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Research Article

Positive Solutions for Systems of Second-Order Difference Equations

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We study the existence and nonexistence of positive solutions of some systems of nonlinear second-order difference equations subject to multipoint boundary conditions which contain some positive constants.

1. Introduction

The mathematical modeling of many nonlinear problems from computer science, economics, mechanical engineering, control systems, biological neural networks, and others leads to the consideration of nonlinear difference equations (see [1, 2]). In the last decades, many authors have investigated such problems by using various methods, such as fixed point theorems, the critical point theory, upper and lower solutions, the fixed point index theory, and the topological degree theory (see, e.g., [3–15]).

In this paper, we consider the system of nonlinear second-order difference equations:

$$\begin{aligned} \Delta^2 u_{n-1} + s_n f(v_n) = 0, \quad n = \overline{1, N-1}, \\ \Delta^2 v_{n-1} + t_n g(u_n) = 0, \quad n = \overline{1, N-1}, \end{aligned} \tag{S}$$

with the multipoint boundary conditions

$$\begin{aligned} u_0 &= \sum_{i=1}^{p} a_i u_{\xi_i} + a_0, \\ u_N &= \sum_{i=1}^{q} b_i u_{\eta_i}, \\ v_0 &= \sum_{i=1}^{r} c_i v_{\zeta_i}, \\ v_N &= \sum_{i=1}^{l} d_i v_{\rho_i} + b_0, \end{aligned} \tag{BC}$$

where $N, p, q, r, l \in \mathbb{N}$, $N \geq 2$, Δ is the forward difference operator with stepsize 1, $\Delta u_n = u_{n+1} - u_n$, $\Delta^2 u_{n-1} = u_{n+1} - 2u_n + u_{n-1}$, $n = \overline{k, m}$ means that $n = k, k+1, \ldots, m$ for $k, m \in \mathbb{N}$, $a_i \in \mathbb{R}$ for all $i = \overline{1, p}$, $b_i \in \mathbb{R}$ for all $i = \overline{1, q}$, $c_i \in \mathbb{R}$ for all $i = \overline{1, r}$, $d_i \in \mathbb{R}$ for all $i = \overline{1, l}$, $\xi_i \in \mathbb{N}$ for all $i = \overline{1, p}$, $\eta_i \in \mathbb{N}$ for all $i = \overline{1, q}$, $\zeta_i \in \mathbb{N}$ for all $i = \overline{1, r}$, $\rho_i \in \mathbb{N}$ for all $i = \overline{1, l}$, $1 \leq \xi_1 < \cdots < \xi_p \leq N-1$, $1 \leq \eta_1 < \cdots < \eta_q \leq N-1$, $1 \leq \zeta_1 < \cdots < \zeta_r \leq N-1$, $1 \leq \rho_1 < \cdots < \rho_l \leq N-1$, and a_0 and b_0 are positive constants.

Under some assumptions on the functions f and g, we will prove the existence of positive solutions of problem (S)-(BC). By a positive solution of (S)-(BC) we mean a pair of sequences $((u_n)_{n=\overline{0,N}}, (v_n)_{n=\overline{0,N}})$ satisfying (S) and (BC) with $u_n, v_n \geq 0$ for all $n = \overline{0, N}$ and $u_n > 0$ for all $n = \overline{0, N-1}$, $v_n > 0$ for all $n = \overline{1, N}$. We will also give sufficient conditions for the nonexistence of positive solutions for this problem. System (S) with the multipoint boundary conditions $\alpha u_0 - \beta \Delta u_0 = 0$, $u_N = \sum_{i=1}^{m-2} a_i u_{\xi_i} + a_0$, $\gamma v_0 - \delta \Delta v_0 = 0$, and

$v_N = \sum_{i=1}^{p-2} b_i v_{\eta_i} + b_0$ $(a_0, b_0 > 0)$ has been investigated in [16]. Some systems of difference equations with parameters, subject to multipoint boundary conditions, were studied in [17, 18], by using the Guo-Krasnosel'skii fixed point theorem. We also mention the paper [19], where the authors investigated the existence and multiplicity of positive solutions for the system $\Delta^2 u_{n-1} + f(n, v_n) = 0$, $\Delta^2 v_{n-1} + g(n, u_n) = 0$, $n = \overline{1, N-1}$, with the multipoint boundary conditions (BC) with $a_0 = b_0 = 0$, by using some theorems from the fixed point index theory.

In Section 2, we present some auxiliary results which investigate a second-order difference equation subject to multipoint boundary conditions. In Section 3, we will prove our main results, and in Section 4, we will present an example which illustrates the obtained theorems. Our main existence result is based on the Schauder fixed point theorem which we present now.

Theorem 1. *Let X be a Banach space and $Y \subset X$ a nonempty, bounded, convex, and closed subset. If the operator $A : Y \to Y$ is completely continuous (continuous and compact, i.e., mapping bounded sets into relatively compact sets), then A has at least one fixed point.*

2. Auxiliary Results

In this section, we present some auxiliary results from [17] related to the following second-order difference equation with the multipoint boundary conditions:

$$\begin{gathered} \Delta^2 u_{n-1} + y_n = 0, \quad n = \overline{1, N-1}, \\ u_0 = \sum_{i=1}^{p} a_i u_{\xi_i}, \\ u_N = \sum_{i=1}^{q} b_i u_{\eta_i}, \end{gathered} \tag{1}$$

where $N, p, q \in \mathbb{N}$, $N \geq 2$, $a_i \in \mathbb{R}$ for all $i = \overline{1, p}$, $b_i \in \mathbb{R}$ for all $i = \overline{1, q}$, $\xi_i \in \mathbb{N}$ for all $i = \overline{1, p}$, $\eta_i \in \mathbb{N}$ for all $i = \overline{1, q}$, $1 \leq \xi_1 < \cdots < \xi_p \leq N-1$, and $1 \leq \eta_1 < \cdots < \eta_q \leq N-1$.

