p-2} + |u_2|^{p/2-2} |u_1|^{p/2} \right) u_2, \\ x &\in \mathbb{R}^N, \end{aligned} \quad (2)$$

where u_1, u_2 are complex value functions.

By the variational method, we know that in order to study the existence of solution for (2) we need to consider the following constrained minimization problem:

$$\begin{aligned} I_q &:= \inf \{ I(u_1, u_2) \mid (u_1, u_2) \in H^\alpha(\mathbb{R}^N) \\ &\quad \times H^\alpha(\mathbb{R}^N), M(u_1, u_2) = q, q > 0 \}, \end{aligned} \quad (3)$$

where

$$\begin{aligned} I(u_1, u_2) &= \frac{1}{2} \int_{\mathbb{R}^N} \left(|(-\Delta)^{\alpha/2} u_1|^2 + |(-\Delta)^{\alpha/2} u_2|^2 \right) dx \\ &\quad - \int_{\mathbb{R}^N} F(x, u_1, u_2) dx, \\ M(u_1, u_2) &= \int_{\mathbb{R}^N} (|u_1|^2 + |u_2|^2) dx, \\ \partial_{u_1} F(x, u_1, u_2) &= \gamma \left(|u_1|^{p-2} + |u_1|^{p/2-2} |u_2|^{p/2} \right) u_1, \\ \partial_{u_2} F(x, u_1, u_2) &= \gamma \left(|u_2|^{p-2} + |u_2|^{p/2-2} |u_1|^{p/2} \right) u_2. \end{aligned} \quad (4)$$

And $H^\alpha(\mathbb{R}^N)$ is the fractional order Sobolev space. We denote

$$\begin{aligned} G_q &:= \{ (u_1, u_2) \mid (u_1, u_2) \in H^\alpha(\mathbb{R}^N) \\ &\quad \times H^\alpha(\mathbb{R}^N), M(u_1, u_2) = q, I(u_1, u_2) = I_q \} \end{aligned} \quad (5)$$

as the set of minimizers of the problem I_q .

Next we recall some basic definitions. Define the fractional order Sobolev space

$$\begin{aligned} H^\alpha(\mathbb{R}^N) &= \left\{ u \mid u \right. \\ &\quad \left. \in L^2(\mathbb{R}^N; \mathbb{C}), F^{-1} \left[(1 + |\xi|^2)^{\alpha/2} Fu \right] \in L^2(\mathbb{R}^N) \right\} \end{aligned} \quad (6)$$

with the norm

$$\|\cdot\|_{\alpha,2} = \left\| F^{-1} (1 + |\xi|^2)^{\alpha/2} Fu \right\|_2. \quad (7)$$

Denote the product space $H^\alpha(\mathbb{R}^N) \times H^\alpha(\mathbb{R}^N) = \{(u_1, u_2) \mid u_1, u_2 \in H^\alpha(\mathbb{R}^N)\}$ with the norm $\|(u_1, u_2)\| = (\|u_1\|_{\alpha,2}^2 + \|u_2\|_{\alpha,2}^2)^{1/2}$. Throughout this paper, we denote the norm of $L^p_{per}(\mathbb{R}^N)$ by $|\cdot|_p$ for all $1 \leq p \leq \infty$. Denote the set $B(y, r) = \{x \mid x, y \in \mathbb{R}^N, r > 0, |x - y| < r\}$. For simplicity and convenience, the letter C will represent a constant, which may be different from one to others. $C(\cdot, \cdot)$, for example, represents the constant C which can be expressed by the parameters appearing in the braces.

Lastly, we give the following some lemmas, which will be used in our paper. For more detail, we can see [21, 23, 24].

Lemma 1. *If $0 < \alpha < 1$, $x \in \mathbb{R}^N$, and $u \in S$, S representing the Schwartz class, then the fractional Laplacian $(-\Delta)^\alpha$ of u is also expressed for the formula*

$$(-\Delta)^\alpha u(x) = C(N, \alpha) P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2\alpha}} dx, \quad (8)$$

where P.V. means the Cauchy principal value on the integral and $C(N, \alpha)$ is some positive normalization constant.

Lemma 2. *For $0 < \alpha < 1$, there are two properties with $H^\alpha(\mathbb{R}^N)$:*

(i) *The norm $\|\cdot\|_{\alpha,2}$ of $H^\alpha(\mathbb{R}^N)$ is equivalent to $\|\cdot\| = |F^{-1}[(1 + |\xi|^\alpha)F\cdot]|_2$.*

(ii) *$\forall u \in H^\alpha(\mathbb{R}^N)$, $c\|u\|_{W^{\alpha,2}(\mathbb{R}^N)} \leq \|u\| \leq C\|u\|_{W^{\alpha,2}(\mathbb{R}^N)}$,*

which implies $H^{\alpha,2}(\mathbb{R}^N) = W^{\alpha,2}(\mathbb{R}^N)$, where $W^{\alpha,2}(\mathbb{R}^N)$ is defined by the method of trace interpolation (see chapter 7 of [24]) and its norm is given by $\|u\|_{W^{\alpha,2}(\mathbb{R}^N)}$. c and C are positive constants.

Remark 3. In the following paper, we denote the norm $\|\cdot\|_{\alpha,2}$ of $H^\alpha(\mathbb{R}^N)$ by $\|\cdot\|$.

Lemma 4 (see [24]). *For $0 < \alpha < 1$, $2 \leq p_1 \leq 2N/(N - 2\alpha)$, $H^\alpha(\mathbb{R}^N) \hookrightarrow L^{p_1}(\mathbb{R}^N)$ and $W^{\alpha,2}(B(y, r)) \hookrightarrow L^{p_1}(B(y, r))$.*

Moreover, if $u_n \rightharpoonup u$ in $H^\alpha(\mathbb{R}^N)$, then $u_n \rightarrow u$ in $L^p_{loc}(\mathbb{R}^N)$, and $u_n \rightarrow u$ almost everywhere in $\mathbb{R}^N$, where $2 \leq p < 2N/(N - 2\alpha)$.

