Research Article

Spanning 3-Ended Trees in Almost Claw-Free Graphs

Xiaodong Chen, Meijin Xu, and Yanjun Liu

College of Science, Liaoning University of Technology, Jinzhou 121001, China

Correspondence should be addressed to Xiaodong Chen; xiaodongchen74@126.com

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We prove that if G is a k -connected ($k \geq 2$) almost claw-free graph of order n and $\sigma_{k+3}(G) \geq n + 2k - 2$, then G contains a spanning 3-ended tree, where $\sigma_k(G) = \min\{\sum_{v \in S} \deg(v) : S \text{ is an independent set of } G \text{ with } |S| = k\}$.

1. Introduction

In this paper, only finite and simple graphs are considered. We refer to [1] for notation and terminology not defined here. If $B \subseteq V(G)$, as well as every edge of G incident with the vertices in B , then B is a *dominating set*. We use $\gamma(G)$ to denote the *domination number* of a graph G , where $\gamma(G) = \min\{|S| : S \text{ is a dominating set of } G\}$. If a graph G has no $K_{1,3}$ subgraph, then G is *claw-free*. The vertex of degree 3 in $K_{1,3}$ is called *claw center* of a claw. Let $N(v) = \{u : uv \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$. If, for any vertex $v \in V(G)$, $\gamma(N(v)) \leq 2$ and the set of all claw centers in $K_{1,3}$ subgraphs of G is an independent set, then G is *almost claw-free*. A claw-free graph is almost claw-free.

$N_H(S) = \{v : v \in V(H) \text{ and } uv \in E(G) \text{ for some vertex } u \in V(S)\}$, and $d_H(S) = |N_H(S)|$. We use $\sigma_k(G)$ to denote the minimum degree sum of all the independent sets with order k in G . apb or $P[a, b]$ denotes a path with a positive orientation from x to y with end vertices a, b . For a path $P[a, b]$, $x, y \in V(P)$, let xPy denote the subpath with end vertices x, y with positive orientation and yP^-x denote the subpath with end vertices y, x with negative orientation. Let $w(G)$ denote the number of components of a graph G .

As we know, graph theory focuses on graphs composed by vertices and edges. The vertices in a graph are considered as discrete points which is usually discussed in control problems. Then there are a lot of results using graph theory to solve control problems [2–8].

A spanning tree in a graph G with no more than k leaves is called a spanning k -ended tree of G . There are many results about the properties of spanning trees [9–13]. Kyaw [14, 15]

gave some degree sum conditions for $K_{1,4}$ -free graphs to contain spanning k -ended trees.

Theorem 1 (see [14]). *If G is a connected $K_{1,4}$ -free graph and $\sigma_4(G) \geq |G| - 1$, then G contains spanning 3-ended trees.*

Theorem 2 (see [15]). *If G is a connected $K_{1,4}$ -free graph, then the following properties hold:*

- (i) G contains a hamiltonian path if $\sigma_3(G) \geq |G|$.
- (ii) G contains spanning k -ended trees if $\sigma_{k+1}(G) \geq |G| - k/2$ for an integer $k \geq 3$.

On the other hand, Kano et al. [16] obtained sharp sufficient conditions for claw-free graphs to contain spanning k -ended trees.

Theorem 3 (see [16]). *If G is a connected claw-free graph of order n and $\sigma_{k+1}(G) \geq n - k$ ($k \geq 2$), then G contains spanning k -ended trees with the maximum degree at most 3.*

Recently, Chen et al. [17] gave some degree sum conditions for k -connected $K_{1,4}$ -free graphs to contain spanning 3-ended trees.

Theorem 4 (see [17]). *If G is a k -connected $K_{1,4}$ -free graph of order n and $\sigma_{k+3}(G) \geq n + 2k - 2$ ($k \geq 2$), then G contains spanning 3-ended trees.*

Inspired by Theorems 4 and 5, in this paper, we further explore sufficient conditions for k -connected almost claw-free graphs to contain spanning 3-ended trees which holds for claw-free graphs.

Theorem 5. *If G is a k -connected almost claw-free graph of order n and $\sigma_{k+3}(G) \geq n + 2k - 2$ ($k \geq 2$), then G contains spanning 3-ended trees.*

Obviously, there are a lot of almost claw-free graphs containing $K_{1,4}$ subgraphs, so to some extent Theorem 5 is a generalization of Theorem 4.

2. Preliminaries

We need the following result given by Ryjáček [18] to prove Theorem 5.