Lemma 2 (see [17]). *If $a_i \in \mathbb{R}$ for all $i = \overline{1, p}$, $b_i \in \mathbb{R}$ for all $i = \overline{1, q}$, $\xi_i \in \mathbb{N}$ for all $i = \overline{1, p}$, $\eta_i \in \mathbb{N}$ for all $i = \overline{1, q}$, $1 \leq \xi_1 < \cdots < \xi_p \leq N-1$, $1 \leq \eta_1 < \cdots < \eta_q \leq N-1$, $\Delta_1 = (1 - \sum_{i=1}^{q} b_i) \sum_{i=1}^{p} a_i \xi_i + (1 - \sum_{i=1}^{p} a_i)(N - \sum_{i=1}^{q} b_i \eta_i) \neq 0$, and $y_n \in \mathbb{R}$ for all $n = \overline{1, N-1}$, then the solution of (1) is given by $u_n = \sum_{j=1}^{N-1} G_1(n, j) y_j$ for all $n = \overline{0, N}$, where Green's function G_1 is defined by*

$$\begin{aligned} G_1(n, j) &= g_0(n, j) \\ &\quad + \frac{1}{\Delta_1} \left[(N-n) \left(1 - \sum_{k=1}^{q} b_k \right) + \sum_{i=1}^{q} b_i (N - \eta_i) \right] \\ &\quad \cdot \sum_{i=1}^{p} a_i g_0(\xi_i, j) + \frac{1}{\Delta_1} \left[n \left(1 - \sum_{k=1}^{p} a_k \right) + \sum_{i=1}^{p} a_i \xi_i \right] \\ &\quad \cdot \sum_{i=1}^{q} b_i g_0(\eta_i, j), \quad n = \overline{0, N}, \ j = \overline{1, N-1}, \end{aligned}$$

$$g_0(n, j) = \frac{1}{N} \begin{cases} j(N-n), & 1 \leq j \leq n \leq N, \\ n(N-j), & 0 \leq n \leq j \leq N-1. \end{cases} \tag{2}$$

Lemma 3 (see [17]). *If $a_i \geq 0$ for all $i = \overline{1, p}$, $\sum_{i=1}^{p} a_i < 1$, $b_i \geq 0$ for all $i = \overline{1, q}$, $\sum_{i=1}^{q} b_i < 1$, $\xi_i \in \mathbb{N}$ for all $i = \overline{1, p}$, $1 \leq \xi_1 < \xi_2 < \cdots < \xi_p \leq N-1$, $\eta_i \in \mathbb{N}$ for all $i = \overline{1, q}$, and $1 \leq \eta_1 < \eta_2 < \cdots < \eta_q \leq N-1$, then Green's function G_1 of problem (1) satisfies $G_1(n, j) \geq 0$ for all $n = \overline{0, N}$, $j = \overline{1, N-1}$. Moreover, if $y_n \geq 0$ for all $n = \overline{1, N-1}$, then the unique solution u_n, $n = \overline{0, N}$, of problem (1) satisfies $u_n \geq 0$ for all $n = \overline{0, N}$.*

Lemma 4 (see [17]). *Assume that $a_i \geq 0$ for all $i = \overline{1, p}$, $\sum_{i=1}^{p} a_i < 1$, $b_i \geq 0$ for all $i = \overline{1, q}$, $\sum_{i=1}^{q} b_i < 1$, $\xi_i \in \mathbb{N}$ for all $i = \overline{1, p}$, $1 \leq \xi_1 < \xi_2 < \cdots < \xi_p \leq N-1$, $\eta_i \in \mathbb{N}$ for all $i = \overline{1, q}$, and $1 \leq \eta_1 < \eta_2 < \cdots < \eta_q \leq N-1$. Then Green's function G_1 of problem (1) satisfies the following inequalities:*

(a) $G_1(n, j) \leq I_1(j)$, $\forall n = \overline{0, N}$, $j = \overline{1, N-1}$, *where*

$$\begin{aligned} I_1(j) &= g_0(j, j) + \frac{1}{\Delta_1} \left(N - \sum_{i=1}^{q} b_i \eta_i \right) \sum_{i=1}^{p} a_i g_0(\xi_i, j) \\ &\quad + \frac{1}{\Delta_1} \left(N - \sum_{i=1}^{p} a_i (N - \xi_i) \right) \sum_{i=1}^{q} b_i g_0(\eta_i, j). \end{aligned} \tag{3}$$

(b) *For every $c \in \{1, \ldots, [\![N/2]\!]\}$, one has*

$$\begin{gathered} \min_{n=\overline{c, N-c}} G_1(n, j) \geq \gamma_1 I_1(j) \geq \gamma_1 G_1(n', j), \\ \forall n' = \overline{0, N}, \ j = \overline{1, N-1}, \end{gathered} \tag{4}$$

where

$$\begin{aligned} \gamma_1 = \min \Bigg\{ & \frac{c}{N-1}, \frac{c(1 - \sum_{k=1}^{q} b_k) + \sum_{i=1}^{q} b_i (N - \eta_i)}{N - \sum_{i=1}^{q} b_i \eta_i}, \\ & \frac{c\left(1 - \sum_{k=1}^{p} a_k\right) + \sum_{i=1}^{p} a_i \xi_i}{N - \sum_{i=1}^{p} a_i (N - \xi_i)} \Bigg\} > 0, \end{aligned} \tag{5}$$

and $[\![N/2]\!]$ is the largest integer not greater than $N/2$.

