Research Article

Positive Solutions for Systems of Second-Order Difference Equations

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We study the existence and nonexistence of positive solutions of some systems of nonlinear second-order difference equations subject to multipoint boundary conditions which contain some positive constants.

1. Introduction

The mathematical modeling of many nonlinear problems from computer science, economics, mechanical engineering, control systems, biological neural networks, and others leads to the consideration of nonlinear difference equations (see [1, 2]). In the last decades, many authors have investigated such problems by using various methods, such as fixed point theorems, the critical point theory, upper and lower solutions, the fixed point index theory, and the topological degree theory (see, e.g., [3–15]).

In this paper, we consider the system of nonlinear second-order difference equations:

\[ \Delta^2 u_{n-1} + s_n f (v_n) = 0, \quad n = 1, N - 1, \]
\[ \Delta^2 v_{n-1} + t_n g (u_n) = 0, \quad n = 1, N - 1, \]  

with the multipoint boundary conditions

\[ u_0 = \sum_{i=1}^{p} a_i u_{\xi_i} + a_0, \]
\[ u_N = \sum_{i=1}^{q} b_i v_{\eta_i}, \]

\[ v_0 = \sum_{i=1}^{l} \zeta_i v_{\rho_i}, \]
\[ v_N = \sum_{i=1}^{l} d_i v_{\rho_i} + b_0, \]

(BC)

where \( N, p, q, r, l \in \mathbb{N}, N \geq 2, \Delta \) is the forward difference operator with stepsize 1, \( \Delta u_n = u_{n+1} - u_n, \Delta^2 u_{n-1} = u_{n+1} - 2u_n + u_{n-1}, n = k, m \) means that \( n = k, k+1, \ldots, m \) for \( k, m \in \mathbb{N} \), \( a_i \in \mathbb{R} \) for all \( i = 1, p, b_i \in \mathbb{R} \) for all \( i = 1, l, c_i \in \mathbb{R} \) for all \( i = 1, l, d_i \in \mathbb{R} \) for all \( i = 1, l, e_i \in \mathbb{R} \) for all \( i = 1, l, f_i \in \mathbb{R} \) for all \( i = 1, l, \xi_i, \eta_i \in \mathbb{N} \) for all \( i = 1, r, \gamma_i \in \mathbb{R} \) for all \( i = 1, l, \delta_i > 0, \epsilon_i \leq \frac{n}{N} \), \( 1 \leq \xi_1 < \cdots < \xi_p \leq N - 1, 1 \leq \eta_1 < \cdots < \eta_q \leq N - 1, 1 \leq \zeta_1 < \cdots < \zeta_r \leq N - 1, 1 \leq \rho_1 < \cdots < \rho_l \leq N - 1 \) and \( a_0 \) and \( b_0 \) are positive constants.

Under some assumptions on the functions \( f \) and \( g \), we will prove the existence of positive solutions of problem (S)-(BC). By a positive solution of (S)-(BC) we mean a pair of sequences \( (u_n)_{n=0}^{N}, (v_n)_{n=0}^{N} \) satisfying (S) and (BC) with \( u_n, v_n \geq 0 \) for all \( n = 0, N \) and \( u_n > 0 \) for all \( n = 0, N - 1 \), \( v_n > 0 \) for all \( n = 1, N \). We will also give sufficient conditions for the nonexistence of positive solutions for this problem.

System (S) with the multipoint boundary conditions \( \alpha u_0 - \beta \Delta u_0 = 0, u_N = \sum_{i=1}^{m-2} a_i u_{\xi_i} + a_0, v_0 - \delta \Delta v_0 = 0, \) and
\[ \nu_N = \sum_{i=1}^{p-1} b_i \eta_i + b_0 \quad (a_0, b_0 > 0) \]

has been investigated in [16]. Some systems of difference equations, with parameters, subject to multipoint boundary conditions, were studied in [17, 18], by using the Guo-Krasnosel'skii fixed point theorem. We also mention the paper [19], where the authors investigated the existence and multiplicity of positive solutions for the system \( \Delta^2 u_{n-1} + f(n,u_n) = 0, \Delta^2 v_{n-1} + g(n,v_n) = 0 \), \( n = 1, N-1 \), with the multipoint boundary conditions (BC) with \( a_k = b_k = 0 \), by using some theorems from the fixed point index theory.

In Section 2, we present some auxiliary results which investigate a second-order difference equation subject to multipoint boundary conditions. In Section 3, we will prove our main results, and in Section 4, we will present an example which illustrates the obtained theorems. Our main existence result is based on the Schauder fixed point theorem which we present now.