Lemma 6 (see [18]). *$\gamma(N(v)) = 2$ for any claw center v in an almost claw-free graph.*

We mainly use the definition and properties of insertible vertex defined in [16] to prove Theorem 5.

Suppose that G is a connected nonhamiltonian graph and C is longest cycle in G with counterclockwise direction as positive orientation. Assume that R is a component of $G - C$ and $N_C(R) = \{u_1, u_2, \dots, u_m\}$ such that $u_1, u_2, \dots, u_m$ are labeled in order along the positive direction of C . Let $S_j = C(u_j, u_{j+1})$, $1 \leq j \leq m - 1$, and $S_m = C(u_m, u_1)$. A vertex u in S_j is an insertible vertex if u has two consecutive neighbors v and v^+ in $C - S_j$.

Chen and Schelp [19] proposed the following two results.

Lemma 7 (see [19]). *For each $S_j, S_j - \{v_{j+1}\}$ contains a noninsertible vertex.*

Assume that v_j is the first noninsertible vertex in $S_j - \{u_{j+1}\}$ for each $j \in [1, m]$.

Lemma 8 (see [19]). *Let $x_i \in C[u_i^+, v_i]$, $x_j \in C[u_j^+, v_j]$ with $1 \leq i < j \leq m$. Then*

- (a) *there is no path $P[x_i, x_j]$ in G such that $P[x_i, x_j] \cap V(C) = \{x_i, x_j\}$,*
- (b) *for any vertex u in $C[x_i^+, x_j^-]$, if $ux_i \in E(G)$, then $u^-x_j \notin E(G)$. By symmetry, for any vertex u in $C[x_j^+, x_i^-]$, if $ux_j \in E(G)$, then $u^-x_i \notin E(G)$,*
- (c) *for any vertex u in $C[x_i, x_j]$, if $ux_i, ux_j \in E(G)$, then $u^-u^+ \notin E(G)$. By symmetry, for any vertex u in $C[x_j, x_i]$, if $ux_i, ux_j \in E(G)$, then $u^-u^+ \notin E(G)$.*

Suppose, for some $i \in [1, m]$, $N(v_i) \cap V(G - C - R) \neq \emptyset$ and v_i' is the second noninsertible vertex in $S_i - \{u_{i+1}\}$. Then Chen et al. [17] gave the following result.

Lemma 9 (see [17]). *Let $1 \leq i < j \leq m$, $x_i \in C[v_i^+, v_i']$ and $x_j \in C[u_j^+, v_j]$. Then the following properties hold:*

- (a) *There does not exist a path $P[x_i, x_j]$ in G such that $P[x_i, x_j] \cap V(C) = \{x_i, x_j\}$.*
- (b) *For every vertex $u \in C[x_i^+, x_j^-]$, if $ux_i \in E(G)$, then $u^-x_j \notin E(G)$; similarly, for every $u \in C[x_j^+, x_i^-]$, if $ux_j \in E(G)$, then $u^-x_i \notin E(G)$.*

- (c) *For every vertex $u \in C[x_i, x_j]$, if $ux_i, ux_j \in E(G)$, then $u^-u^+ \notin E(G)$; by symmetry, for any vertex u in $C[x_j, x_i]$, if $ux_i, ux_j \in E(G)$, then $u^-u^+ \notin E(G)$.*

3. Proof of Theorem 5

To the contrary, suppose that G satisfies the conditions of Theorem 5 and any spanning tree in G contains more than 3 leaves. Let $P = P[x, y]$ be longest path in G such that P satisfies the following two conditions:

(T1) $w(G - P)$ is minimum.

(T2) $|P[x, u_1]|$ is minimum such that u_1 is the first vertex in P with $N(u_1) \cap V(G - P) \neq \emptyset$, subject to (T1).

Suppose that R is a component in $G - P$, and $\{u_1, \dots, u_m\} = N_P(R)$ with $u_1, \dots, u_m$ in order along the positive direction of P . Let R_i denote an independent set in R .

Let G' denote a graph with $V(G') = V(G) \cup \{u_0\}$, $E(G') = E(G) \cup \{u_0u : u \in V(G)\}$. Then $C = u_0P[x, y]u_0$ is a maximal cycle in G' . We define the counterclockwise orientation as the positive direction of C . Let S_i denote the segment $C(u_i, u_{i+1})$ for $0 \leq i \leq m - 1$, and $S_m = C(u_m, u_0)$. By Lemma 7, let v_i denote the first noninsertible vertex in S_i , for $i \in [0, m]$, and $U = \{v_0, v_1, \dots, v_m\}$. By Lemma 8(a), U is an independent set.