Lemma 5 (see [17]). *Assume that $a_i \geq 0$ for all $i = \overline{1, p}$, $\sum_{i=1}^{p} a_i < 1$, $b_i \geq 0$ for all $i = \overline{1, q}$, $\sum_{i=1}^{q} b_i < 1$, $\xi_i \in \mathbb{N}$ for all $i = \overline{1, p}$, $1 \leq \xi_1 < \xi_2 < \cdots < \xi_p \leq N-1$, $\eta_i \in \mathbb{N}$ for all $i = \overline{1, q}$, $1 \leq \eta_1 < \eta_2 < \cdots < \eta_q \leq N-1$, $c \in \{1, \ldots, [\![N/2]\!]\}$, and $y_n \geq 0$ for all $n = \overline{1, N-1}$. Then the solution u_n, $n = \overline{0, N}$, of problem (1) satisfies the inequality $\min_{n=\overline{c, N-c}} u_n \geq \gamma_1 \max_{m=\overline{0, N}} u_m$.*

We can also formulate similar results as Lemmas 2–5 above for the discrete boundary value problem

$$\begin{aligned} \Delta^2 v_{n-1} + h_n &= 0, \quad n = \overline{1, N-1}, \\ v_0 &= \sum_{i=1}^{r} c_i v_{\zeta_i}, \\ v_N &= \sum_{i=1}^{l} d_i v_{\rho_i}, \end{aligned} \tag{6}$$

where $N, r, l \in \mathbb{N}$, $N \geq 2$, $c_i \in \mathbb{R}$ for all $i = \overline{1, r}$, $d_i \in \mathbb{R}$ for all $i = \overline{1, l}$, $\zeta_i \in \mathbb{N}$ for all $i = \overline{1, r}$, $\rho_i \in \mathbb{N}$ for all $i = \overline{1, l}$, $1 \leq \zeta_1 < \cdots < \zeta_r \leq N-1$, $1 \leq \rho_1 < \cdots < \rho_l \leq N-1$, and $h_n \in \mathbb{R}$ for all $n = \overline{1, N-1}$. We denote by Δ_2, γ_2, G_2, and I_2 the corresponding constants and functions for problem (6) defined in a similar manner as Δ_1, γ_1, G_1, and I_1, respectively.

3. Main Results

We present first the assumptions that we will use in the sequel:

($H1$) $\xi_i \in \mathbb{N}$ for all $i = \overline{1, p}$, $1 \leq \xi_1 < \cdots < \xi_p \leq N-1$, $a_i \geq 0$ for all $i = \overline{1, p}$, $\sum_{i=1}^{p} a_i < 1$, $\eta_i \in \mathbb{N}$ for all $i = \overline{1, q}$, $1 \leq \eta_1 < \cdots < \eta_q \leq N-1$, $b_i \geq 0$ for all $i = \overline{1, q}$, $\sum_{i=1}^{q} b_i < 1$, $\zeta_i \in \mathbb{N}$ for all $i = \overline{1, r}$, $1 \leq \zeta_1 < \cdots < \zeta_r \leq N-1$, $c_i \geq 0$ for all $i = \overline{1, r}$, $\sum_{i=1}^{r} c_i < 1$, $\rho_i \in \mathbb{N}$ for all $i = \overline{1, l}$, $1 \leq \rho_1 < \cdots < \rho_l \leq N-1$, and $d_i \geq 0$ for all $i = \overline{1, l}$, $\sum_{i=1}^{l} d_i < 1$.

($H2$) The constants $s_n, t_n \geq 0$ for all $n = \overline{1, N-1}$, and there exist $i_0, j_0 \in \{1, \ldots, N-1\}$ such that $s_{i_0} > 0$, $t_{j_0} > 0$.

($H3$) $f, g : [0, \infty) \to [0, \infty)$ are continuous functions and there exists $c_0 > 0$ such that $f(u) < c_0/L$, $g(u) < c_0/L$ for all $u \in [0, c_0]$, where $L = \max\{\sum_{i=1}^{N-1} s_i I_1(i), \sum_{i=1}^{N-1} t_i I_2(i)\}$ and I_1, I_2 are defined in Section 2.

($H4$) $f, g : [0, \infty) \to [0, \infty)$ are continuous functions and satisfy the conditions $\lim_{u \to \infty}(f(u)/u) = \infty$, $\lim_{u \to \infty}(g(u)/u) = \infty$.

Our first theorem is the following existence result for problem (S)-(BC).