**Theorem 1.** Let \( X \) be a Banach space and \( Y \subset X \) a nonempty, bounded, convex, and closed subset. If the operator \( A : Y \rightarrow Y \) is completely continuous (continuous and compact, i.e., mapping bounded sets into relatively compact sets), then \( A \) has at least one fixed point.

### 2. Auxiliary Results

In this section, we present some auxiliary results from [17] related to the following second-order difference equation with the multipoint boundary conditions:

\[ \Delta^2 u_{n-1} + y_n = 0, \quad n = 1, N-1, \]

\[ u_0 = \sum_{i=1}^{p} a_i u_i, \quad u_N = \sum_{i=1}^{p} b_i u_i, \quad (1) \]

where \( N, p, q \in \mathbb{N}, N \geq 2, a_i \in \mathbb{R} \) for all \( i = 1, p, b_i \in \mathbb{R} \) for all \( i = 1, q, \xi_i \in \mathbb{R} \) for all \( i = 1, p, \eta_i \in \mathbb{N} \) for all \( i = 1, q, 1 \leq \xi_1 < \cdots < \xi_p \leq N - 1, \text{ and } 1 \leq \eta_1 < \cdots < \eta_q \leq N - 1 \).

**Lemma 2** (see [17]). If \( a_i \in \mathbb{R} \) for all \( i = 1, p, b_i \in \mathbb{R} \) for all \( i = 1, q, \xi_i \in \mathbb{N} \) for all \( i = 1, p, \eta_i \in \mathbb{N} \) for all \( i = 1, q, 1 \leq \xi_1 < \cdots < \xi_p \leq N - 1, 1 \leq \eta_1 < \cdots < \eta_q \leq N - 1, \Delta_1 = (1 - \sum_{i=1}^{q} b_i) \sum_{i=1}^{p} a_i \xi_i + (1 - \sum_{i=1}^{p} a_i)(N - \sum_{i=1}^{q} b_i \eta_i) \neq 0, \) and \( y_n \in \mathbb{R} \) for all \( n = 1, N-1, \) then the solution of (1) is given by \( u_n = \sum_{i=1}^{N-1} G_1(n,j) y_j \) for all \( n = 0, N, \) where Green's function \( G_1 \) is defined by

\[
G_1(n,j) = g_0(n,j) + \frac{1}{\Delta_1} \left[ (N-n) \left( 1 - \sum_{k=1}^{q} b_k \right) + \sum_{k=1}^{q} b_k (N-\eta_k) \right] \cdot \sum_{i=1}^{p} a_i g_0(\xi_i,j) + \frac{1}{\Delta_1} \left[ n \left( 1 - \sum_{k=1}^{p} a_k \right) + \sum_{k=1}^{p} a_k \xi_k \right] \cdot \sum_{i=1}^{q} b_i g_0(\eta_i,j), \quad n = 0, N, \quad j = 1, N-1.
\]

**Lemma 3** (see [17]). If \( a_i \geq 0 \) for all \( i = 1, p, b_i \geq 0 \) for all \( i = 1, q, \xi_i \leq 1, \eta_i \in \mathbb{N} \) for all \( i = 1, q, 1 \leq \xi_1 < \cdots < \xi_p \leq N - 1, \text{ and } 1 \leq \eta_1 < \cdots < \eta_q \leq N - 1, \) then Green's function \( G_1 \) of problem (1) satisfies \( G_1(n,j) \geq 0 \) for all \( n = 0, N, j = 1, N-1. \) Moreover, if \( y_n \geq 0 \) for all \( n = 1, N-1, \) then the unique solution \( u_n, n = 0, N, \) of problem (1) satisfies \( u_n \geq 0 \) for all \( n = 0, N. \)

**Lemma 4** (see [17]). Assume that \( a_i \geq 0 \) for all \( i = 1, p, b_i \geq 0 \) for all \( i = 1, q, \xi_i \leq 1, \eta_i \in \mathbb{N} \) for all \( i = 1, q, 1 \leq \xi_1 < \cdots < \xi_p \leq N - 1, \text{ and } 1 \leq \eta_1 < \cdots < \eta_q \leq N - 1. \) Then Green's function \( G_1 \) of problem (1) satisfies the following inequalities:

\[
G_1(n,j) \leq \sum_{i=1}^{q} b_i g_0(\eta_i,j), \quad n = 0, N, \quad j = 1, N-1.
\]