Lemma 5. If $\{(u_{1n}, u_{2n})\}$ is bounded in $H^\alpha(\mathbb{R}^N) \times H^\alpha(\mathbb{R}^N)$ and $\lim_{n \rightarrow +\infty} \sup_{y \in \mathbb{R}^N} \int_{B(y, r)} (|u_{1n}|^2 + |u_{2n}|^2) dx = 0$ for some $r > 0$, then $\lim_{n \rightarrow +\infty} (u_{1n}, u_{2n}) = (0, 0)$ in $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$ for $2 < p < 2N/(N - 2\alpha)$.

Remark 6. The proof of Lemma 5 is obtained easily according to Lemma 2.4 (see [21]), so we will omit it.

Lemma 7 (commutator estimates). If $0 < \alpha < 1$, $f, g \in S$, the Schwartz class, then

$$\begin{aligned} & |(-\Delta)^{\alpha/2} (fg) - f(-\Delta)^{\alpha/2} g|_2 \\ & \leq C \left(|\nabla f|_{\bar{p}_1} |(-\Delta)^{(\alpha-1)/2} g|_{\bar{q}_1} + |(-\Delta)^{\alpha/2} f|_{\bar{p}_2} |g|_{\bar{q}_2} \right), \end{aligned} \quad (9)$$

where $\bar{p}_2, \bar{q}_1 \in [2, +\infty)$ satisfying $1/\bar{p}_1 + 1/\bar{q}_1 = 1/\bar{p}_2 + 1/\bar{q}_2 = 1/2$.

In [23], the authors have obtained the existence and uniqueness of the global solution of a class of systems for fractional nonlinear Schrödinger equations with periodic boundary condition by using the Faedo-Galerkin method. In the proof of Theorem 4.1 in the references, only needing to let the period $2\pi\epsilon_i \rightarrow +\infty$, we can obtain the following theorem.

Theorem 8. Let $0 < \alpha < 1$. If $\Psi_0 = (\Psi_{10}, \Psi_{20}) \in H^\alpha(\mathbb{R}^N) \times H^\alpha(\mathbb{R}^N)$ then there exists a global weak solution $\Psi(x, t) = (\Psi_1(x, t), \Psi_2(x, t)) \in C(\mathbb{R}, H^\alpha(\mathbb{R}^N) \times H^\alpha(\mathbb{R}^N))$ to the initial value problem of coupled nonlinear fractional Schrödinger equations (1) with initial data $\Psi_{10} = \Psi_1(x, 0)$, $\Psi_{20} = \Psi_2(x, 0)$.

3. Uniform Estimates of the Solution

In the section, we consider coupled nonlinear equations (2). Since the scaling $u_{i\gamma} = \gamma^{1/(p-2)} u_i(x)$ ($i = 1, 2$), we can transform (2) to the following equations:

$$\begin{aligned} & (-\Delta)^\alpha u_{1\gamma} - \omega_1 u_{1\gamma} \\ & = \left(|u_{1\gamma}|^{p-2} + |u_{1\gamma}|^{p/2-2} |u_{2\gamma}|^{p/2} \right) u_{1\gamma}, \quad x \in \mathbb{R}^N, \\ & (-\Delta)^\alpha u_{2\gamma} - \omega_2 u_{2\gamma} \\ & = \left(|u_{2\gamma}|^{p-2} + |u_{1\gamma}|^{p/2} |u_{2\gamma}|^{p/2-2} \right) u_{2\gamma}, \quad x \in \mathbb{R}^N. \end{aligned} \quad (10)$$

So we only consider coupled nonlinear equations (10). So we have

$$F(x, u_1, u_2) = \frac{1}{p} (|u_1|^p + |u_2|^p) + \frac{2}{p} |u_1|^{p/2} |u_2|^{p/2}. \quad (11)$$

For any $q > 0$, let $\tilde{I}_q := I_q$ and $\tilde{G} := G_q$.

Under the case we have the following main results.

Theorem 9. Assuming $\{(u_{1n}, u_{2n})\}$ is a minimizing sequence of problem $\tilde{I}_q$, then there exists a sequence $\{y_n\} \in \mathbb{R}^N$ such that $\{(u_{1n}(\cdot + y_n), u_{2n}(\cdot + y_n))\}$ contains a convergent subsequence in $H^\alpha(\mathbb{R}^N) \times H^\alpha(\mathbb{R}^N)$. In particular, there exists a minimizer for problem $(\tilde{I}_q)$ (i.e., $\tilde{G}_q \neq \emptyset$), and

$$\lim_{n \rightarrow +\infty} \inf_{(g_1, g_2) \in \tilde{G}} (\|u_{1n} - g_1\| + \|u_{2n} - g_2\|) = 0. \quad (12)$$

Theorem 10. The set $\tilde{G}_q$ is $H^\alpha(\mathbb{R}^N) \times H^\alpha(\mathbb{R}^N)$ -stable with respect to (1); that is, $\forall \varepsilon > 0$, there exists $\delta > 0$ such that if $(u_1, u_2) \in C([0, T], H^\alpha(\mathbb{R}^N) \times H^\alpha(\mathbb{R}^N))$ is a solution of (1) with initial value (u_{10}, u_{20}) satisfying

$$\inf_{(g_1, g_2) \in \tilde{G}_q} (\|u_{10} - g_1\| + \|u_{20} - g_2\|) < \delta, \quad (13)$$

then, $\forall t \in [0, T]$,

$$\inf_{(g_1, g_2) \in \tilde{G}_q} (\|u_1(t, \cdot) - g_1\| + \|u_2(t, \cdot) - g_2\|) < \varepsilon. \quad (14)$$

Before we give the proof of Theorems 9 and 10, we firstly prove the following some lemmas.

Lemma 11. For any $q > 0$, one has the following:

- (i) $-\infty < \tilde{I}_q < 0$,
- (ii) every minimizing sequence problem $\tilde{I}_q$ is bounded in $H^\alpha(\mathbb{R}^N) \times H^\alpha(\mathbb{R}^N)$. And for sufficiently large n , there exists a positive constant δ such that $|u_{1n}|_p + |u_{2n}|_p \geq \delta$.