Theorem 6. *Assume that assumptions* ($H1$)–($H3$) *hold. Then problem* (S)-(BC) *has at least one positive solution for* $a_0 > 0$ *and* $b_0 > 0$ *sufficiently small.*

Proof. We consider the problems

$$\begin{aligned} \Delta^2 h_{n-1} &= 0, \\ n &= \overline{1, N-1}, \\ h_0 &= \sum_{i=1}^{p} a_i h_{\xi_i} + 1, \\ h_N &= \sum_{i=1}^{q} b_i h_{\eta_i}, \end{aligned} \tag{7}$$

$$\begin{aligned} \Delta^2 k_{n-1} &= 0, \\ n &= \overline{1, N-1}, \\ k_0 &= \sum_{i=1}^{r} c_i k_{\zeta_i}, \\ k_N &= \sum_{i=1}^{l} d_i k_{\rho_i} + 1. \end{aligned} \tag{8}$$

Problems (7) and (8) have the solutions

$$\begin{aligned} h_n &= \frac{1}{\Delta_1}\left[-n\left(1 - \sum_{i=1}^{q} b_i\right) + \left(N - \sum_{i=1}^{q} b_i \eta_i\right)\right], \\ & \qquad n = \overline{0, N}, \\ k_n &= \frac{1}{\Delta_2}\left[n\left(1 - \sum_{i=1}^{r} c_i\right) + \sum_{i=1}^{r} c_i \zeta_i\right], \quad n = \overline{0, N}, \end{aligned} \tag{9}$$

respectively, where Δ_1 and Δ_2 are defined in Section 2. By assumption ($H1$) we obtain $h_n > 0$ for all $n = \overline{0, N-1}$ and $k_n > 0$ for all $n = \overline{1, N}$.

We define the sequences $(x_n)_{n=\overline{0,N}}$, $(y_n)_{n=\overline{0,N}}$ by

$$\begin{aligned} x_n &= u_n - a_0 h_n, \\ y_n &= v_n - b_0 k_n, \\ & \qquad n = \overline{0, N}, \end{aligned} \tag{10}$$

where $((u_n)_{n=\overline{0,N}}, (v_n)_{n=\overline{0,N}})$ is a solution of (S)-(BC). Then (S)-(BC) can be equivalently written as

$$\begin{aligned} \Delta^2 x_{n-1} + s_n f\left(y_n + b_0 k_n\right) &= 0, \quad n = \overline{1, N-1}, \\ \Delta^2 y_{n-1} + t_n g\left(x_n + a_0 h_n\right) &= 0, \quad n = \overline{1, N-1}, \end{aligned} \tag{11}$$

with the boundary conditions

$$\begin{aligned} x_0 &= \sum_{i=1}^{r} a_i x_{\xi_i}, \\ x_N &= \sum_{i=1}^{q} b_i x_{\eta_i}, \\ y_0 &= \sum_{i=1}^{r} c_i y_{\zeta_i}, \\ y_N &= \sum_{i=1}^{l} d_i y_{\rho_i}. \end{aligned} \tag{12}$$

Using Green's functions G_1 and G_2 from Section 2, a pair $((x_n)_{n=\overline{0,N}}, (y_n)_{n=\overline{0,N}})$ is a solution of problem (11)-(12) if and only if it is a solution for the problem

$$x_n = \sum_{i=1}^{N-1} G_1(n,i) \cdot s_i f\left(\sum_{j=1}^{N-1} G_2(i,j)\, t_j g\left(x_j + a_0 h_j\right) + b_0 k_i\right), \quad n = \overline{0,N},$$
$$y_n = \sum_{i=1}^{N-1} G_2(n,i)\, t_i g\left(x_i + a_0 h_i\right), \quad n = \overline{0,N}, \tag{13}$$

where $(h_n)_{n=\overline{0,N}}$, $(k_n)_{n=\overline{0,N}}$ are given in (9).

We consider the Banach space $X = \mathbb{R}^{N+1}$ with the norm $\|u\| = \max_{n=\overline{0,N}} |u_n|$, $u = (u_n)_{n=\overline{0,N}}$, and we define the set $M = \{(x_n)_{n=\overline{0,N}},\ 0 \le x_n \le c_0,\ \forall n = \overline{0,N}\} \subset X$.

We also define the operator $\mathscr{E} : M \to X$ by

$$\mathscr{E}(x) = \left(\sum_{i=1}^{N-1} G_1(n,i) \cdot s_i f\left(\sum_{j=1}^{N-1} G_2(i,j)\, t_j g\left(x_j + a_0 h_j\right) + b_0 k_i\right)\right)_{n=\overline{0,N}}, \quad x = (x_n)_{n=\overline{0,N}} \in M. \tag{14}$$

For sufficiently small $a_0 > 0$ and $b_0 > 0$, by $(H3)$, we deduce

$$f(y_n + b_0 k_n) \le \frac{c_0}{L}, \quad g(x_n + a_0 h_n) \le \frac{c_0}{L}, \quad \forall n = \overline{0,N},\ \forall (x_n)_n, (y_n)_n \in M. \tag{15}$$

Then, by using Lemma 3, we obtain $\mathscr{E}(x)_n \ge 0$ for all $n = \overline{0,N}$ and $x = (x_n)_{n=\overline{0,N}} \in M$. By Lemma 4, for all $x \in M$, we have

$$\begin{aligned}
&\sum_{j=1}^{N-1} G_2(i,j)\, t_j g\left(x_j + a_0 h_j\right) \\
&\quad \le \sum_{j=1}^{N-1} I_2(j)\, t_j g\left(x_j + a_0 h_j\right) \le \frac{c_0}{L} \sum_{j=1}^{N-1} t_j I_2(j) \le c_0, \quad \forall i = \overline{1, N-1}, \\
&\mathscr{E}(x)_n \\
&\quad \le \sum_{i=1}^{N-1} I_1(i)\, s_i f\left(\sum_{j=1}^{N-1} G_2(i,j)\, t_j g\left(x_j + a_0 h_j\right) + b_0 k_i\right) \\
&\quad \le \frac{c_0}{L} \sum_{i=1}^{N-1} s_i I_1(i) \le c_0, \quad \forall n = \overline{0,N}.
\end{aligned} \tag{16}$$

Therefore $\mathscr{E}(M) \subset M$.