**Lemma 5** (see [17]). Assume that \( a_i \geq 0 \) for all \( i = 1, p, b_i \geq 0 \) for all \( i = 1, q, \xi_i \leq 1, \eta_i \in \mathbb{N} \) for all \( i = 1, q, 1 \leq \xi_1 < \cdots < \xi_p \leq N - 1, \text{ and } 1 \leq \eta_1 < \cdots < \eta_q \leq N - 1. \) Then Green's function \( G_1 \) of problem (1) satisfies the inequality \( \min_{n=N/2} G_1(n,j) \geq \gamma_1 G_1(n',j), \) \( \forall n' \in \mathbb{N}, j = 1, N-1, \)

where

\[
\gamma_1 = \min \left\{ \frac{c}{N-1}, \frac{c(1 - \sum_{k=1}^{q} b_k) + \sum_{k=1}^{q} b_k (N-\eta_k)}{N - \sum_{k=1}^{q} b_k \eta_k}, \frac{c(1 - \sum_{i=1}^{p} a_i) + \sum_{i=1}^{p} a_i \xi_i}{N - \sum_{i=1}^{p} a_i (N-\xi_i)} \right\} > 0,
\]

and \( \lceil N/2 \rceil \) is the largest integer not greater than \( N/2. \)
Discrete Dynamics in Nature and Society

We can also formulate similar results as Lemmas 2–5 above for the discrete boundary value problem

\[
\Delta^2 v_{n-1} + h_n = 0, \quad n = 1, N - 1,
\]

\[
v_0 = \sum_{i=1}^{r} c_i v_{\xi_i},
\]

\[
v_N = \sum_{i=1}^{l} d_i v_{\rho_i},
\]

where \( N, r, l \in \mathbb{N}, N \geq 2, c_i \in \mathbb{R} \) for all \( i = 1, r, d_i \in \mathbb{R} \) for all \( i = 1, l \), \( 1 \leq \xi_i < \cdots < \xi_r \leq N - 1, a_i \geq 0 \) for all \( i = 1, q \), \( 1 \leq \eta_i < \cdots < \eta_1 \leq N - 1, b_i \geq 0 \) for all \( i = 1, q \), \( \sum_{i=1}^{\eta_i} b_i < 1, \xi_i \in \mathbb{N} \) for all \( i = 1, r \), \( 1 \leq \varsigma_i < \cdots < \varsigma_r \leq N - 1, c_i \geq 0 \) for all \( i = 1, q \), \( \sum_{i=1}^{\eta_i} c_i < 1, \rho_i \in \mathbb{N} \) for all \( i = 1, q \), \( 1 \leq \varsigma_i < \cdots < \varsigma_r \leq N - 1, d_i \geq 0 \) for all \( i = 1, q \).

3. Main Results

We present first the assumptions that we will use in the sequel:

(H1) \( \xi_i \in \mathbb{N} \) for all \( i = 1, q \), \( 1 \leq \xi_i < \cdots < \xi_q \leq N - 1, a_i \geq 0 \) for all \( i = 1, q \),

(H2) The constants \( s_n, t_n \geq 0 \) for all \( n = 1, N - 1 \), and there exist \( i_0, j_0 \in \{1, \ldots, N - 1\} \) such that \( s_{i_0} > 0, t_{j_0} > 0 \).

(H3) \( f, g : [0, \infty) \to [0, \infty) \) are continuous functions and there exists \( c_0 > 0 \) such that \( f(u) < c_0/L \), \( g(u) < c_0/L \) for all \( u \in [0, c_0] \), where \( L = \max \{ \sum_{i=1}^{N-1} s_i \}, \sum_{i=1}^{N-1} t_i \} \) and \( I_1, I_2 \) are defined in Section 2.

(H4) \( f, g : [0, \infty) \to [0, \infty) \) are continuous functions and satisfy the conditions \( \lim_{u \to \infty} (f(u)/u) = c_0 \), \( \lim_{u \to \infty} (g(u)/u) = c_0 \).

Our first theorem is the following result for problem (S)-(BC).

**Theorem 6.** Assume that assumptions (H1)-(H3) hold. Then problem (S)-(BC) has at least one positive solution for \( a_0 > 0 \) and \( b_0 > 0 \) sufficiently small.

Proof. We consider the problems

\[
\Delta^2 h_{n-1} = 0, \quad n = 1, N - 1,
\]

\[
h_0 = \sum_{i=1}^{p} a_i h_{\xi_i} + 1,
\]

\[
h_N = \sum_{i=1}^{q} b_i h_{\rho_i},
\]

Problems (7) and (8) have the solutions

\[
h_n = \frac{1}{\Delta_1} \left[ -n \left( 1 - \sum_{i=1}^{q} b_i \right) + \left( N - \sum_{i=1}^{q} b_i \right) \right], \quad n = 0, N,
\]

\[
k_n = \frac{1}{\Delta_2} \left[ n \left( 1 - \sum_{i=1}^{r} c_i \right) + \sum_{i=1}^{r} c_i \right], \quad n = 0, N,
\]

respectively, where \( \Delta_1 \) and \( \Delta_2 \) are defined in Section 2. By assumption (H1) we obtain \( h_n > 0 \) for all \( n = 0, N - 1 \) and \( k_n > 0 \) for all \( n = 1, N \).