C can be divided into disjoint intervals $S = C[a, b]$ with $a, b^+ \notin N(U)$ and $C[a^+, b] \subseteq N(U)$. We call the intervals U -segments. If $a = b$, then $C[a^+, b] = \emptyset$; that is, if $|S| = 1$, then $d_U(S) = 0$. By the definition of U -segment, for any U -segment S , there exists $l \in [0, m]$ such that $S \subseteq C[v_l, v_{l+1}^-]$ (subscripts expressed modulo $m + 1$).

Claim 1. $x = v_0$ and $y \notin N(v_i)$ for any $i \in [0, m - 1]$.

Proof. Suppose that x is an insertible vertex such that $xu, xu^+ \in E(G)$, where $u, u^+ \in C - S_0$. If $u \neq y$, then we can get a path $P' = P[x^+, u]xP[u^+, y]$, a contradiction to (T2). If $u = y$, then let $P' = P[x^+, y]x$, a contradiction to (T2). Thus $x = v_0$. Suppose $v_iy \in E(G)$, for some $i \in [0, m - 1]$. Obviously, $u_0 = y^+$. Since $yv_i, y^+v_i \in E(G)$ and $y \in C - S_i, v_i$ is an insertible vertex, a contradiction. $\square$

Claim 2. For any vertex $u \in V(P)$, if $N[u]$ is claw-free, then $d_U(u) \leq 1$.

Proof. Suppose that u is in some U -segment S with $S \subseteq C[v_i, v_{i+1}^-]$, $i \in [0, m]$, and $v_{i_1}, v_{i_2} \in N_U(u)$ with $0 \leq i_1 < i_2 \leq m$. Then by Lemma 8(c), $u^-u^+ \notin E(G)$. Since $G[u, u^-, u^+, v_{i_1}] \neq K_{1,3}$, $v_{i_1}u^- \in E(G)$ or $v_{i_1}u^+ \in E(G)$. Similarly, $v_{i_2}u^- \in E(G)$ or $v_{i_2}u^+ \in E(G)$. Obviously, at least one vertex in $\{v_{i_1}, v_{i_2}\}$ is not in $C[v_i, v_{i+1}^-]$. Without loss of generality, suppose $v_{i_1} \notin C[v_i, v_{i+1}^-]$ and $v_{i_1}u^- \in E(G)$. Then by $v_{i_1}u^-, v_{i_1}u \in E(G)$, v_{i_1} is an insertible vertex, a contradiction. $\square$

Claim 3. $d_U(u) \leq 2$ for any vertex $u \in V(P)$, and if $d_U(u) = 2$, then u is a claw center.

Proof. Without loss of generality, suppose that u is in U -segment S and $S \subseteq C[v_0, v_1^-]$. If $|S| = 1$, then $d_U(u) = 0$.

Suppose $|S| \geq 2$ and $S = \{x_0, x_1, x_2, \dots, x_h\}$, where $x_0, x_1, x_2, \dots, x_h$ are in order along the positive direction of C . Then $x_0 \notin N(U)$, $x_i \in N(U)$ for $i \in [1, h]$. For some $i \in [1, h]$, suppose $v_{i_1}, v_{i_2}, v_{i_3} \in N_U(x_i)$ with $0 \leq i_1 < i_2 < i_3 \leq m$. Then x_i is a claw center. By Lemma 6, suppose y_1, y_2 are two distinct domination vertices of $N(x_i)$. Then $N[y_1], N[y_2]$ are claw-free and at least two vertices in $\{v_{i_1}, v_{i_2}, v_{i_3}\}$ are incident with y_1 or y_2 . Without loss of generality, suppose $v_{i_1}y_1, v_{i_2}y_1 \in E(G)$. Then $y_1 \in V(P)$, and $y_1^-y_1^+ \notin E(G)$ by Lemma 8(c). Suppose $S_j = C[u_j, u_{j+1}]$ containing y_1 , $0 \leq j \leq m$. Obviously, at least one vertex in $\{v_{i_1}, v_{i_2}\}$ is not in S_j . Without loss of generality, suppose $v_{i_1} \notin S_j$. Since v_{i_1} is a noninsertible vertex and $v_{i_1}y_1 \in E(G)$, $y_1^-v_{i_1}, y_1^+v_{i_1} \notin E(G)$. Thus $G[y_1, y_1^-, y_1^+, v_{i_1}] = K_{1,3}$, a contradiction. If $d_U(u) = 2$, then by Claim 2, u is a claw center. $\square$

Claim 4. For any U -segment S not containing y , S contains at most one vertex u with $d_U(u) = 2$, and $d_U(S) \leq |S|$.