Using standard arguments, we deduce that $\mathscr{E}$ is completely continuous. By Theorem 1, we conclude that $\mathscr{E}$ has a fixed point $x = (x_n)_{n=\overline{0,N}} \in M$. This element together with $y = (y_n)_{n=\overline{0,N}}$ given by (13) represents a solution for (11)-(12). This shows that our problem (S)-(BC) has a positive solution $((u_n)_{n=\overline{0,N}}, (v_n)_{n=\overline{0,N}})$ with $u_n = x_n + a_0 h_n$, $v_n = y_n + b_0 k_n$, $n = \overline{0,N}$, ($u_n > 0$ for all $n = \overline{0, N-1}$ and $v_n > 0$ for all $n = \overline{1,N}$) for sufficiently small $a_0 > 0$ and $b_0 > 0$. □

In what follows, we present sufficient conditions for the nonexistence of positive solutions of (S)-(BC).

Theorem 7. *Assume that assumptions $(H1)$, $(H2)$, and $(H4)$ hold. Then problem (S)-(BC) has no positive solution for a_0 and b_0 sufficiently large.*

Proof. We suppose that $((u_n)_{n=\overline{0,N}}, (v_n)_{n=\overline{0,N}})$ is a positive solution of (S)-(BC). Then $((x_n)_{n=\overline{0,N}}, (y_n)_{n=\overline{0,N}})$ with $x_n = u_n - a_0 h_n$, $y_n = v_n - b_0 k_n$, $n = \overline{0,N}$, is a solution for (11)-(12), where $(h_n)_{n=\overline{0,N}}$ and $(k_n)_{n=\overline{0,N}}$ are the solutions of problems (7) and (8), respectively (given by (9)). By $(H2)$ there exists $c \in \{1, 2, \ldots, [\![N/2]\!]\}$ such that $i_0, j_0 \in \{c, \ldots, N-c\}$ and then $\sum_{i=c}^{N-c} s_i I_1(i) > 0$ and $\sum_{i=c}^{N-c} t_i I_2(i) > 0$. By using Lemma 3 we have $x_n \ge 0$, $y_n \ge 0$ for all $n = \overline{0,N}$, and by Lemma 5, we obtain $\min_{n=\overline{c,N-c}} x_n \ge \gamma_1 \|x\|$ and $\min_{n=\overline{c,N-c}} y_n \ge \gamma_2 \|y\|$, where γ_1 and γ_2 are defined in Section 2.

Using now (9), we deduce that

$$\min_{n=\overline{c,N-c}} h_n = h_{N-c} = \frac{h_{N-c}}{h_0} \|h\|, \quad \min_{n=\overline{c,N-c}} k_n = k_c = \frac{k_c}{k_N} \|k\|. \tag{17}$$

Therefore, we obtain

$$\begin{aligned}
\min_{n=\overline{c,N-c}} (x_n + a_0 h_n) &\ge \gamma_1 \|x\| + \frac{a_0 h_{N-c}}{h_0} \|h\| \\
&\ge r_1 (\|x\| + a_0 \|h\|) \\
&\ge r_1 \|x + a_0 h\|, \\
\min_{n=\overline{c,N-c}} (y_n + b_0 k_n) &\ge \gamma_2 \|y\| + \frac{b_0 k_c}{k_N} \|k\| \\
&\ge r_2 (\|y\| + b_0 \|k\|) \ge r_2 \|y + b_0 k\|,
\end{aligned} \tag{18}$$

where $r_1 = \min\{\gamma_1, h_{N-c}/h_0\}$ and $r_2 = \min\{\gamma_2, k_c/k_N\}$.

We now consider

$$R = \left(\min\left\{\gamma_2 r_1 \sum_{i=c}^{N-c} t_i I_2(i), \gamma_1 r_2 \sum_{i=c}^{N-c} s_i I_1(i)\right\}\right)^{-1} > 0. \tag{19}$$

By using $(H4)$, for R defined above, we conclude that there exists $M_0 > 0$ such that $f(u) > 2Ru$, $g(u) > 2Ru$ for

all $u \geq M_0$. We consider $a_0 > 0$ and $b_0 > 0$ sufficiently large such that

$$\begin{aligned} \min_{n=\overline{c,N-c}} (x_n + a_0 h_n) &\geq M_0, \\ \min_{n=\overline{c,N-c}} (y_n + b_0 k_n) &\geq M_0. \end{aligned} \tag{20}$$

By $(H2)$, (11), (12), and the above inequalities, we deduce that $\|x\| > 0$ and $\|y\| > 0$.