We define the sequences \( (x_n)_{n=0}^{N}, (y_n)_{n=0}^{N} \) by

\[
x_n = u_n - a_0 h_n,
\]

\[
y_n = v_n - b_0 k_n,
\]

where \( ((u_n)_{n=0}^{N}, (v_n)_{n=0}^{N}) \) is a solution of (S)-(BC). Then (S)-(BC) can be equivalently written as

\[
\Delta^2 x_{n-1} + s_n f (x_n + b_0 k_n) = 0, \quad n = 1, N - 1,
\]

\[
\Delta^2 y_{n-1} + t_n g (x_n + a_0 h_n) = 0, \quad n = 1, N - 1,
\]

with the boundary conditions

\[
x_0 = \sum_{i=1}^{p} a_i x_{\xi_i},
\]

\[
x_N = \sum_{i=1}^{q} b_i x_{\rho_i},
\]

(12)
Using Green's functions $G_1$ and $G_2$ from Section 2, a pair $(x_n)_{n=0}^{N}$, $(y_n)_{n=0}^{N}$ is a solution of problem (11)-(12) if and only if it is a solution for the problem

$$x_n = \sum_{i=1}^{N-1} G_1 (n, i) \cdot s_i f \left( \sum_{j=1}^{N-1} G_2 (i, j) t_j g (x_j + a_0 h_j) + b_0 k_i \right),$$

$$y_n = \sum_{i=1}^{N-1} G_2 (n, i) t_i g (x_i + a_0 h_i), \quad n = 0, N,$$

where $(h_n)_{n=0}^{N}$, $(k_n)_{n=0}^{N}$ are given in (9).

We consider the Banach space $X = \mathbb{R}^{N+1}$ with the norm $||u|| = \max_{n=0, N} |u_n|$, $u = (u_n)_{n=0}^{N}$, and we define the set $M = \{(x_n)_{n=0}^{N} \in \mathbb{R}^{N+1}, 0 \leq x_n \leq c_0, \quad \forall n = 0, N \} \subset X$.

We also define the operator $\mathcal{E} : M \to X$ by

$$\mathcal{E}(x) = \left( \sum_{i=1}^{N-1} G_1 (n, i) \cdot s_i f \left( \sum_{j=1}^{N-1} G_2 (i, j) t_j g (x_j + a_0 h_j) + b_0 k_i \right) \right)_{n=0, N}.$$

For sufficiently small $a_0 > 0$ and $b_0 > 0$, by (H3), we deduce

$$f (y_n + b_0 k_n) \leq \frac{c_0}{L}, \quad g (x_n + a_0 h_n) \leq \frac{c_0}{L},$$

$$\forall n = 0, N, \quad \forall (x_n)_{n=0}^{N}, (y_n)_{n=0}^{N} \in M.$$

Then, by using Lemma 3, we obtain $\mathcal{E}(x)_n \geq 0$ for all $n = 0, N$ and $x = (x_n)_{n=0}^{N} \in M$. By Lemma 4, for all $x \in M$, we have

$$\sum_{j=1}^{N-1} I_j (j) t_j g (x_j + a_0 h_j) 
\leq \sum_{j=1}^{N-1} I_j (j) t_j g (x_j + a_0 h_j) \leq \frac{c_0}{L} \sum_{j=1}^{N-1} I_j (j) \leq c_0,$$

$$\forall i = 1, N - 1,$$

$$\mathcal{E}(x)_n 
\leq \sum_{i=1}^{N-1} I_i (i) s_i f \left( \sum_{j=1}^{N-1} G_2 (i, j) t_j g (x_j + a_0 h_j) + b_0 k_i \right) 
\leq \frac{c_0}{L} \sum_{i=1}^{N-1} s_i I_i (i) \leq c_0, \quad \forall n = 0, N.$$

Therefore $\mathcal{E}(M) \subset M$.

Using standard arguments, we deduce that $\mathcal{E}$ is completely continuous. By Theorem 1, we conclude that $\mathcal{E}$ has a fixed point $x = (x_n)_{n=0}^{N} \in M$. This element together with $y = (y_n)_{n=0}^{N}$ given by (13) represents a solution for (11)-(12).

This shows that our problem (S)-(BC) has a positive solution $(u_n)_{n=0}^{N}$, $(v_n)_{n=0}^{N}$ with $u_n = x_n + a_0 h_n$, $v_n = y_n + b_0 k_n$, $n = 0, N$, $(u_n > 0$ for all $n = 0, N - 1$ and $v_n > 0$ for all $n = 1, N$) for sufficiently small $a_0 > 0$ and $b_0 > 0$.