Proof. (i) For given $(u_1, u_2) \in H^\alpha(\mathbb{R}^N) \times H^\alpha(\mathbb{R}^N)$, $|u_1|_2^2 + |u_2|_2^2 = q$, $|u_1|_p^p + |u_2|_q^q \neq 0$. And set $u_{i\theta}(x) = \theta^{1/2} u_i(\theta^{1/N} x)$ ($i = 1, 2$); then $|u_{1\theta}|_2^2 + |u_{2\theta}|_2^2 = |u_1|_2^2 + |u_2|_2^2 = q$.

By Lemma 1, we have that

$$\begin{aligned} & \tilde{I}(u_{1\theta}, u_{2\theta}) \\ & = \frac{1}{2} \theta^{2\alpha/N} \int_{\mathbb{R}^N} (|(-\Delta)^{\alpha/2} u_1|^2 + |(-\Delta)^{\alpha/2} u_2|^2) dx \\ & \quad - \frac{1}{p} \theta^{p/2-1} \int_{\mathbb{R}^N} (|u_1|^p + |u_2|^p) dx \\ & \quad - \frac{2}{p} \theta^{p/2-1} \int_{\mathbb{R}^N} |u_1|^{p/2} |u_2|^{p/2} dx \\ & \leq \frac{1}{2} \theta^{2\alpha/N} \int_{\mathbb{R}^N} (|(-\Delta)^{\alpha/2} u_1|^2 + |(-\Delta)^{\alpha/2} u_2|^2) dx \\ & \quad - \frac{1}{p} \theta^{p/2-1} \int_{\mathbb{R}^N} (|u_1|^p + |u_1|^p) dx. \end{aligned} \quad (15)$$

By $0 < p/2 - 1 < 2\alpha/N$, there exists $\tilde{I}(u_{1\theta}, u_{2\theta}) < 0$ taking sufficiently small θ . Therefore, we obtain that $\tilde{I} \leq \tilde{I}(u_{1\theta}, u_{2\theta}) < 0$.

By the Hölder inequality and the Sobolev inequality, we can derive

$$|u_i|_p \leq |u_i|_2^{(1-\lambda)} |u_i|_{2N/(N-2\alpha)}^\lambda \leq C |u_i|_2^{1-\lambda} \|u_i\|^\lambda, \quad (16)$$

$i = 1, 2$,

where $p(1-\lambda)/2 + \lambda p/(2N/(N-2\alpha)) = 1$. Because of the Young inequality and $0 < \lambda p < 2$, for $\varepsilon > 0$, from the above inequality we can derive that

$$|u_i|_p^p \leq \varepsilon \|u_i\|^2 + C(\varepsilon) |u_i|_2^{2p(1-\lambda)/(2-\lambda p)}, \quad i = 1, 2. \quad (17)$$

So for $(u_1, u_2) \in H^\alpha(\mathbb{R}^N) \times H^\alpha(\mathbb{R}^N)$, setting $|u_i|_2^2 = q_i$ ($i = 1, 2$), $q_1 + q_2 = q$ by the Young inequality and (17), we have that

$$\begin{aligned} \tilde{I}(u_1, u_2) &\geq \frac{1}{2} (\|u_1\|^2 + \|u_2\|^2) - \frac{1}{2} q \\ &\quad - \frac{1}{p} \int_{\mathbb{R}^N} (|u_1|^p + |u_2|^p) dx \\ &\quad - \frac{2}{p} \int_{\mathbb{R}^N} |u_1|^{p/2} |u_2|^{p/2} dx \\ &\geq \frac{1}{2} (\|u_1\|^2 + \|u_2\|^2) - \frac{1}{2} q - \frac{\varepsilon}{p} (\|u_1\|^2 + \|u_2\|^2) \\ &\quad - \frac{C(\varepsilon)}{p} (|u_1|_2^{2p(1-\lambda)/(2-\lambda p)} + |u_2|_2^{2p(1-\lambda)/(2-\lambda p)}) \\ &\quad - \frac{1}{p} \int_{\mathbb{R}^N} (|u_1|^p + |u_2|^p) dx \\ &\geq \frac{1}{2} (\|u_1\|^2 + \|u_2\|^2) - \frac{1}{2} q - \frac{2\varepsilon}{p} (\|u_1\|^2 + \|u_2\|^2) \\ &\quad - \frac{2C(\varepsilon)}{p} (q_1^{p(1-\lambda)/(2-\lambda p)} + q_2^{p(1-\lambda)/(2-\lambda p)}). \end{aligned} \quad (18)$$

Therefore for sufficiently small ε , we have that

$$\begin{aligned} \tilde{I}(u_1, u_2) &\geq -\frac{1}{2} q \\ &\quad - C(\varepsilon, p) (q_1^{p(1-\lambda)/(2-\lambda p)} + q_2^{p(1-\lambda)/(2-\lambda p)}) \\ &> -\infty. \end{aligned} \quad (19)$$

So $-\infty < \tilde{I}_q < 0$.

(ii) Let $\{(u_{1n}, u_{2n})\}$ be minimizing sequence for problem $(\tilde{I}_q)$.

By (16), we have that

$$\begin{aligned} &\frac{1}{2} (\|u_{1n}\|^2 + \|u_{2n}\|^2) \\ &\leq \tilde{I}(u_{1n}, u_{2n}) + \frac{q}{2} + \frac{1}{p} \int_{\mathbb{R}^N} (|u_{1n}|^p + |u_{2n}|^p) dx \\ &\quad + \frac{2}{p} \int_{\mathbb{R}^N} |u_{1n}|^{p/2} |u_{2n}|^{p/2} dx \end{aligned}$$

$$\begin{aligned} &\leq \tilde{I}(u_{1n}, u_{2n}) + \frac{q}{2} + C |u_{1n}|_2^{p(1-\lambda)} \|u_{1n}\|^{p\lambda} \\ &\quad + C |u_{2n}|_2^{p(1-\lambda)} \|u_{2n}\|^{p\lambda} \\ &\leq \tilde{I}(u_{1n}, u_{2n}) + \frac{q}{2} \\ &\quad + C (q_1^{p(1-\lambda)/2} \|u_{1n}\|^{p\lambda} + q_2^{p(1-\lambda)/2} \|u_{2n}\|^{p\lambda}). \end{aligned} \quad (20)$$

By $0 < \lambda p < 2$, we get that $\|u_{1n}\|^2 + \|u_{2n}\|^2 < C$; that is, $\{(u_{1n}, u_{2n})\}$ is bounded in $H^\alpha(\mathbb{R}^N) \times H^\alpha(\mathbb{R}^N)$.