Now, by using Lemma 4 and the above considerations, we have

$$\begin{aligned} y_c &= \sum_{i=1}^{N-1} G_2(c,i)\, t_i g(x_i + a_0 h_i) \\ &\geq \gamma_2 \sum_{i=1}^{N-1} I_2(i)\, t_i g(x_i + a_0 h_i) \\ &\geq \gamma_2 \sum_{i=c}^{N-c} I_2(i)\, t_i g(x_i + a_0 h_i) \\ &\geq 2R\gamma_2 \sum_{i=c}^{N-c} I_2(i)\, t_i (x_i + a_0 h_i) \\ &\geq 2R\gamma_2 \sum_{i=c}^{N-c} I_2(i)\, t_i \min_{j=\overline{c,N-c}} \{x_j + a_0 h_j\} \\ &\geq 2R\gamma_2 r_1 \sum_{i=c}^{N-c} I_2(i)\, t_i \|x + a_0 h\| \geq 2\|x + a_0 h\| \\ &\geq 2\|x\|. \end{aligned} \tag{21}$$

Therefore, we obtain

$$\|x\| \leq \frac{y_c}{2} \leq \frac{\|y\|}{2}. \tag{22}$$

In a similar manner, we deduce

$$\begin{aligned} x_c &= \sum_{i=1}^{N-1} G_1(c,i)\, s_i f(y_i + b_0 k_i) \\ &\geq \gamma_1 \sum_{i=1}^{N-1} I_1(i)\, s_i f(y_i + b_0 k_i) \\ &\geq \gamma_1 \sum_{i=c}^{N-c} I_1(i)\, s_i f(y_i + b_0 k_i) \\ &\geq 2R\gamma_1 \sum_{i=c}^{N-c} I_1(i)\, s_i (y_i + b_0 k_i) \\ &\geq 2R\gamma_1 \sum_{i=c}^{N-c} I_1(i)\, s_i \min_{j=\overline{c,N-c}} \{y_j + b_0 k_j\} \\ &\geq 2R\gamma_1 r_2 \sum_{i=c}^{N-c} I_1(i)\, s_i \|y + b_0 k\| \geq 2\|y + b_0 k\| \\ &\geq 2\|y\|. \end{aligned} \tag{23}$$

So, we obtain

$$\|y\| \leq \frac{x_c}{2} \leq \frac{\|x\|}{2}. \tag{24}$$

By (22) and (24), we obtain $\|x\| \leq \|y\|/2 \leq \|x\|/4$, which is a contradiction, because $\|x\| > 0$. Then, for a_0 and b_0 sufficiently large, problem (S)-(BC) has no positive solution. □

Similar results as Theorems 6 and 7 can be obtained if instead of boundary conditions (BC) we have

$$\begin{aligned} u_0 &= \sum_{i=1}^{p} a_i u_{\xi_i}, \\ u_N &= \sum_{i=1}^{q} b_i u_{\eta_i} + a_0, \\ v_0 &= \sum_{i=1}^{r} c_i v_{\zeta_i} + b_0, \\ v_N &= \sum_{i=1}^{l} d_i v_{\rho_i} \end{aligned} \tag{BC$_1$}$$

or

$$\begin{aligned} u_0 &= \sum_{i=1}^{p} a_i u_{\xi_i} + a_0, \\ u_N &= \sum_{i=1}^{q} b_i u_{\eta_i}, \\ v_0 &= \sum_{i=1}^{r} c_i v_{\zeta_i} + b_0, \\ v_N &= \sum_{i=1}^{l} d_i v_{\rho_i} \end{aligned} \tag{BC$_2$}$$

or

$$\begin{aligned} u_0 &= \sum_{i=1}^{p} a_i u_{\xi_i}, \\ u_N &= \sum_{i=1}^{q} b_i u_{\eta_i} + a_0, \\ v_0 &= \sum_{i=1}^{r} c_i v_{\zeta_i}, \\ v_N &= \sum_{i=1}^{l} d_i v_{\rho_i} + b_0, \end{aligned} \tag{BC$_3$}$$

where a_0 and b_0 are positive constants.

For problem (S)–(BC_1), instead of sequences $(h_n)_{n=\overline{0,N}}$ and $(k_n)_{n=\overline{0,N}}$ from the proof of Theorem 6, the solutions of problems

$$\begin{aligned} \Delta^2\widetilde{h}_{n-1} &= 0, \quad n=\overline{1,N-1}, \\ \widetilde{h}_0 &= \sum_{i=1}^{p} a_i \widetilde{h}_{\xi_i}, \\ \widetilde{h}_N &= \sum_{i=1}^{q} b_i \widetilde{h}_{\eta_i} + 1, \end{aligned} \tag{25}$$

$$\begin{aligned} \Delta^2\widetilde{k}_{n-1} &= 0, \quad n=\overline{1,N-1}, \\ \widetilde{k}_0 &= \sum_{i=1}^{r} c_i \widetilde{k}_{\zeta_i} + 1, \\ \widetilde{k}_N &= \sum_{i=1}^{l} d_i \widetilde{k}_{\rho_i} \end{aligned} \tag{26}$$

are

$$\begin{aligned} \widetilde{h}_n &= \frac{1}{\Delta_1}\left[n\left(1-\sum_{i=1}^{p} a_i\right)+\sum_{i=1}^{p} a_i\xi_i\right], \quad n=\overline{0,N}, \\ \widetilde{k}_n &= \frac{1}{\Delta_2}\left[-n\left(1-\sum_{i=1}^{l} d_i\right)+\left(N-\sum_{i=1}^{l} d_i\rho_i\right)\right], \\ & \qquad n=\overline{0,N}, \end{aligned} \tag{27}$$

respectively. By assumption $(H1)$ we obtain $\widetilde{h}_n > 0$, for all $n=\overline{1,N}$, and $\widetilde{k}_n > 0$ for all $n=\overline{0,N-1}$.