Proof. If $y \notin N(U)$, then there exists a U -segment S with $S = \{y, u_0\}$. Suppose $y \in N(U)$ and y in U -segment S , then $u_0 \in S$ by $u_0 = y^+$ and $u_0 \in N(U)$. Thus if $y \notin S$, then $u_0 \notin S_0$. Without loss of generality, suppose $S = \{x_0, x_1, x_2, \dots, x_h\} \subseteq C[v_i, v_{i+1}]$, $0 \leq i \leq m$, where $x_0, x_1, x_2, \dots, x_h$ are in order along the positive direction of C . By Claim 3, suppose that x_j is the first vertex in S with $d_U(x_j) = 2$, $1 \leq j \leq h$, and $\{v_{i_1}, v_{i_2}\} = N_U(x_j)$, where $0 \leq i_1 < i_2 \leq m$. Then $v_{i_2} \neq v_i$, and, by Claim 3, x_j is a claw center. Then $N[x_j^+]$ is claw-free, and, by Claim 2, $d_U(x_j^+) \leq 1$. Thus if $j \leq h \leq j+1$, then we are done.

Suppose $h > j+1$ and $N_U(x_{j+1}) = \{v_{i_3}\}$, $i_3 \in [0, m]$. Since $G[x_{j+1}, x_{j+2}, x_j, v_{i_3}] \neq K_{1,3}$, $E(G)$ contains at least one edge in $\{x_jv_{i_3}, x_{j+2}v_{i_3}, x_jx_{j+2}\}$. Since v_{i_3} is a noninsertible vertex and $v_{i_3}x_{j+1} \in E(G)$, $v_{i_3} = v_i$ if $x_jv_{i_3}$ or $x_{j+2}v_{i_3} \in E(G)$, a contradiction to Lemma 8(b) since $x_jv_{i_1}, x_jv_{i_2} \in E(G)$. Thus $v_{i_3} \neq v_i$, and $x_jx_{j+2} \in E(G)$. Since x_j is a claw center, $N[x_{j+2}]$ is claw-free and, by Claim 2, $d_U(x_{j+2}) \leq 1$. Thus if $h = j+2$, then we are done.

Suppose $h > j+2$ and $\{v_{i_4}\} = N_U(x_{j+2})$, $i_4 \in [0, m]$. Since $x_{j+1}v_{i_3}, x_{j+2}v_{i_4} \in E(G)$ and $v_{i_3} \neq v_i$, by Lemma 8(b) $v_{i_4} \neq v_i$. Then $v_{i_4}x_{j+3} \notin E(G)$. Since $G[x_{j+2}, x_j, x_{j+3}, v_{i_4}] \neq K_{1,3}$, $v_{i_4}x_j$ or $x_jx_{j+3} \in E(G)$. If $v_{i_4}x_j \in E(G)$, then $v_{i_4} \in \{v_{i_1}, v_{i_2}\}$. Then, by Lemma 8(b), $v_{i_3} = v_{i_4}$ and $v_{i_4} = v_{i_2}$ since $x_{j+1}v_{i_3}, x_{j+2}v_{i_4} \in E(G)$. Then $v_{i_2}x_{j+1}, v_{i_2}x_{j+2} \in E(G)$, a contradiction to the noninsertibility of v_{i_2} . Thus $x_jx_{j+3} \in E(G)$, and then $N[x_{j+3}]$ is claw-free. By Claim 2, $d_U(x_{j+3}) \leq 1$. Thus if $h = j+3$, then we are done. If $h > j+3$, then, proceeding in the above manners to the set $L = \{x_{j+4}, \dots, x_h\}$, we can get that $N[u]$ is claw-free for any vertex u in L , and then, by Claim 2, $d_U(u) \leq 1$. It follows that S has exactly one vertex x_j with $d_U(x_j) = 2$ for $j \in [1, h]$. Thus the claim holds. $\square$

Claim 5. Suppose that the U -segment S_0 contains y . Then $d_U(u) \leq 1$ for any vertex $u \in S_0 - \{u_0\}$.