Using an argument by contradiction, we prove the second part of (ii). Assuming δ does not exist such that $|u_{1n}|_p + |u_{2n}|_p \geq \delta > 0$, then

$$\liminf_{n \rightarrow +\infty} (|u_{1n}|_p + |u_{2n}|_p) \leq 0. \quad (21)$$

Now we have that

$$\begin{aligned} &\lim_{n \rightarrow +\infty} \tilde{I}(u_{1n}, u_{2n}) \\ &= \lim_{n \rightarrow +\infty} \left(\int_{\mathbb{R}^N} (|(-\Delta)^{\alpha/2} u_{1n}|^2 + |(-\Delta)^{\alpha/2} u_{2n}|^2) dx \right. \\ &\quad - \frac{1}{p} \int_{\mathbb{R}^N} (|u_{1n}|^p + |u_{2n}|^p) dx \\ &\quad - \frac{2}{p} \int_{\mathbb{R}^N} |u_{1n}|^{p/2} |u_{2n}|^{p/2} dx \Big) \geq -\frac{2}{p} \\ &\quad \cdot \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^N} (|u_{1n}|^p + |u_{2n}|^p) dx \geq 0, \end{aligned} \quad (22)$$

which contradicts $-\infty < \tilde{I}_q < 0$. Therefore for sufficiently large n , there exists a positive constant δ such that $|u_{1n}|_p + |u_{2n}|_p \geq \delta$. We have completed the proof of the lemma. $\square$

Now, for minimizing sequence $\{(u_{1n}, u_{2n})\}$ of problem $(\tilde{I}_q)$ we introduce the Lévy concentration function $Q_n(r) = \sup_{y \in \mathbb{R}^N} \int_{B(y,r)} (|u_{1n}|^2 + |u_{2n}|^2) dx$. Apparently, each function Q_n is nondecreasing on $(0, +\infty)$. Using Helly's selection theorem, $\{Q_n\}$ has a convergent subsequence $\{Q_m\}$, such that $\lim_{m \rightarrow +\infty} Q_m(r) = Q(r)$, for all $r > 0$, where $Q(r)$ is a nondecreasing function. By $0 \leq Q_m(r) \leq q$, there exists β ($0 \leq \beta \leq q$) satisfying $\lim_{r \rightarrow +\infty} Q(r) = \beta$.

Next, we will prove $\beta = q$.

Lemma 12. Assuming $q > 0$ and $\{(u_{1n}, u_{2n})\}$ is a minimizing sequence for $(\tilde{I}_q)$, then $\beta > 0$.

Proof. By the definition of β , we know $\beta \geq 0$. Using an argument by contradiction, assume $\beta = 0$; then there exist n_k and r_0 , satisfying

$$\lim_{n_k \rightarrow +\infty} \sup_{y \in \mathbb{R}^N} \int_{B(y, r_0)} (|u_{1n_k}|^2 + |u_{2n_k}|^2) dx \rightarrow 0. \quad (23)$$

By Lemma 5, we have $(u_{1n_k}, u_{2n_k}) \rightarrow (0, 0)$ in $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$. According to Lemma 11(ii), $(u_{1n_k}, u_{2n_k}) \rightharpoonup (0, 0)$

in $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$. They are contradictory. Therefore, we obtain $\beta > 0$. $\square$

Lemma 13. For $\gamma_1, \gamma_2 > 0$, then $\tilde{I}_{\gamma_1+\gamma_2} < \tilde{I}_{\gamma_1} + \tilde{I}_{\gamma_2}$.

Proof. For given $(u_1, u_2) \in H^\alpha(\mathbb{R}^N) \times H^\alpha(\mathbb{R}^N)$, $|u_1|_2^2 + |u_2|_2^2 = \beta > 0$, let $u_{i\theta}(x) = \theta^{\gamma_i} u_i(\theta^{\gamma_i} x)$ ($i = 1, 2$), where $\theta > 0$, $\gamma_1 = 2\alpha/(4\alpha - (p-2)N)$, $\gamma_2 = (p-2)/(4\alpha - (p-2)N)$. Then $|u_{1\theta}|_2^2 + |u_{2\theta}|_2^2 = \theta\beta$ and

$$\begin{aligned} \tilde{I}(u_{1\theta}, u_{2\theta}) &= \theta^{(2\alpha p - N(p-2))/(4\alpha - (p-2)N)} \left(\frac{1}{2} \right. \\ &\quad \cdot \int_{\mathbb{R}^N} \left(|(-\Delta)^{\alpha/2} u_1|^2 + |(-\Delta)^{\alpha/2} u_2|^2 \right) dx - \frac{1}{p} \\ &\quad \cdot \int_{\mathbb{R}^N} (|u_1|^p + |u_2|^p) dx - \frac{2}{p} \\ &\quad \cdot \int_{\mathbb{R}^N} |u_1|^{p/2} |u_2|^{p/2} dx \Big), \end{aligned} \quad (24)$$

where we apply Lemma 1.