For problem (S)–(BC_2), instead of sequences $(h_n)_{n=\overline{0,N}}$ and $(k_n)_{k=\overline{0,N}}$ from Theorem 6, the solutions of problems (7) and (26) are $(h_n)_{n=\overline{0,N}}$ and $(\widetilde{k}_n)_{n=\overline{0,N}}$, respectively, which satisfy $h_n > 0$, for all $n=\overline{0,N-1}$, and $\widetilde{k}_n > 0$ for all $n=\overline{0,N-1}$. For problem (S)–(BC_3), instead of sequences $(h_n)_{n=\overline{0,N}}$ and $(k_n)_{k=\overline{0,N}}$ from Theorem 6, the solutions of problems (25) and (8) are $(\widetilde{h}_n)_{n=\overline{0,N}}$ and $(k_n)_{n=\overline{0,N}}$, respectively, which satisfy $\widetilde{h}_n > 0$, for all $n=\overline{1,N}$, and $k_n > 0$ for all $n=\overline{1,N}$.

Therefore we also obtain the following results.

Theorem 8. *Assume that assumptions $(H1)$–$(H3)$ hold. Then problem (S)–(BC_1) has at least one positive solution ($u_n > 0$, for all $n=\overline{1,N}$, and $v_n > 0$ for all $n=\overline{0,N-1}$) for $a_0 > 0$ and $b_0 > 0$ sufficiently small.*

Theorem 9. *Assume that assumptions $(H1)$, $(H2)$, and $(H4)$ hold. Then problem (S)–(BC_1) has no positive solution ($u_n > 0$, for all $n=\overline{1,N}$, and $v_n > 0$ for all $n=\overline{0,N-1}$) for a_0 and b_0 sufficiently large.*

Theorem 10. *Assume that assumptions $(H1)$–$(H3)$ hold. Then problem (S)–(BC_2) has at least one positive solution ($u_n > 0$, for all $n=\overline{0,N-1}$, and $v_n > 0$ for all $n=\overline{0,N-1}$) for $a_0 > 0$ and $b_0 > 0$ sufficiently small.*

Theorem 11. *Assume that assumptions $(H1)$, $(H2)$, and $(H4)$ hold. Then problem (S)–(BC_2) has no positive solution ($u_n > 0$, for all $n=\overline{0,N-1}$, and $v_n > 0$ for all $n=\overline{0,N-1}$) for a_0 and b_0 sufficiently large.*

Theorem 12. *Assume that assumptions $(H1)$–$(H3)$ hold. Then problem (S)–(BC_3) has at least one positive solution ($u_n > 0$, for all $n=\overline{1,N}$, and $v_n > 0$ for all $n=\overline{1,N}$) for $a_0 > 0$ and $b_0 > 0$ sufficiently small.*

Theorem 13. *Assume that assumptions $(H1)$, $(H2)$, and $(H4)$ hold. Then problem (S)–(BC_3) has no positive solution ($u_n > 0$, for all $n=\overline{1,N}$, and $v_n > 0$ for all $n=\overline{1,N}$) for a_0 and b_0 sufficiently large.*

4. An Example

We consider $N=20$, $s_n=\widehat{c}/(n+1)$, $t_n=\widehat{d}/n$ for all $n=\overline{1,19}$, $\widehat{c}>0$, $\widehat{d}>0$, $p=2$, $q=3$, $r=1$, $l=2$, $a_1=1/2$, $a_2=1/3$, $\xi_1=4$, $\xi_2=16$, $b_1=1/3$, $b_2=1/4$, $b_3=1/5$, $\eta_1=5$, $\eta_2=10$, $\eta_3=15$, $c_1=3/4$, $\zeta_1=10$, $d_1=1/3$, $d_2=1/5$, $\rho_1=3$, and $\rho_2=18$. We also consider the functions $f,g:[0,\infty)\to[0,\infty)$, $f(x)=\widetilde{a}x^{\alpha_1}/(2x+1)$, and $g(x)=\widetilde{b}x^{\alpha_2}/(3x+2)$, for all $x\in[0,\infty)$, with $\widetilde{a},\widetilde{b}>0$ and $\alpha_1,\alpha_2>2$. We have $\lim_{x\to\infty}f(x)/x=\lim_{x\to\infty}g(x)/x=\infty$.

Therefore, we consider the system of second-order difference equations

$$\begin{aligned} \Delta^2 u_{n-1} + \frac{\widehat{c}\widetilde{a}v_n^{\alpha_1}}{(n+1)(2v_n+1)} &= 0, \quad n=\overline{1,19}, \\ \Delta^2 v_{n-1} + \frac{\widehat{d}\widetilde{b}u_n^{\alpha_2}}{n(3u_n+2)} &= 0, \quad n=\overline{1,19}, \end{aligned} \tag{S_0}$$

with the multipoint boundary conditions

$$\begin{aligned} u_0 &= \frac{1}{2}u_4 + \frac{1}{3}u_{16} + a_0, \\ u_{20} &= \frac{1}{3}u_5 + \frac{1}{4}u_{10} + \frac{1}{5}u_{15}, \\ v_0 &= \frac{3}{4}v_{10}, \\ v_{20} &= \frac{1}{3}v_3 + \frac{1}{5}v_{18} + b_0, \end{aligned} \tag{BC_0}$$

where a_0 and b_0 are positive constants.