In what follows, we present sufficient conditions for the nonexistence of positive solutions of (S)-(BC).

**Theorem 7.** Assume that assumptions (H1), (H2), and (H4) hold. Then problem (S)-(BC) has no positive solution for $a_0$ and $b_0$ sufficiently large.

**Proof.** We suppose that $((u_n)_{n=0}^{N}, (v_n)_{n=0}^{N})$ is a positive solution of (S)-(BC). Then $(x_n)_{n=0}^{N}, (y_n)_{n=0}^{N})$ with $x_n = u_n - a_0 h_n$, $y_n = v_n - b_0 k_n$, $n = 0, N$, is a solution for (11)-(12), where $(h_n)_{n=0}^{N}$ and $(k_n)_{n=0}^{N}$ are the solutions of problems (7) and (8), respectively (given by (9)). By (H2) there exists $c \in \{1, 2, \ldots, \lfloor N/2 \rfloor \}$ such that $i_j, j, k \in \{c, \ldots, N - c\}$ and then $\sum_{i=c}^{N-c} s_i I_i (i) > 0$ and $\sum_{i=c}^{N-c} t_i I_i (i) > 0$. By using Lemma 3 we have $x_n \geq 0$, $y_n \geq 0$ for all $n = 0, N$, and by Lemma 5, we obtain $\min_{n=0, N-c} |y_n| \geq y_1 \|x\|$ and $\min_{n=0, N-c} |y_n| \geq y_1 \|y\|$, where $y_1$ and $y_2$ are defined in Section 2.

Using now (9), we deduce that

$$\min_{n=0, N-c} h_n = h_{N-c} = \frac{h_{N-c}}{b_0} \|h\|,$$

$$\min_{n=0, N-c} k_n = k_c = k_c \|k\|.$$

Therefore, we obtain

$$\min_{n=0, N-c} (x_n + a_0 h_n) \geq y_1 \|x\| + \frac{a_0 h_{N-c}}{b_0} \|h\| \geq r_1 (\|x\| + a_0 \|h\|),$n$$

$$\min_{n=0, N-c} (y_n + b_0 k_n) \geq y_2 \|y\| + b_0 k_N \|k\| \geq r_2 (\|y\| + b_0 \|k\|),$$

where $r_1 = \min \{y_1, h_{N-c}/b_0\}$ and $r_2 = \min \{y_2, k_N/k_c\}$.

We now consider

$$R = \left( \min_{n=0, N-c} \left( y_1 r_1 \sum_{i=c}^{N-c} s_i I_i (i), y_1 r_2 \sum_{i=c}^{N-c} s_i I_i (i) \right) \right)^{-1} > 0.$$

By using (H4), for $R$ defined above, we conclude that there exists $M_0 > 0$ such that $f(u) > 2Ru$, $g(u) > 2Ru$ for
all \( u \geq M_0 \). We consider \( a_0 > 0 \) and \( b_0 > 0 \) sufficiently large such that

\[
\min_{n \in \mathbb{N}} (x_n + a_0 h_n) \geq M_0,
\]

\[
\min_{n \in \mathbb{N}} (y_n + b_0 k_n) \geq M_0. \tag{20}
\]

By (H2), (11), (12), and the above inequalities, we deduce that \( ||x|| > 0 \) and \( ||y|| > 0 \).

Now, by using Lemma 4 and the above considerations, we have

\[
y_x = \sum_{i=1}^{N-1} G_2 (c, i) \cdot t_i \cdot g(x_i + a_0 h_i)
\]

\[
\geq \gamma_2 \sum_{i=1}^{N-1} I_2 (i) \cdot t_i \cdot g(x_i + a_0 h_i)
\]

\[
\geq \gamma_2 \sum_{i=1}^{N-1} I_2 (i) \cdot t_i \cdot g(x_i + a_0 h_i)
\]

\[
\geq 2\gamma_2 \sum_{i=1}^{N-1} I_2 (i) \cdot t_i \cdot g(x_i + a_0 h_i)
\]

\[
\geq 2\gamma_2 \sum_{i=1}^{N-1} I_2 (i) \cdot t_i \cdot g(x_i + a_0 h_i)
\]

\[
\geq 2\gamma_2 \sum_{i=1}^{N-1} I_2 (i) \cdot t_i \cdot g(x_i + a_0 h_i)
\]

\[
\geq 2 \sum_{i=1}^{N-1} I_2 (i) \cdot t_i \cdot g(x_i + a_0 h_i)
\]

\[
\geq 2 \| x \|.
\]

Therefore, we obtain

\[
\| x \| \leq \frac{\gamma_x}{2} \leq \frac{\| y \|}{2}. \tag{22}
\]

