Research Article

Merrifield-Simmons Index in Random Phenylene Chains and Random Hexagon Chains

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The author obtains explicit expressions for the expected value of the Merrifield-Simmons index of a random phenylene chain and a random hexagon chain, respectively. The author also computes the corresponding entropy constants and obtains the maximum and minimum values in both random systems, respectively.

1. Introduction

Let $G = (V, E)$ be a simple undirected graph on $n$ vertices. Two vertices of $G$ are said to be independent if they are not adjacent in $G$. A $k$-independent set of $G$ is a set of $k$ mutually independent vertices. Denote by $i(G, k)$ the number of the $k$-independent sets of $G$. By definition, the empty vertex set is an independent set. Then $i(G, 0) = 1$ for any graph $G$. The Merrifield-Simmons index of $G$, denoted by $i(G)$, is defined as $i(G) = \sum_{k=0}^{n} i(G, k)$. So $i(G)$ is equal to the total number of the independent sets of $G$. The Merrifield-Simmons index was introduced in 1982 by Prodinger and Tichy [1], where it was called the Fibonacci number of a graph. The Merrifield-Simmons index is one of the most popular topological indices in chemistry intensively studied, as seen in the monograph [2]. Recently, there have been many papers studying the Merrifield-Simmons index for a graph. For more details see [3–8], among others.

Phenylenes are a class of conjugated hydrocarbons composed of six- and four-membered rings, where the six-membered rings (hexagons) are adjacent only to four-membered rings, and every four-membered ring is adjacent to a pair of nonadjacent hexagons. If each six-membered ring of a phenylene is adjacent only to two four-membered rings, we say that is a phenylene chain. Due to their aromatic and antiaromatic rings, phenylenes exhibit unique physicochemical properties. In Figure 1, some examples of phenylene chains are presented. The unique phenylene chains for $n = 1$ and $n = 2$ are shown in Figure 1. More generally, a phenylene chain with $n + 1$ hexagons (see Figure 2) can be regarded as a phenylene chain $PH_n$, with $n$ hexagons to which a new terminal hexagon has been adjoined by a four-membered ring. But, for $n \geq 3$, the terminal hexagon can be attached in three ways, which results in the local arrangements we describe as $PH_{1, n+1}$, $PH_{2, n+1}$, and $PH_{3, n+1}$ (see Figure 3). Naturally, we define a random phenylene chain $PH(n, p)$ with $n$ hexagons as a phenylene chain obtained by stepwise addition of terminal hexagons. At each step $k (=3, 4, \ldots, n)$ a random selection is made from one of the three possible constructions: (1) $PH_{k-1} \rightarrow PH_{1, k}$, with probability $p$, (2) $PH_{k-1} \rightarrow PH_{2, k}$, with probability $p$, or (3) $PH_{k-1} \rightarrow PH_{3, k}$, with probability $q = 1 - 2p$. We assume that the probability $p$ is a constant, invariant to the step parameter $k$. That is, the process described is a zeroth-order Markov process.

By eliminating, “squeezing out,” the squares from a phenylene, a catacondensed hexagonal system (which may be jammed) is obtained, called the hexagonal squeeze of the respective phenylene (see Figure 4). Clearly, there is a one-to-one correspondence between a phenylene (PH) and its hexagonal squeeze (HS). Both possess the same number of hexagons. The respective hexagonal squeeze of a random phenylene chain $PH(n, p)$ is a random hexagonal chain, and we denote it by $HS_{n, p}$.

The Wiener index and the number of perfect matchings of a random hexagonal chain $HS_{n, p}$ have been studied.
2. Merrifield-Simmons Index of a Random Phenylene Chain

Firstly, let us recall some results in [14], useful to this paper.

**Lemma 1** (see [14]). Consider $i(G_1 \cup G_2) = i(G_1)(G_2)$.

**Lemma 2** (see [14]). Let $u$ be a vertex of $G$ and let $N_u$ be the subset of $V(G)$ consisting of the vertex $u$ and its neighbors. Then

$$i(G) = i(G - u) + i(G - N_u). \quad (1)$$

As described above, the phenylene chain $PH_n$ can be obtained by adjoining to $PH_{n-1}$ a hexagon by a 4-membered ring. For this construction, the following relations are easily obtained by Lemmas 1 and 2.