Now, we can derive that

$$\begin{aligned} \tilde{I}_{\theta\beta} &= \inf_{(u_1, u_2) \in H^\alpha(\mathbb{R}^N) \times H^\alpha(\mathbb{R}^N), |u_1|_2^2 + |u_2|_2^2 = \theta\beta} \left\{ \frac{1}{2} \right. \\ &\quad \cdot \int_{\mathbb{R}^N} \left(|(-\Delta)^{\alpha/2} u_1|^2 + |(-\Delta)^{\alpha/2} u_2|^2 \right) dx - \frac{1}{p} \\ &\quad \cdot \int_{\mathbb{R}^N} (|u_1|^p + |u_2|^p) dx - \frac{2}{p} \\ &\quad \cdot \int_{\mathbb{R}^N} |u_1|^{p/2} |u_2|^{p/2} dx \Big\} = \theta^{(2\alpha p - N(p-2))/(4\alpha - (p-2)N)} \\ &\quad \cdot \inf_{(u_1, u_2) \in H^\alpha(\mathbb{R}^N) \times H^\alpha(\mathbb{R}^N), |u_1|_2^2 + |u_2|_2^2 = \beta} \left\{ \frac{1}{2} \right. \\ &\quad \cdot \int_{\mathbb{R}^N} \left(|(-\Delta)^{\alpha/2} u_1|^2 + |(-\Delta)^{\alpha/2} u_2|^2 \right) dx - \frac{1}{p} \\ &\quad \cdot \int_{\mathbb{R}^N} (|u_1|^p + |u_2|^p) dx - \frac{2}{p} \\ &\quad \cdot \int_{\mathbb{R}^N} |u_1|^{p/2} |u_2|^{p/2} dx \Big\} \\ &= \theta^{(2\alpha p - N(p-2))/(4\alpha - (p-2)N)} \tilde{I}_\beta. \end{aligned} \quad (25)$$

Therefore by Lemma 11(ii) and $(2\alpha p - N(p-2))/(4\alpha - (p-2)N) > 1$, we have that

$$\begin{aligned} \tilde{I}_{\gamma_1+\gamma_2} &= (\gamma_1 + \gamma_2)^{(2\alpha p - N(p-2))/(4\alpha - (p-2)N)} \tilde{I}_1 \\ &< (\gamma_1^{(2\alpha p - N(p-2))/(4\alpha - (p-2)N)} \\ &\quad + \gamma_2^{(2\alpha p - N(p-2))/(4\alpha - (p-2)N)}) \tilde{I}_1 = \tilde{I}_{\gamma_1} + \tilde{I}_{\gamma_2}. \end{aligned} \quad (26)$$

$\square$

Lemma 14. Assuming $q > 0$ and $0 < \beta < q$, then $\tilde{I}_q \geq \tilde{I}_\beta + \tilde{I}_{q-\beta}$.

Proof. For given minimizing sequence $\{(u_{1n}, u_{2n})\}$ for problem $(\tilde{I}_q)$, $r > 0$, define $Q_n(r) = \sup_{y \in \mathbb{R}^N} \int_{B(y,r)} (|u_{1n}|^2 + |u_{2n}|^2) dx$. Then Q_n is nondecreasing on $(0, +\infty)$; moreover for all $r > 0$, $\{Q_n\}$ has subsequences denoted again by $\{Q_n\}$, existing nondecreasing function $Q(r)$ such that $\lim_{n \rightarrow +\infty} Q_n(r) = Q(r)$, $\lim_{r \rightarrow +\infty} Q(r) = \beta$. Therefore $\forall \varepsilon > 0$, $\exists r_\varepsilon, n_\varepsilon$ such that $r > r_\varepsilon, n > n_\varepsilon$ satisfy

$$\begin{aligned} \beta - \frac{\varepsilon}{2} &< Q(r) \leq Q(2r) \leq \beta, \\ \beta - \frac{3\varepsilon}{4} &< Q_n(r) \leq Q_n(2r) \leq \beta + \frac{3\varepsilon}{4}. \end{aligned} \quad (27)$$

Apparently, there exists $\{y_n\} \subset \mathbb{R}^N$, satisfying

$$\begin{aligned} \int_{B(y_n, r)} (|u_{1n}|^2 + |u_{2n}|^2) dx &> \beta - \varepsilon, \\ \int_{B(y_n, 2r)} (|u_{1n}|^2 + |u_{2n}|^2) dx &< \beta + \varepsilon, \end{aligned} \quad (28)$$

where because of definition of Q_n and (27).

Setting $v_{in} = \Psi_{ir}(x - y_n) u_{in}$, $w_{in} = \tilde{\Psi}_{ir}(x - y_n) u_{in}$, $i = 1, 2$, they satisfy that $\Psi_{ir}(x) = \Psi_i(x/r)$, $\tilde{\Psi}_{ir}(x) = \tilde{\Psi}_i(x/r)$; $\forall x \in \bar{B}(0, 1)$, $\Psi_i(x) = 1$; for $|x| \geq 2$, $\Psi_i(x) = 0$; $\Psi_i(x) \in C_0^\infty(B(0, 2))$, $\tilde{\Psi}_i \in C^\infty(\mathbb{R}^N)$; and $\forall x \in \mathbb{R}^N$, $\Psi_i^2(x) + \tilde{\Psi}_i^2(x) = 1$. Then we have that

$$\beta - \varepsilon < \int_{\mathbb{R}^N} (|v_{1n}|^2 + |v_{2n}|^2) dx < \beta + \varepsilon, \quad (29)$$

$$q - \beta - \varepsilon < \int_{\mathbb{R}^N} (|w_{1n}|^2 + |w_{2n}|^2) dx < q - \beta + \varepsilon.$$

Therefore (29) imply that there exist (μ_{1n}, μ_{2n}) and (v_{1n}, v_{2n}) , $|\mu_{in} - 1| < \varepsilon, |v_{in} - 1| < \varepsilon$, $i = 1, 2$, satisfying $|\mu_{1n} v_{1n}|_2^2 + |\mu_{2n} v_{2n}|_2^2 = \beta$, $|v_{1n} w_{1n}|_2^2 + |v_{2n} w_{2n}|_2^2 = q - \beta$. Then we have that

$$\tilde{I}_\beta \leq \tilde{I}(\mu_{1n} v_{1n}, \mu_{2n} v_{2n}) \leq \tilde{I}(v_{1n}, v_{2n}) + C\varepsilon, \quad (30)$$

$$\tilde{I}_{q-\beta} \leq \tilde{I}(v_{1n} w_{1n}, v_{2n} w_{2n}) \leq \tilde{I}(w_{1n}, w_{2n}) + C\varepsilon.$$