In a similar manner, we deduce

\[
x_x = \sum_{i=1}^{N-1} G_1 (c, i) \cdot s_i \cdot f(y_i + b_0 k_i)
\]

\[
\geq \gamma_1 \sum_{i=1}^{N-1} I_1 (i) \cdot s_i \cdot f(y_i + b_0 k_i)
\]

\[
\geq \gamma_1 \sum_{i=1}^{N-1} I_1 (i) \cdot s_i \cdot f(y_i + b_0 k_i)
\]

\[
\geq 2\gamma_1 \sum_{i=1}^{N-1} I_1 (i) \cdot s_i \cdot f(y_i + b_0 k_i)
\]

\[
\geq 2\gamma_1 \sum_{i=1}^{N-1} I_1 (i) \cdot s_i \cdot f(y_i + b_0 k_i)
\]

\[
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\]

\[
\geq 2\gamma_1 \sum_{i=1}^{N-1} I_1 (i) \cdot s_i \cdot f(y_i + b_0 k_i)
\]

\[
\geq 2\gamma_1 \sum_{i=1}^{N-1} I_1 (i) \cdot s_i \cdot f(y_i + b_0 k_i)
\]

\[
\geq 2 \| y \|.
\]

So, we obtain

\[
\| y \| \leq \frac{\gamma_y}{2} \leq \frac{\| x \|}{2}. \tag{24}
\]

By (22) and (24), we obtain \( \| x \| \leq \| y \| / 2 \leq \| x \| / 4 \), which is a contradiction, because \( \| x \| > 0 \). Then, for \( a_0 \) and \( b_0 \) sufficiently large, problem (S)-(BC) has no positive solution. \( \square \)

Similar results as Theorems 6 and 7 can be obtained if instead of boundary conditions (BC) we have

\[
u_0 = \sum_{i=1}^{p} a_i u_{c_i},
\]

\[
u_N = \sum_{i=1}^{q} b_i u_{n_i} + a_0,
\]

\[
u_0 = \sum_{i=1}^{r} c_i v_{c_i} + b_0,
\]

\[
u_N = \sum_{i=1}^{l} d_i v_{p_i},
\]

where \( a_0 \) and \( b_0 \) are positive constants.
For problem (S)–(BC1), instead of sequences \((h_n)_{n=1,N-1}\) and \((k_n)_{n=1,N-1}\) from the proof of Theorem 6, the solutions of problems

\[
\Delta^2 \tilde{h}_{n-1} = 0, \quad n = 1, N - 1, \\
\tilde{h}_0 = \sum_{i=1}^{p} a_i \tilde{h}_i, \\
\tilde{h}_N = \sum_{i=1}^{q} b_i \tilde{h}_i + 1, \\
\Delta^2 \tilde{k}_{n-1} = 0, \quad n = 1, N - 1, \\
\tilde{k}_0 = \sum_{i=1}^{r} c_i \tilde{k}_i + 1, \\
\tilde{k}_N = \sum_{i=1}^{s} d_i \tilde{k}_i,
\]

are

\[
\tilde{h}_n = \frac{1}{\Delta_1} \left[ n \left( 1 - \sum_{i=1}^{p} a_i \right) + \sum_{i=1}^{p} a_i \xi_i \right], \quad n = 0, N, \\
\tilde{k}_n = \frac{1}{\Delta_2} \left[ -n \left( 1 - \sum_{i=1}^{r} d_i \right) + \left( N - \sum_{i=1}^{r} d_i \rho_i \right) \right],
\]

respectively. By assumption (H1) we obtain \(\tilde{h}_n > 0\), for all \(n = 1, N\), and \(\tilde{k}_n > 0\) for all \(n = 0, N - 1\).

For problem (S)–(BC2), instead of sequences \((h_n)_{n=1,N}\) and \((k_n)_{n=0,N}\) from Theorem 6, the solutions of problems (7) and (26) are \((h_n)_{n=0,N}\) and \((\tilde{k}_n)_{n=0,N}\), respectively, which satisfy \(h_n > 0\), for all \(n = 0, N - 1\), and \(\tilde{k}_n > 0\) for all \(n = 0, N - 1\). For problem (S)–(BC3), instead of sequences \((h_n)_{n=0,N}\) and \((k_n)_{n=0,N}\) from Theorem 6, the solutions of problems (25) and (8) are \((h_n)_{n=0,N}\) and \((k_n)_{n=0,N}\), respectively, which satisfy \(h_n > 0\), for all \(n = 1, N\), and \(k_n > 0\) for all \(n = 1, N\).

Therefore we also obtain the following results.