Proof. If $y \notin N(U)$, then $S_0 = \{y, u_0\}$ and $d_U(y) = 0$. Suppose $y \in N(U)$. Then, by Claim 1, $N_U(y) = \{v_i\}$. By Lemma 8(b), $N_U(u) \subseteq \{v_i\}$ and then $d_U(u) \leq 1$ for any vertex $u \in S_0 - \{u_0\}$. $\square$

Claim 6. For any vertex u_i ($1 \leq i \leq m$) in $N_P(R) = \{u_1, \dots, u_m\}$, $d_{R_i}(u_i) \leq 2$.

Proof. Suppose $z_1, z_2, z_3 \in N_{R_i}(u_i)$, where z_1, z_2, z_3 are distinct vertices, $i \in [1, m]$. Since $G[u_i, z_1, z_2, z_3] = K_{1,3}$, u_i is a claw center. By Lemma 6, suppose y_1, y_2 are the two distinct domination vertices in $N(u_i)$. Then $u_i^-, u_i^+ \in N(y_1) \cup N(y_2)$, and $N[y_1], N[y_2]$ are claw-free. Since $u_i^-, u_i^+ \notin N_P(R)$, $\{y_1, y_2\} \cap V(C) \neq \emptyset$. Suppose y_1, y_2 are both in $V(C)$. Then at least two vertices of z_1, z_2, z_3 are adjacent to y_1 or y_2 . Without loss of generality, suppose $z_1, z_2 \in N(y_1)$, and then $G[y_1, z_1, z_2, y_1^+] = K_{1,3}$, a contradiction. Thus one vertex of y_1, y_2 is in $V(C)$, and the other vertex is in $V(R)$. Without loss of generality, suppose $y_1 \in V(C)$, $y_2 \in V(R)$. Then $u_i^+y_1 \in E(G)$. If $z_1, z_2, z_3 \in N(y_2)$, then $G[y_2, z_1, z_2, z_3] = K_{1,3}$, a contradiction, and, by the preceding proof, there is exactly one vertex of z_1, z_2, z_3 adjacent to y_1 . Thus $y_1 \in N_P(R) - \{u_i\}$. Without loss of generality, suppose $z_1 \in N(y_1)$. Since $G[y_1, y_1^-, y_1^+, z_1] \neq K_{1,3}$, $y_1^-y_1^+ \in E(G)$. Then we can get a longer cycle $C' = y_1z_1C^-[u_i, y_1^+]C^-[y_1^-, u_i^+]y_1$ than C , a contradiction. $\square$

Claim 7. For any vertex u_i ($1 \leq i \leq m$) in $N_P(R) = \{u_1, \dots, u_m\}$, if $d_U(u_i) = 1$ and $d_{R_i}(u_i) = 2$, then $d_U(u) \leq 1$ and $d_U(S) = |S| - 1$, for any vertex $u \in S$, where S is the U -segment containing u_i .

Proof. Suppose $d_U(u_i) = 1$ and $N_{R_i}(u_i) = \{z_1, z_2\}$, $i \in [1, m]$. Since $G[u_i, u_i^-, z_1, z_2] = K_{1,3}$, u_i is a claw center. Assume $S = \{x_1, x_2, \dots, x_h\}$ and $x_j = u_i$, $2 \leq j \leq h$. If $h > j$, then, by Lemma 8(a), $N_U(u) = \{v_i\}$ for any vertex $u \in \{x_{j+1}, \dots, x_h\}$. Thus if $u \in S$ with $d_U(u) = 2$, then $u \in \{x_2, x_3, \dots, x_{j-1}\}$. Suppose $d_U(x_p) = 2$, $p \in [2, j-1]$. By the proof of Claim 4, $N[u]$ is claw-free for any vertex u in $\{x_{p+1}, \dots, x_h\}$, which contradicts that x_j is a claw center. It follows that $d_U(S) = |S| - 1$. $\square$

Claim 8. For any vertex u_i ($1 \leq i \leq m$) in $N_P(R) = \{u_1, \dots, u_m\}$, if $d_U(u_i) = 2$, then $d_{R_i}(u_i) \leq 1$.