Using Lemma 7, we derive that

$$\begin{aligned} &|(-\Delta)^{\alpha/2} (\Psi_{ir}(x - y_n) u_{in}) - \Psi_{ir}(x - y_n) (-\Delta)^{\alpha/2} u_{in}|_2 \\ &\leq C \left(|\nabla \Psi_{ir}|_\infty |(-\Delta)^{(\alpha-1)/2} u_{in}|_2 \right. \\ &\quad \left. + |(-\Delta)^{\alpha/2} \Psi_{ir}(x - y_n)|_{\tilde{p}} |u_{in}|_p \right), \end{aligned} \quad (31)$$

where $1/\tilde{p} + 1/p = 1/2$, $i = 1, 2$. By $|\nabla \Psi_{ir}|_\infty = (1/r) |\nabla \Psi_i|_\infty \leq C/r$, the Sobolev inequality, Lemma 11, and (31), we obtain that

$$\begin{aligned} &\frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2} (\Psi_{ir}(x - y_n) u_{in})|^2 dx \\ &\leq \frac{1}{2} \int_{\mathbb{R}^N} \Psi_{ir}^2(x - y_n) |(-\Delta)^{\alpha/2} u_{in}|^2 dx + C\varepsilon, \end{aligned} \quad (32)$$

$i = 1, 2$,

for sufficiently large r . Similarly, for sufficiently large r we have that

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}^N} \left| (-\Delta)^{\alpha/2} (\widetilde{\Psi}_{ir}(x - y_n) u_{in}) \right|^2 dx \\ & \leq \frac{1}{2} \int_{\mathbb{R}^N} \widetilde{\Psi}_{ir}^2(x - y_n) \left| (-\Delta)^{\alpha/2} u_{in} \right|^2 dx + C\varepsilon, \end{aligned} \quad (33)$$

$i = 1, 2.$

So we have that

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}^N} \left(\left| (-\Delta)^{\alpha/2} (\Psi_{1r}(x - y_n) u_{1n}) \right|^2 \right. \\ & \quad \left. + \left| (-\Delta)^{\alpha/2} (\Psi_{2r}(x - y_n) u_{2n}) \right|^2 \right) dx + \frac{1}{2} \\ & \quad \cdot \int_{\mathbb{R}^N} \left(\left| (-\Delta)^{\alpha/2} (\widetilde{\Psi}_{1r}(x - y_n) u_{1n}) \right|^2 \right. \\ & \quad \left. + \left| (-\Delta)^{\alpha/2} (\widetilde{\Psi}_{2r}(x - y_n) u_{2n}) \right|^2 \right) dx \leq \frac{1}{2} \\ & \quad \cdot \int_{\mathbb{R}^N} \left(\left| (-\Delta)^{\alpha/2} u_{1n} \right|^2 + \left| (-\Delta)^{\alpha/2} u_{2n} \right|^2 \right) dx \\ & \quad + C\varepsilon. \end{aligned} \quad (34)$$

By Lemma 11, (16), (27), $2 < p < 2 + 4\alpha/N$, and the definitions of $\Psi_{ir}, \widetilde{\Psi}_{ir}$ ($i = 1, 2$), we can derive that

$$\begin{aligned} & -\frac{1}{p} \int_{\mathbb{R}^N} (|\Psi_{1r}(x - y_n) u_{1n}|^p + |\Psi_{2r}(x - y_n) u_{2n}|^p) dx \\ & + \frac{1}{p} \int_{\mathbb{R}^N} (|u_{1n}|^p + |u_{2n}|^p) dx - \frac{1}{p} \\ & \cdot \int_{\mathbb{R}^N} (|\widetilde{\Psi}_{1r}(x - y_n) u_{1n}|^p \\ & + |\widetilde{\Psi}_{2r}(x - y_n) u_{2n}|^p) dx = -\frac{1}{p} \\ & \cdot \int_{\mathbb{R}^N} (|\Psi_{1r}(x - y_n) u_{1n}|^p \\ & + |\Psi_{2r}(x - y_n) u_{2n}|^p) dx + \frac{1}{p} \\ & \cdot \int_{\mathbb{R}^N} (\Psi_{1r}^2(x - y_n) + \widetilde{\Psi}_{1r}^2(x - y_n)) |u_{1n}|^p dx \\ & + \frac{1}{p} \int_{\mathbb{R}^N} (\Psi_{2r}^2(x - y_n) + \widetilde{\Psi}_{2r}^2(x - y_n)) |u_{2n}|^p dx \\ & - \frac{1}{p} \int_{\mathbb{R}^N} (|\widetilde{\Psi}_{1r}(x - y_n) u_{1n}|^p \\ & + |\widetilde{\Psi}_{2r}(x - y_n) u_{2n}|^p) dx = \frac{1}{p} \int_{\mathbb{R}^N} (-|\Psi_{1r}|^p \\ & + \Psi_{1r}^2)(x - y_n) |u_{1n}(x)|^p dx + \frac{1}{p} \int_{\mathbb{R}^N} (-|\Psi_{2r}|^p \end{aligned}$$