**Theorem 8.** Assume that assumptions (H1)–(H3) hold. Then problem (S)–(BC1) has at least one positive solution \((u_n > 0, \forall n = 1, N, \text{ and } v_n > 0, \forall n = 0, N - 1\) for \(a_0 > 0\) and \(b_0 > 0\) sufficiently small.

**Theorem 9.** Assume that assumptions (H1), (H2), and (H4) hold. Then problem (S)–(BC1) has no positive solution \((u_n > 0, \forall n = 1, N, \text{ and } v_n > 0, \forall n = 0, N - 1\) for \(a_0 > 0\) and \(b_0 > 0\) sufficiently large.

**Theorem 10.** Assume that assumptions (H1)–(H3) hold. Then problem (S)–(BC2) has at least one positive solution \((u_n > 0, \forall n = 0, N - 1, \text{ and } v_n > 0, \forall n = 0, N - 1)\) for \(a_0 > 0\) and \(b_0 > 0\) sufficiently small.

**Theorem 11.** Assume that assumptions (H1), (H2), and (H4) hold. Then problem (S)–(BC2) has no positive solution \((u_n > 0, \forall n = 0, N - 1, \text{ and } v_n > 0, \forall n = 0, N - 1\) for \(a_0 > 0\) and \(b_0 > 0\) sufficiently large.

**Theorem 12.** Assume that assumptions (H1)–(H3) hold. Then problem (S)–(BC3) has at least one positive solution \((u_n > 0, \forall n = 1, N, \text{ and } v_n > 0, \forall n = 1, N)\) for \(a_0 > 0\) and \(b_0 > 0\) sufficiently small.

**Theorem 13.** Assume that assumptions (H1), (H2), and (H4) hold. Then problem (S)–(BC3) has no positive solution \((u_n > 0, \forall n = 1, N, \text{ and } v_n > 0, \forall n = 1, N\) for \(a_0 > 0\) and \(b_0 > 0\) sufficiently large.

### 4. An Example

We consider \(N = 20, s_n = \bar{c}/(n + 1), a_n = \bar{d}/n\) for all \(n = 1, 19, \bar{c} > 0, \bar{d} > 0, p = 2, q = 3, r = 1, l = 2, a_1 = 1/2, a_2 = 1/3, \xi_1 = 4, \xi_2 = 16, b_1 = 1/3, b_2 = 1/4, b_3 = 1/5, \eta_1 = 5, \eta_2 = 10, \eta_3 = 15, c_1 = 3/4, \xi_1 = 10, d_1 = 1/3, d_2 = 1/5, \rho_1 = 3, \rho_2 = 18\). We also consider the functions \(f, g : [0, \infty) \to [0, \infty), f(x) = \bar{a}x^{\alpha_1}/(2x + 1), \text{ and } g(x) = \bar{b}x^{\alpha_2}/(3x + 2), \) for all \(x \in [0, \infty)\), with \(\bar{a}, \bar{b} > 0\) and \(\alpha_1, \alpha_2 > 2\). We have \(\lim_{x \to \infty} f(x)/x = \lim_{x \to \infty} g(x)/x = \infty\).

Therefore, we consider the system of second-order difference equations

\[
\Delta^2 u_{n-1} + \frac{\bar{c} \bar{a} u_{n}^{\alpha_1}}{(n + 1) (2v_{n} + 1)} = 0, \quad n = 1, 19, \\
\Delta^2 v_{n-1} + \frac{\bar{d} \bar{b} v_{n}^{\alpha_2}}{n (3u_{n} + 2)} = 0, \quad n = 1, 19,
\]

with the multipoint boundary conditions

\[
u_0 = \frac{1}{2} u_4 + \frac{1}{3} u_{16} + a_0, \\
u_{20} = \frac{1}{3} u_5 + \frac{1}{4} u_{10} + \frac{1}{5} u_{15}, \\
v_0 = \frac{3}{4} v_{10}, \\
v_{20} = \frac{1}{3} v_3 + \frac{1}{5} v_{19} + b_0,
\]

where \(a_0\) and \(b_0\) are positive constants.
We have $\sum_{i=1}^{2} a_i = 5/6 < 1$, $\sum_{i=1}^{3} b_i = 47/60 < 1$, $\sum_{i=1}^{4} c_i = 3/4 < 1$, and $\sum_{i=1}^{5} d_i = 8/15 < 1$. The functions $I_1$ and $I_2$ are given by

$$I_1(j) = \begin{cases} 
2565 j - \frac{j^2}{61} & 1 \leq j \leq 2, \\
7820 + 70 j - \frac{j^2}{20} & 2 < j \leq 5, \\
12620 + \frac{290 j}{671} - \frac{j^2}{20}, & 5 < j \leq 10, \\
30700 - \frac{864 j}{671} - \frac{j^2}{20}, & 10 < j \leq 16, \\
\end{cases} \quad (28)$$

and $I_2(j) = \begin{cases} 
145 j - \frac{j^2}{20}, & 1 \leq j \leq 2, \\
250 + 85 j - \frac{j^2}{20}, & 2 < j \leq 3, \\
147 + \frac{8 j}{20}, & 3 < j \leq 9, \\
2560 + \frac{8 j}{20}, & 9 < j \leq 17, \\
3460 - \frac{26 j}{147} + \frac{j^2}{20}, & 17 < j \leq 19, \\
\end{cases}$