Proof. Without loss of generality, we consider u_2 . Suppose $\{v_i, v_j\} = N_U(u_2)$ with $0 \leq i < j \leq m$ and $z_1, z_2 \in N_{R_i}(u_2)$, where z_1, z_2 are two distinct vertices. By Claim 3, u_2 is a claw center. By Lemma 6, suppose y_1, y_2 are the two distinct domination vertices in $N(u_2)$. Then $N[y_1], N[y_2]$ are claw-free and $v_i, v_j \in N(y_1) \cup N(y_2)$. Thus $\{y_1, y_2\} \cap V(C) \neq \emptyset$. Without loss of generality, suppose $y_1 \in V(C)$. If $y_2 \in V(R)$, then $v_iy_1, v_jy_1 \in E(G)$, a contradiction to Claim 2. Suppose $y_2 \in V(C)$. By Claim 2, v_i, v_j can not both be incident with y_1 or y_2 . Without loss of generality, suppose $z_1y_1, z_2y_2, v_iy_1, v_jy_2 \in E(G)$. Obviously, $y_1 \in N_P(R) - \{u_2\}$. Suppose $y_1 = u_j$, $j \in \{1, 3, \dots, m\}$. If $v_i \neq v_{j-1}$, then $G[y_1, y_1^+, v_i, z_1] = K_{1,3}$, a contradiction. Suppose $v_i = v_{j-1}$. Then, by $G[y_1, y_1^+, v_i, z_1] \neq K_{1,3}$, $y_1^+v_i \in E(G)$. Obviously, all the vertices in $C(u_{j-1}, v_{j-1}^-)$ can be inserted into $C[y_1, u_{j-1}]$. Let $y_1P_1u_{j-1}$ denote the path by the above inserting process with $V(P_1) = V(C)$ and $u_{j-1}P_Ry_1$ denote the path connecting u_{j-1} and y_1 with internal vertices in R . Then we can get a cycle $C' = y_1^-P_1^-v_{j-1}y_1^+P_1u_{j-1}P_Ry_1$ longer than C , a contradiction. $\square$

Claim 9. Consider the following: $\sum_{i=0}^m d_P(v_i) \leq |P| - 1$ and $\sum_{i=0}^m d_P(v_i) + \sum_{z \in R_I} d_P(z) \leq |P| + m - 1$.

Proof. Obviously, $V(P) = \bigcup_{i=0}^{m-1} V(P[v_i, v_{i+1}^-]) \cup V(P[v_m, y])$ and $\sum_{i=0}^m d_P(v_i) = d_U(P)$. Then $\sum_{i=0}^m d_P(v_i) = \sum_{i=0}^{m-1} d_U(P[v_i, v_{i+1}^-]) + d_U(P[v_m, y])$. By Claim 5, $d_U(P[v_m, y]) \leq |P[v_m, y]| - 1$. By Claim 4, $d_U(P[v_i, v_{i+1}^-]) \leq |P[v_i, v_{i+1}^-]|$ for $1 \leq i \leq m - 1$. Thus $\sum_{i=0}^m d_P(v_i) \leq \sum_{i=0}^{m-1} |P[v_i, v_{i+1}^-]| + |P[v_m, y]| - 1 = |P| - 1$.

Obviously, $N_P(R_I) \subseteq \{u_1, u_2, \dots, u_m\}$. Then $\sum_{i=0}^m d_P(v_i) + \sum_{z \in R_I} d_P(z) = d_U(P) + \sum_{z \in R_I} d_{N_P(R_I)}(z) = \sum_{i=0}^{m-1} (d_U(P[v_i, v_{i+1}^-]) + \sum_{z \in R_I} d_{u_i}(z)) + d_U(P[v_m, y])$. Without loss of generality, we consider $P[v_1, v_2^-]$. Let S_{u_2} denote the U -segment containing u_2 . By Claim 4, $d_U(S) \leq |S|$ for any U -segment in $P[v_1, v_2^-]$. Obviously, $\sum_{z \in R_I} d_{u_2}(z) = d_{R_I}(u_2)$. By Claim 6, $d_{R_I}(u_2) \leq 2$.

Suppose $d_U(u_2) = 0$. Then u_2 is the first vertex in S_{u_2} and $S_{u_2} = C[u_2, v_2]$. By Lemma 8(a), $N_U(S_{u_2}) \subseteq \{v_2\}$. Thus $d_U(S_{u_2}) = |S_{u_2}| - 1$ and $d_U(P[v_1, v_2^-]) + \sum_{z \in R_I} d_{u_2}(z) = \sum_{S \neq S_{u_2}} d_U(S) + d_U(S_{u_2}) + d_{R_I}(u_2) \leq \sum_{S \neq S_{u_2}} |S| + (|S_{u_2}| - 1) + 2 = |P[v_1, v_2^-]| + 1$.