We have $\sum_{i=1}^{2} a_i = 5/6 < 1$, $\sum_{i=1}^{3} b_i = 47/60 < 1$, $\sum_{i=1}^{1} c_i = 3/4 < 1$, and $\sum_{i=1}^{2} d_i = 8/15 < 1$. The functions I_1 and I_2 are given by

$$I_1(j) = \begin{cases} \dfrac{2565j}{671} - \dfrac{j^2}{20}, & 1 \le j \le 4, \\ \dfrac{7820}{671} + \dfrac{70j}{61} - \dfrac{j^2}{20}, & 5 \le j \le 9, \\ \dfrac{12620}{671} + \dfrac{290j}{671} - \dfrac{j^2}{20}, & 10 \le j \le 15, \\ \dfrac{30700}{671} - \dfrac{864j}{671} - \dfrac{j^2}{20}, & 16 \le j \le 19. \end{cases}$$

$$I_2(j) = \begin{cases} \dfrac{145j}{63} - \dfrac{j^2}{20}, & 1 \le j \le 2, \\ \dfrac{250}{147} + \dfrac{85j}{49} - \dfrac{j^2}{20}, & 3 \le j \le 9, \\ \dfrac{2560}{147} + \dfrac{8j}{49} - \dfrac{j^2}{20}, & 10 \le j \le 17, \\ \dfrac{3460}{147} - \dfrac{26j}{147} - \dfrac{j^2}{20}, & 18 \le j \le 19. \end{cases} \tag{28}$$

Hence, we deduce that assumptions $(H1)$, $(H2)$, and $(H4)$ are satisfied. In addition, by using the above functions I_1 and I_2, we obtain $\widetilde{A} := \sum_{i=1}^{19} I_1(i)/(i+1) \approx 30.1784002$, $\widetilde{B} := \sum_{i=1}^{19} I_2(i)/i \approx 23.63831254$, and then $L = \max\{\widehat{c}\widetilde{A}, \widehat{d}\widetilde{B}\}$. We choose $c_0 = 1$ and if we select $\widetilde{a}$, $\widetilde{b}$ satisfying the conditions $\widetilde{a} < 3/L = 3\min\{1/(\widehat{c}\widetilde{A}), 1/(\widehat{d}\widetilde{B})\}$, $\widetilde{b} < 5/L = 5\min\{1/(\widehat{c}\widetilde{A}), 1/(\widehat{d}\widetilde{B})\}$, then we conclude that $f(x) \le \widetilde{a}/3 < 1/L$, $g(x) \le \widetilde{b}/5 < 1/L$ for all $x \in [0,1]$. For example, if $\widehat{c} = 1$, $\widehat{d} = 2$, then for $\widetilde{a} \le 0.063$ and $\widetilde{b} \le 0.105$ the above conditions for f and g are satisfied. So, assumption $(H3)$ is also satisfied. By Theorems 6 and 7 we deduce that problem (S_0)-(BC_0) has at least one positive solution (here $u_n > 0$ and $v_n > 0$ for all $n = \overline{0,20}$) for sufficiently small $a_0 > 0$ and $b_0 > 0$ and no positive solution for sufficiently large a_0 and b_0.

By the proofs of Theorems 6 and 7 we can find some intervals for a_0 and b_0 such that problem (S_0)-(BC_0) has at least one positive solution, or it has no positive solution. We consider $\widetilde{a} = 0.063$, $\widetilde{b} = 0.105$, $\widehat{c} = 1$, $\widehat{d} = 2$, $c_0 = 1$, $L = \max\{\widetilde{A}, 2\widetilde{B}\} = 2\widetilde{B}$ (as above), $\alpha_1 = 3$, and $\alpha_2 = 4$. Then $\Delta_1 = 671/180$, $\Delta_2 = 147/20$, and the sequences $(h_n)_{n=\overline{0,20}}$ and $(k_n)_{n=\overline{0,20}}$ from (9) are $h_n = (-39n + 2310)/671$ and $k_n = (5n + 150)/147$ for all $n = \overline{0,20}$. We also obtain $h_{\max} = h_0 = 210/61$ and $k_{\max} = k_{20} = 250/147$. If we choose $b_0 \le (f^{-1}(1/L) - 1)/k_{\max}$ and $a_0 \le (g^{-1}(1/L) - 1)/h_{\max}$, then inequalities (15) are satisfied. Because $f^{-1}(1/L) \approx 1.0031$ and $g^{-1}(1/L) \approx 1.00213$, for $a_0 \le 6.17 \cdot 10^{-4}$ and $b_0 \le 18.21 \cdot 10^{-4}$ problem (S_0)-(BC_0) has at least one positive solution.

Now we choose $c = 5$ (the constant from the beginning of the proof of Theorem 7), and then we obtain $\gamma_1 = \gamma_2 = 5/19$, $h_{15} = 1725/671$, $k_5 = 175/147$, $r_1 = \min\{\gamma_1, h_{15}/h_0\} = 5/19$, $r_2 = \min\{\gamma_2, k_5/k_{20}\} = 5/19$, $\sum_{i=5}^{15} t_i I_2(i) \approx 27.88321391$, $\sum_{i=5}^{15} s_i I_1(i) \approx 18.41083231$, and $R \approx 0.78432087$ (given by (19)). For $\widetilde{R} := 2R + 0.1$, the inequalities $f(x)/x \ge \widetilde{R}$ and $g(y)/y \ge \widetilde{R}$ are satisfied for $x \ge M_0' \approx 53.468121231$ and $y \ge M_0'' \approx 7.2166269$, respectively. We consider $M_0 = \max\{M_0', M_0''\} = M_0'$, and then for $a_0 \ge M_0/h_{15}$ and $b_0 \ge M_0/k_5$, inequalities (20) are satisfied. Therefore, if $a_0 \ge 20.7984$ and $b_0 \ge 44.9133$, problem (S_0)-(BC_0) has no positive solution.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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