Hence, we deduce that assumptions (H1), (H2), and (H4) are satisfied. In addition, by using the above functions $I_1$ and $I_2$, we obtain $\bar{a} := \sum_{i=1}^{19} I_1(i)/(i+1) = 30.1784002$, $\bar{b} := \sum_{i=1}^{19} I_2(i)/(i+1) = 23.63831254$, and then $L = \max\{\bar{a}, \bar{b}\}$.

We choose $c_0 = 1$ and if we select $\tilde{a}, \tilde{b}$ satisfying the conditions $\tilde{a} < 3/L = 3\min\{1/(\tilde{a}), 1/(\tilde{d}\bar{b})\}$, $\tilde{b} < 5/L = 5\min\{1/(\tilde{c}\bar{a}), 1/(\tilde{d}\bar{b})\}$, then we conclude that $f(x) \leq \tilde{a}/3 < 1/L$, $g(x) \leq \tilde{b}/5 < 1/L$ for all $x \in [0,1]$. For example, if $\tilde{c} = 1$, $\tilde{d} = 2$, then for $\tilde{a} \leq 0.063$ and $\tilde{b} \leq 0.105$ the above conditions for $f$ and $g$ are satisfied. So, assumption (H3) is also satisfied. By Theorems 6 and 7 we deduce that problem $(S_0)$-(BC0) has at least one positive solution (here $u_n > 0$ and $v_n > 0$ for all $n = 0, 20$) for sufficiently small $a_0$ and $b_0 > 0$ and no positive solution for sufficiently large $a_0$ and $b_0$.

By the proofs of Theorems 6 and 7 we can find some intervals for $a_0$ and $b_0$ such that problem $(S_0)$-(BC0) has at least one positive solution, or it has no positive solution. We consider $\bar{a} = 0.063$, $\bar{b} = 0.105$, $\tilde{c} = 1$, $\tilde{d} = 2$, $c_0 = 1$, $L = \max\{\bar{a}, \bar{b}\} = 2\bar{b}$ (as above), $\alpha_1 = 3$, and $\alpha_2 = 4$. Then $\Delta_1 = 671/180$, $\Delta_2 = 147/20$, and the sequences $(h_n)_{n=0}^{20}$ and $(k_n)_{n=0}^{20}$ from (9) are $h_n = (39n + 2310)/671$ and $k_n = (5n + 150)/147$ for all $n = 0, 20$. We also obtain $h_{\max} = h_{20} = 210/61$ and $k_{\max} = k_{20} = 250/147$. If we choose $b_0 \leq (g^{-1}/(1/L) - 1)/k_{\max}$ and $a_0 \leq (f^{-1}(1/L) - 1)/h_{\max}$, then inequalities (15) are satisfied. Because $f^{-1}(1/L) \approx 1.0031$ and $g^{-1}(1/L) \approx 1.00213$, for $a_0 \leq 6.17 \cdot 10^{-4}$ and $b_0 \leq 18.21 \cdot 10^{-4}$ problem $(S_0)$-(BC0) has at least one positive solution.

Now we choose $c = 5$ (the constant from the beginning of the proof of Theorem 7), and then we obtain $y_1 = y_2 = 5/19$, $h_{15} = 1725/671$, $k_5 = 175/147$, $r_1 = \min\{y_1, h_{15}/h_{15}\} = 5/19$, $r_2 = \min\{y_2, h_{15}/h_{15}\} = 5/19$, $\sum_{i=1}^{5} I_1(i) = 27.88323191$, $\sum_{i=5}^{15} I_2(i) = 18.41083231$, and $R = 0.78432087$ (given by (19)). For $R = 2R + 0.1$, the inequalities $f(x)/x \geq R$ and $g(y)/y \geq R$ are satisfied for $x \geq M_0$ and $y \geq M_0''$, respectively. We consider $M_0 = \max\{M_0', M_0''\} = M_0'$, and then for $a_0 \geq M_0/h_{15}$ and $b_0 \geq M_0/k_{15}$, inequalities (20) are satisfied. Therefore, if $a_0 \geq 20.7984$ and $b_0 \geq 44.9133$, problem $(S_0)$-(BC0) has no positive solution.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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