Suppose $d_U(u_2) = 1$. If $d_{R_I}(u_2) \leq 1$, then $d_U(P[v_1, v_2^-]) + \sum_{z \in R_I} d_{u_2}(z) = \sum_{S \neq S_{u_2}} d_U(S) + d_U(S_{u_2}) + d_{R_I}(u_2) \leq \sum_{S \neq S_{u_2}} |S| + |S_{u_2}| + 1 = |P[v_1, v_2^-]| + 1$. If $d_{R_I}(u_2) = 2$, then, by Claim 7, $d_U(S_{u_2}) = |S_{u_2}| - 1$. Thus $d_U(P[v_1, v_2^-]) + \sum_{z \in R_I} d_{u_2}(z) = \sum_{S \neq S_{u_2}} d_U(S) + d_U(S_{u_2}) + d_{R_I}(u_2) \leq \sum_{S \neq S_{u_2}} |S| + (|S_{u_2}| - 1) + 2 = |P[v_1, v_2^-]| + 1$.

Suppose $d_U(u_2) = 2$. Then, by Claim 8, $d_{R_I}(u_2) \leq 1$. Thus $d_U(P[v_1, v_2^-]) + \sum_{z \in R_I} d_{u_2}(z) = \sum_{S \neq S_{u_2}} d_U(S) + d_U(S_{u_2}) + d_{R_I}(u_2) \leq \sum_{S \neq S_{u_2}} |S| + |S_{u_2}| + 1 = |P[v_1, v_2^-]| + 1$.

It follows that $d_U(P[v_1, v_2^-]) + \sum_{z \in R_I} d_{u_2}(z) \leq |P[v_1, v_2^-]| + 1$. Similarly, for any $i \in [1, m - 1]$, $d_U(P[v_i, v_{i+1}^-]) \leq |P[v_i, v_{i+1}^-]| + 1$. By Claim 5, $d_U(P[v_m, y]) \leq |P[v_m, y]| - 1$. Thus $d_U(P) + \sum_{z \in R_I} d_P(z) \leq \sum_{i=0}^{m-1} (|P[v_i, v_{i+1}^-]| + 1) + |P[v_m, y]| - 1 \leq |P| + m - 1$. $\square$

Claim 10. $d_P(R) = k$ and R is hamiltonian-connected for each component R of $G - P$.

Proof. $N_{G-P}(v_i) \cap N_{G-P}(v_j) = \emptyset$ from Lemma 8(a), for $0 \leq i \neq j \leq m$, and then $\sum_{i=0}^m d_{G-P}(v_i) \leq n - |P| - |R|$. By the connectedness of G , $m \geq k$. If $m \geq k + 2$, then we get an independent set $\{v_0, v_1, \dots, v_m\}$ of order at least $k + 3$. By Claim 9, $\sum_{i=0}^m d(v_i) = \sum_{i=0}^m d_{G-P}(v_i) + \sum_{i=0}^m d_P(v_i) \leq (n - |P| - |R|) + (|P| - 1) = n - 1 - |R|$, a contradiction to $\sigma_{k+3}(G) \geq n + 2k - 2$. Suppose $m = k + 1$ and $z \in V(R)$. Then we can get an independent set $\{z, v_0, v_1, \dots, v_m\}$ of order $k + 3$. By Claim 9, $d_P(z) + \sum_{i=0}^{k+1} d_P(v_i) \leq (k + 1) - 1 + |P| = |P| + k$. Then $d(z) + \sum_{i=0}^{k+1} d(v_i) = \sum_{i=0}^{k+1} d_{G-P}(v_i) + d_R(z) + \sum_{i=0}^{k+1} d_P(v_i) + d_P(z) \leq (n - |P| - |R|) + (|R| - 1) + (|P| + k) = n + k - 1$, a contradiction to $\sigma_{k+3}(G) \geq n + 2k - 2$ by $k \geq 2$. Thus $m = k$.

Suppose that R is not hamiltonian-connected. Then by Ore's theorem [20], suppose $z_1, z_2 \in V(R)$ with $z_1 z_2 \notin E(G)$ and $d_R(z_1) + d_R(z_2) \leq |R|$. We can get an independent set $\{z_1, z_2, v_0, v_1, \dots, v_k\}$ with order $k + 3$. By Claim 9,

$d(z_1) + d(z_2) + \sum_{i=0}^k d(v_i) = d_R(z_1) + d_R(z_2) + \sum_{i=0}^k d_{G-P}(v_i) + (\sum_{i=0}^k d_P(v_i) + d_P(z_1) + d_P(z_2)) \leq |R| + (n - |P| - |R|) + (|P| + k - 1) = n + k - 1$, a contradiction to $\sigma_{k+3}(G) \geq n + 2k - 2$ by $k \geq 2$. $\square$

If $G - P$ contains only one component R , then, by Claim 10, there is a spanning 3-ended tree in G . Thus we only consider the case that $G - P$ contains at least two components and suppose R' is a component in $G - P - R$.