**Lemma 3.** Let $PH_n$ be denoted as a phenylene chain as in Figure 2; then for $n \geq 2$ one has

$$i(PH_n) = 8i(PH_{n-1}) + 5[i(PH_{n-1} - a_{n-1}) + i(PH_{n-1} - b_{n-1})] + 3i(PH_{n-1} - b_{n-1}), \quad (2)$$

for $a = t_n$,

$$6s(PH_{n-1}) + 4i(PH_{n-1} - a_{n-1}) + 3i(PH_{n-1} - b_{n-1}), \quad (2)$$

for $a = u_n$,

$$5i(PH_{n-1}) + 3i(PH_{n-1} - a_{n-1}) + 5i(PH_{n-1} - b_{n-1}), \quad (2)$$

for $a = v_n$,

$$5i(PH_{n-1}) + 3i(PH_{n-1} - a_{n-1}) + 5i(PH_{n-1} - b_{n-1}), \quad (2)$$

for $a = w_n$.

**Proof.** Consider

$$i(PH_n) = i(PH_{n-1} - a_{n-1}) + i(PH_{n-1} - b_{n-1})] + 5[i(PH_{n-1} - a_{n-1}) + i(PH_{n-1} - b_{n-1})] + 3i(PH_{n-1} - b_{n-1}),$$

for $a = t_n$,

$$6i(PH_{n-1}) + 4i(PH_{n-1} - a_{n-1}) + 3i(PH_{n-1} - b_{n-1}), \quad (2)$$

for $a = u_n$,

$$5i(PH_{n-1}) + 3i(PH_{n-1} - a_{n-1}) + 5i(PH_{n-1} - b_{n-1}), \quad (2)$$

for $a = v_n$,

$$5i(PH_{n-1}) + 3i(PH_{n-1} - a_{n-1}) + 5i(PH_{n-1} - b_{n-1}), \quad (2)$$

for $a = w_n$.

$$i(PH_n - t_n) = i(PH_{n-1} - t_n) + i((PH_{n-1} - a_{n-1}) \cup P_3)$$

$$= i(PH_{n-1} - u_n) + i((PH_{n-1} - a_{n-1}) \cup P_2)$$

$$= i((PH_{n-1} - u_n) - a_{n-1}) \cup P_3)$$

$$= i(PH_{n-1} - a_{n-1}) \cup P_3$$

$$= i(PH_{n-1} - a_{n-1}) \cup P_3$$

$$= i(PH_{n-1} - u_n) + i((PH_{n-1} - a_{n-1}) \cup P_2)$$

$$= i((PH_{n-1} - u_n) - a_{n-1}) \cup P_3)$$

$$= i(PH_{n-1} - a_{n-1}) \cup P_3$$

$$= i(PH_{n-1} - u_n) + i((PH_{n-1} - a_{n-1}) \cup P_2)$$

$$= i((PH_{n-1} - u_n) - a_{n-1}) \cup P_3)$$

$$= i(PH_{n-1} - a_{n-1}) \cup P_3$$

$$= i(PH_{n-1} - u_n) + i((PH_{n-1} - a_{n-1}) \cup P_2)$$

$$= i((PH_{n-1} - u_n) - a_{n-1}) \cup P_3)$$

$$= i(PH_{n-1} - a_{n-1}) \cup P_3$$

$$= i(PH_{n-1} - u_n) + i((PH_{n-1} - a_{n-1}) \cup P_2)$$

$$= i((PH_{n-1} - u_n) - a_{n-1}) \cup P_3)$$

$$= i(PH_{n-1} - a_{n-1}) \cup P_3$$

$$= i(PH_{n-1} - u_n) + i((PH_{n-1} - a_{n-1}) \cup P_2)$$

$$= i((PH_{n-1} - u_n) - a_{n-1}) \cup P_3)$$

$$= i(PH_{n-1} - a_{n-1}) \cup P_3$$
\[
\begin{align*}
+ i (P_2) & i (PH_{n-1} - a_{n-1}) \\
= 6i (PH_{n-1}) + 3i (PH_n - a_{n-1}) \\
+ 4i (PH_{n-1} - b_{n-1}).
\end{align*}
\]

(3)