$$\begin{aligned} & + \Psi_{2r}^2)(x - y_n) |u_{2n}(x)|^p dx + \frac{1}{p} \int_{\mathbb{R}^N} (-|\widetilde{\Psi}_{1r}|^p \\ & + \widetilde{\Psi}_{1r}^2)(x - y_n) |u_{1n}(x)|^p dx + \frac{1}{p} \int_{\mathbb{R}^N} (-|\widetilde{\Psi}_{2r}|^p \\ & + \widetilde{\Psi}_{2r}^2)(x - y_n) |u_{2n}(x)|^p dx \leq \frac{4}{p} \int_{\Omega} (|u_{1n}(x)|^p \\ & + |u_{2n}(x)|^p) dx \leq \frac{C}{p} \left(\|u_{1n}\|_{L^2(\Omega)}^{p(1-\lambda)} \right. \\ & \cdot \|u_{1n}\|_{W^{\alpha,2}(\Omega)}^{\lambda p} + \|u_{2n}\|_{L^2(\Omega)}^{p(1-\lambda)} \|u_{2n}\|_{W^{\alpha,2}(\Omega)}^{\lambda p} \Big), \\ & - \int_{\mathbb{R}^N} |\Psi_{1r}(x - y_n) u_{1n}|^{p/2} |\Psi_{2r}(x - y_n) u_{2n}|^{p/2} dx \\ & - \int_{\mathbb{R}^N} |\widetilde{\Psi}_{1r}(x - y_n) u_{1n}|^{p/2} \\ & \cdot |\widetilde{\Psi}_{2r}(x - y_n) u_{2n}|^{p/2} dx + \int_{\mathbb{R}^N} |u_{1n}|^{p/2} \\ & \cdot |u_{2n}|^{p/2} dx = \int_{\mathbb{R}^N} \left(1 \right. \\ & - |\Psi_{1r}(x - y_n)|^{p/2} |\Psi_{2r}(x - y_n)|^{p/2} \\ & - |\widetilde{\Psi}_{1r}(x - y_n)|^{p/2} |\widetilde{\Psi}_{2r}(x - y_n)|^{p/2} \Big) |u_1|^{p/2} \\ & \cdot |u_2|^{p/2} dx \leq \frac{3}{p} \int_{\Omega} |u_{1n}|^{p/2} |u_{2n}|^{p/2} dx \\ & \leq \frac{3}{2p} \left(\int_{\Omega} |u_{1n}|^p dx + \int_{\Omega} |u_{2n}|^p dx \right) \\ & \leq \frac{C}{p} \left(\|u_{1n}\|_{L^2(\Omega)}^{p(1-\lambda)} \|u_{1n}\|_{W^{\alpha,2}(\Omega)}^{\lambda p} + \|u_{2n}\|_{L^2(\Omega)}^{p(1-\lambda)} \right. \\ & \cdot \|u_{2n}\|_{W^{\alpha,2}(\Omega)}^{\lambda p} \Big) \leq C\varepsilon, \end{aligned} \quad (35)$$

where $\Omega = B(y_n, 2r) - B(y_n, r)$ and $\lambda = N(p - 2)/2\alpha p$. Therefore by the two inequations, we obtain that

$$\begin{aligned} & \tilde{I}(v_{1n}, v_{2n}) + \tilde{I}(w_{1n}, w_{2n}) \\ & \leq \frac{1}{2} \int_{\mathbb{R}^N} \left(\left| (-\Delta)^{\alpha/2} u_{1n} \right|^2 + \left| (-\Delta)^{\alpha/2} u_{2n} \right|^2 \right) dx \\ & - \frac{1}{p} \int_{\mathbb{R}^N} (|u_{1n}|^p + |u_{2n}|^p) dx \\ & - \frac{2}{p} \int_{\mathbb{R}^N} |u_{1n}|^{p/2} |u_{2n}|^{p/2} dx + C\varepsilon \\ & = \tilde{I}(u_{1n}, u_{2n}) + C\varepsilon. \end{aligned} \quad (36)$$

By (30) and (36), we obtain

$$\begin{aligned} \tilde{I}_\beta + \tilde{I}_{q-\beta} - 3C\varepsilon & \leq \tilde{I}(v_{1n}, v_{2n}) + \tilde{I}(w_{1n}, w_{2n}) - C\varepsilon \\ & \leq \tilde{I}(u_{1n}, u_{2n}) \longrightarrow \tilde{I}_q. \end{aligned} \quad (37)$$

So, we prove that $\tilde{I}_q \geq \tilde{I}_\beta + \tilde{I}_{q-\beta}$. $\square$

Obviously, using Lemmas 11–14, we obtain $\beta = q$.

Proof of Theorem 9. From the above parts; we know that $\beta = q$, that is, there exists a subsequence $\{(u_{1m}, u_{2m})\}$ of $\{(u_{1n}, u_{2n})\}$ satisfying

$$\lim_{r \rightarrow +\infty} \lim_{m \rightarrow +\infty} \sup_{y \in \mathbb{R}^N} \int_{B(y, r)} (|u_{1m}|^2 + |u_{2m}|^2) dx = \beta = q. \quad (38)$$

Therefore, for all $k \in \mathbb{N}$, there exist $r_k > 0$, $M_k \in \mathbb{N}$, such that, $\forall m > M_k, \exists y_m \in \mathbb{R}^N$, having

$$\int_{B(y_m, r_k)} (|u_{1m}|^2 + |u_{2m}|^2) dx > q - \frac{1}{k}. \quad (39)$$

For the above inequality, with a Cantor diagonalization argument, using Lemma 11(ii), $|u_{1m}|_2^2 + |u_{2m}|_2^2 = q$, we know that there exists $(g_1, g_2) \in H^\alpha(\mathbb{R}^N) \times H^\alpha(\mathbb{R}^N)$ satisfying $\int_{\mathbb{R}^N} (|g_1|^2 + |g_2|^2) dx = q$, such that going if necessary to exist a subsequence $\{(u_{1j}(\cdot + y_j), u_{2j}(\cdot + y_j))\}$ of $\{(u_{1m}(\cdot + y_m), u_{2m}(\cdot + y_m))\}$ satisfying

$$\begin{aligned} (u_{1j}(\cdot + y_j), u_{2j}(\cdot + y_j)) &\rightharpoonup (g_1, g_2) \\ &\text{in } H^\alpha(\mathbb{R}^N) \times H^\alpha(\mathbb{R}^N), \\ (u_{1j}(\cdot + y_j), u_{2j}(\cdot + y_j)) &\longrightarrow (g_1, g_2) \\ &\text{in } L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N). \end{aligned} \quad (40)$$

Then by the interpolation theorem, we obtain that

$$\begin{aligned} (u_{1j}(\cdot + y_j), u_{2j}(\cdot + y_j)) &\longrightarrow (g_1, g_2) \\ &\text{in } L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N). \end{aligned} \quad (41)$$

And by the weak low semicontinuity of the norm in $H^\alpha(\mathbb{R}^N)$, we also obtain that

$$\begin{aligned} (u_{1j}(\cdot + y_j), u_{2j}(\cdot + y_j)) &\longrightarrow (g_1, g_2) \\ &\text{in } H^\alpha(\mathbb{R}^N) \times H^\alpha(\mathbb{R}^N), \end{aligned} \quad (42)$$

and (g_1, g_2) is a minimizer of problem $(\tilde{I}_q)$.