Claim 11. Consider the following: $N_{R'}(v_i) \neq \emptyset$ for some $i \in [1, k]$.

Proof. Suppose $N_{R'}(v_i) = \emptyset$ for any $i \in [1, k]$. Then $\sum_{i=0}^k d_{G-P}(v_i) \leq n - |P| - |R| - |R'|$. Let $z_1 \in V(R)$, $z_2 \in V(R')$. Then we get an independent set $\{z_1, z_2, v_0, v_1, \dots, v_k\}$ of order $k + 3$. By Claim 10, $d_P(z_2) \leq k$ and, by Claim 9, $\sum_{i=0}^k d(v_i) + d_P(z_1) \leq |P| + k - 1$. Thus $\sum_{i=0}^k d(v_i) + d(z_1) + d(z_2) = d_P(z_2) + d_R(z_1) + d_{R'}(z_2) + (\sum_{i=0}^k d_P(v_i) + d_P(z_1)) + \sum_{i=0}^k d_{G-P}(v_i) \leq k + (|R| - 1) + (|R'| - 1) + (|P| + k - 1) + (n - |P| - |R| - |R'|) = n + 2k - 3$, a contradiction to $\sigma_{k+3}(G) \geq n + 2k - 2$. $\square$

By Claim 11, we assume $N_{R'}(v_i) \neq \emptyset$ for some $i \in [1, k]$. By Lemma 8(a), $N_{R'}(v_j) = \emptyset$ for $j \in [0, i - 1] \cup [i + 1, k]$.

Claim 12. $S_i - \{u_{i+1}\}$ contains a second noninsertible vertex v'_i .

Proof. Suppose $S_i - \{u_{i+1}\}$ contains only one noninsertible vertex v_i . Then we can get a path $P_1[u_{i+1}, u_i]$ such that $V(P_1) = V(C) - \{v_i\}$ by inserting all the vertices in $S_i - \{v_i\}$ to $C[u_{i+1}, u_i]$. Suppose $|V(R)| = \{u\}$. Then we get a cycle $C' = P_1[u_{i+1}, u_i] u u_{i+1}$. Let $P' = V(C') - \{u_0\}$. Then $w(G - P') < w(G - P)$, a contradiction to (T1). Suppose $|V(R)| \geq 2$. If $N_R(u_i) \cup N_R(u_{i+1}) = \{z\}$, then $N_P(R) \cup \{z\} - \{u_i, u_{i+1}\}$ is a vertex cut of G with order $k - 1$, a contradiction to Claim 10. Thus $|N_R(u_i) \cup N_R(u_{i+1})| \geq 2$. By Claim 10, there is a hamiltonian path $u_i P_2 u_{i+1}$ of $R \cup \{u_i, u_{i+1}\}$. Thus we can get a cycle $C_1 = u_{i+1} P_1 u_i P_2 u_{i+1}$ longer than C , a contradiction. $\square$

Now, we complete the proof of Theorem 5. By Claim 12 and Lemma 9, $U' = \{v_0, \dots, v_{i-1}, v'_i, v_{i+1}, \dots, v_k\}$ is an independent set. Then, by the preceding proof, U' has the same properties as U . By Lemma 9, $N_{G-P}(v_i) \cap N_{G-P}(v_j) = \emptyset$, for any two distinct vertices $v_i, v_j \in U'$. Thus $\sum_{v \in U'} d_{G-P}(v) \leq n - |P| - |R| - |R'|$. By Claim 9, $\sum_{v \in U'} d_P(v) \leq |P| - 1$. Let $z_1 \in V(R)$, $z_2 \in V(R')$. Then $U' \cup \{z_1, z_2\}$ is an independent set of order $k + 3$ in G . Thus $\sum_{v \in U'} d(v) = \sum_{v \in U'} d_P(v) + \sum_{v \in U'} d_{G-P}(v) \leq (|P| - 1) + (n - |P| - |R| - |R'|) = n - 1 - |R| - |R'|$. By Claim 10, $d(z_1) \leq |R| - 1 + k$, $d(z_2) \leq |R'| - 1 + k$. Thus $d(z_1) + d(z_2) + \sum_{v \in U'} d(v) \leq (|R| - 1 + k) + (|R'| - 1 + k) + (n - 1 - |R| - |R'|) = n + 2k - 3$, a contradiction to $\sigma_{k+3}(G) \geq n + 2k - 2$. It follows that Theorem 5 holds.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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