By the symmetry,
\[
i (PH_n - v_n) = 6i (PH_{n-1}) + 4i (PH_{n-1} - a_{n-1}) \\
+ 3i (PH_{n-1} - b_{n-1}),
\]
\[
i (PH_n - w_n) = 5i (PH_{n-1}) + 3i (PH_{n-1} - a_{n-1}) \\
+ 5i (PH_{n-1} - b_{n-1}).
\]

(4)

For a random phenylene chain PH\(_{n,p}\), the Merrifield-Simmons indices \(i(PH_{n,p})\), \(i(PH_{n,p} - a_n)\), and \(i(PH_{n,p} - b_n)\) are random variables and we denote their expected values by \(I_n = E[i(PH_{n,p})], U_n = E[i(PH_{n,p} - a_n)],\) and \(V_n = E[i(PH_{n,p} - b_n)]\), respectively.

By the definition of PH\(_{n,p}\), we immediately have that, for \(n \geq 2\),
\[
\begin{align*}
I_n &= 8I_{n-1} + 5 [U_{n-1} + V_{n-1}], \\
U_n &= pE \left[ 5i \left( PH_{n-1,p} - a_{n-1} \right) \\
+ 3i \left( PH_{n-1,p} - b_{n-1} \right) \right] \\
+ pE \left[ 6i \left( PH_{n-1,p} - a_{n-1} \right) \\
+ 4i \left( PH_{n-1,p} - b_{n-1} \right) \\
+ 3i \left( PH_{n-1,p} - b_{n-1} \right) \right] \\
&= (6 - p) I_{n-1} + (4 - 2p) U_{n-1} + (3 + 3p) V_{n-1}.
\end{align*}
\]

(5)

By the symmetry,
\[
V_n = (6 - p) I_{n-1} + (4 - 2p) U_{n-1} + (3 + 3p) V_{n-1}.
\]

(6)

To solve the recursion equation, we use the method of the generating functions. Set
\[
I(t) = \sum_{n \geq 1} I_n t^n, \quad U(t) = \sum_{n \geq 1} U_n t^n, \quad V(t) = \sum_{n \geq 1} V_n t^n.
\]

(7)

Then we have that
\[
\begin{align*}
I(t) - 18t &= 8I(t) + 5t [U(t) + V(t)], \\
U(t) - 13t &= (6 - p) I(t) + (4 - 2p) tV(t) \\
&\quad + (3 + 3p) tU(t),
\end{align*}
\]
\[ V(t) - 13t = (6 - p) t I(t) + (3 + 3p) t V(t) + (4 - 2p) t U(t). \]

Recall that \( I_1 = 18, I_2 = 274, \) and \( U_1 = V_1 = 13. \) Solving the equations, we have that
\[
I(t) = \frac{2t(9 + 2t - 9pt)}{1 - 15t - pt - 4t^2} + \frac{18pt}{2}.
\]

So we have the following result.

**Theorem 4.** If \( 0 \leq p \leq 1/2, \) then for \( n \geq 2 \) one has that
\[
E[i(\text{PH}_{n,p})] = \left(9 + \frac{204\sqrt{511}}{511}\right)\left(\frac{23 + \sqrt{511}}{3}\right)^{n-1} + \left(9 - \frac{204\sqrt{511}}{511}\right)\left(\frac{23 - \sqrt{511}}{3}\right)^{n-1},
\]
\[
E[i(\text{PH}_{n,0})] = \left(9 + \frac{139\sqrt{241}}{241}\right)\left(\frac{15 + \sqrt{241}}{2}\right)^{n-1} + \left(9 - \frac{139\sqrt{241}}{241}\right)\left(\frac{15 - \sqrt{241}}{2}\right)^{n-1}.
\]

The following corollary is easily obtained from Theorem 4 which gives the limits of the entropy constant \( \log E[i(\text{PH}_{n,p})]/|V(\text{PH}_{n,p})| \) as \( n \to +\infty, \) where \( V(\text{PH}_{n,p}) \) is the vertex set of \( \text{PH}_{n,p}. \)

**Corollary 5.** If \( 0 \leq p \leq 1/2, \) then for \( n \geq 2, \) one has
\[
\lim_{n \to +\infty} \frac{\log E[i(\text{PH}_{n,p})]}{6n} = \frac{15 + p + \sqrt{241 - 42p + p^2}}{12}.
\]