Using an argument by contradiction, we prove

$$\lim_{n \rightarrow +\infty} \inf_{(g_1, g_2) \in \tilde{G}_q} (\|u_{1n} - g_1\| + \|u_{2n} - g_2\|) = 0. \quad (43)$$

Assume that there exist $\varepsilon_0 > 0$ and a subsequence $\{(u_{1n_j}, u_{2n_j})\}$ of $\{(u_{1n}, u_{2n})\}$ satisfying

$$\inf_{(g_1, g_2) \in \tilde{G}_q} (\|u_{1n_j} - g_1\| + \|u_{2n_j} - g_2\|) \geq \varepsilon_0 > 0. \quad (44)$$

By the above proof, we know that there exist a subsequence $\{(u_{1j}, u_{2j})\}$ of $\{(u_{1n_j}, u_{2n_j})\}$, $\{y_j\} \in \mathbb{R}^N$, and $(g_1, g_2) \in \tilde{G}_q$ satisfying

$$\begin{aligned} (u_{1j}(\cdot + y_j), u_{2j}(\cdot + y_j)) &\longrightarrow (g_1, g_2) \\ &\text{in } H^\alpha(\mathbb{R}^N) \times H^\alpha(\mathbb{R}^N). \end{aligned} \quad (45)$$

So we derive

$$\begin{aligned} \lim_{j \rightarrow +\infty} (\|u_{1j}(\cdot + y_j) - g_1\| + \|u_{2j}(\cdot + y_j) - g_2\|) \\ = 0, \end{aligned} \quad (46)$$

where since $(g_1(\cdot - y_j), g_2(\cdot - y_j)) \in \tilde{G}_q$, we have that

$$\begin{aligned} \lim_{j \rightarrow +\infty} \inf_{(g_1, g_2) \in \tilde{G}_q} (\|u_{1j}(\cdot + y_j) - g_1\| \\ + \|u_{2j}(\cdot + y_j) - g_2\|) = 0. \end{aligned} \quad (47)$$

Then there exists a contradiction. We completed the proof of Theorem 9. $\square$

Proof of Theorem 10. Now using an argument by contradiction, we complete the proof of Theorem 10. Assume that the set $\tilde{G}_q$ is not $H^\alpha(\mathbb{R}^N) \times H^\alpha(\mathbb{R}^N)$ -stable. Then there exist $\varepsilon_0 > 0$, $(u_{1k}^0, u_{2k}^0) \in H^\alpha(\mathbb{R}^N) \times H^\alpha(\mathbb{R}^N)$, and $t_k \in [0, T)$ such that

$$\inf_{(g_1, g_2) \in \tilde{G}_q} (\|u_{1k}^0 - g_1\| + \|u_{2k}^0 - g_2\|) \leq \frac{1}{k}, \quad k \in \mathbb{N}, \quad (48)$$

and for $(u_{1k}, u_{2k}) \in C([0, T], H^\alpha(\mathbb{R}^N) \times H^\alpha(\mathbb{R}^N))$ satisfying (1) with initial value (u_{1k}^0, u_{2k}^0) , we have

$$\inf_{(g_1, g_2) \in \tilde{G}_q} (\|u_{1k}(t_k, \cdot) - g_1\| + \|u_{2k}(t_k, \cdot) - g_2\|) \geq \varepsilon_0. \quad (49)$$

By (48), we know that

$$\lim_{k \rightarrow +\infty} \tilde{I}(u_{1k}^0, u_{2k}^0) = \tilde{I}_q, \quad (50)$$

$$\lim_{k \rightarrow +\infty} |u_{1k}^0|_2^2 + |u_{2k}^0|_2^2 = q.$$

Then there exist $\{\mu_{1k}\}, \{\mu_{2k}\} \in \mathbb{R}$ satisfying $\lim_{k \rightarrow +\infty} \mu_{1k} = 1$ and $\lim_{k \rightarrow +\infty} \mu_{2k} = 1$ such that $\{(\mu_{1k} u_{1k}^0, \mu_{2k} u_{2k}^0)\}$ is a minimizing sequence of the problem $(\tilde{I}_q)$. Since $\tilde{I}(u_{1k}^0, u_{2k}^0) = \tilde{I}(u_{1k}(t_k), u_{2k}(t_k))$, $|u_{1k}^0|_2^2 = |u_{1k}(t_k)|_2^2$ and $|u_{2k}^0|_2^2 = |u_{2k}(t_k)|_2^2$, we easily derive that $\{(\mu_{1k} u_{1k}(t_k), \mu_{2k} u_{2k}(t_k))\}$ is a minimizing sequence of the problem $(\tilde{I}_q)$. By Theorem 9, there exists $\{(g_{1k_m}, g_{2k_m})\} \subset \tilde{G}_q$, for sufficiently large k_m satisfying

$$\begin{aligned} \|\mu_{1k_m} u_{1k_m}(t_{k_m}) - g_{1k_m}\| + \|\mu_{2k_m} u_{2k_m}(t_{k_m}) - g_{2k_m}\| \\ < \frac{\varepsilon_0}{2}. \end{aligned} \quad (51)$$

Therefore by Lemma 11(ii), (49), $\lim_{k \rightarrow +\infty} \mu_{1k} = 1$, $\lim_{k \rightarrow +\infty} \mu_{2k} = 1$, and the above inequality, for sufficiently large k_m , we derive that

$$\begin{aligned} \varepsilon_0 &\leq \|u_{1k_m}(t_{k_m}) - g_{1k_m}\| + \|u_{2k_m}(t_{k_m}) - g_{2k_m}\| \\ &\leq \|u_{1k_m}(t_{k_m}) - \mu_{1k_m} u_{1k_m}(t_{k_m})\| \\ &\quad + \|\mu_{1k_m} u_{1k_m}(t_{k_m}) - g_{1k_m}\| \\ &\quad + \|u_{2k_m}(t_{k_m}) - \mu_{2k_m} u_{2k_m}(t_{k_m})\| \\ &\quad + \|\mu_{2k_m} u_{2k_m}(t_{k_m}) - g_{2k_m}\| \leq \frac{3\varepsilon_0}{4}. \end{aligned} \quad (52)$$

Then obtaining a contradiction, the set $\tilde{G}_q$ is $H^\alpha(\mathbb{R}^N) \times H^\alpha(\mathbb{R}^N)$ -stable with respect to (1). The proof is completed. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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