It is easy to check that \( f(p) = \frac{(15 + p + \sqrt{241 - 42p + p^2})}{12} \) is a monotonic decreasing function on \( p, \) so the limit of \( \log E[i(\text{PH}_{n,p})]/6n \) has the maximum value \((15 + \sqrt{241})/12 \approx 2.5437 \) at \( p = 0 \) and the minimum value \((31 + \sqrt{881})/24 \approx 2.5284 \) at \( p = 1/2. \) That is say, for different \( p \) \((0 \leq p \leq 1/2), \) the limit of \( \log E[i(\text{PH}_{n,p})]/6n \) has little difference.

### 3. Merrifield-Simmons Index of a Random Hexagon Chain

Similar to the phenylene chain \( \text{PH}_n, \) the hexagon chain \( \text{HS}_n \) can be obtained by adjoining to \( \text{HS}_{n-1} \) a hexagon. For this construction the following relations are easily obtained by Lemmas 1 and 2.

**Lemma 6.** Let \( \text{HS}_n \) be denoted as a hexagon squeeze of a phenylene chain \( \text{PH}_n \) as in Figure 2; then for \( n \geq 2 \) one has
\[
i(\text{HS}_n) = 3i(\text{HS}_{n-1}) + 2 \left[i(\text{HS}_{n-1} - a_{n-1}) + i(\text{HS}_{n-1} - b_{n-1})\right] + i(\text{HS}_{n-1} - a_{n-1} - b_{n-1}),
\]
\[
\begin{align*}
  i(HS_n - a - b) &= \\
  &= \begin{cases} 
    2i(HS_{n-1}) + i(HS_{n-1} - b_{n-1}), & \text{if } a = t_n, b = u_n, \\
    i(HS_{n-1}) + i(HS_{n-1} - a_{n-1}), & \text{if } a = t_n, b = v_n, \\
    i(HS_{n-1}) + i(HS_{n-1} - a_{n-1}), & \text{if } a = v_n, b = w_n, \\
    2i(HS_{n-1}) + i(HS_{n-1} - a_{n-1}), & \text{if } a = v_n, b = w_n. 
  \end{cases}
\end{align*}
\]

Proof. Consider

\[
\begin{align*}
i(HS_n) &= i(HS_n - t_n) + i(HS_n - t_n - a_{n-1} - u_n) \\
  &= i(HS_n - t_n) + i((HS_n - t_n - a_{n-1}) \cup P_2) \\
  & \quad + i((HS_n - a_{n-1}) \cup P_1) \\
  & \quad + i(HS_n - a_{n-1} - b_{n-1}) \\
  &= i(P_2) i(HS_{n-1}) \\
  & \quad + i(P_1) [i(HS_{n-1} - b_{n-1}) + i(HS_{n-1} - a_{n-1})] \\
  & \quad + i(HS_{n-1} - a_{n-1} - b_{n-1}) \\
  &= 3i(HS_{n-1}) \\
  & \quad + 2 [i(HS_{n-1} - a_{n-1}) + i(HS_{n-1} - b_{n-1})] \\
  & \quad + i(HS_{n-1} - a_{n-1} - b_{n-1}), \\
i(HS_n - t_n) &= i(HS_n - t_n) + i((HS_n - t_n - a_{n-1}) \cup P_2) \\
  &= i(P_2) i(HS_{n-1}) + i(P_1) i(HS_{n-1} - b_{n-1}) \\
  &= 3i(HS_{n-1}) + 2i(HS_{n-1} - b_{n-1}), \\
i(HS_n - u_n) &= i(HS_n - u_n) \\
  &= i(HS_n - u_n - t_n) \\
  & \quad + i(HS_{n-1} - u_n - t_n - a_{n-1}) \\
  &= i(HS_n - u_n - t_n) + i(HS_{n-1} - b_{n-1}),
\end{align*}
\]

By the symmetry,

\[
\begin{align*}
i(HS_n - v_n) &= 2i(HS_{n-1}) + i(HS_n - a_n) \\
  & \quad + 2i(HS_{n-1} - b_{n-1}) + i(HS_n - a_n - b_{n-1}), \quad (17) \\
i(HS_n - w_n) &= 3i(HS_{n-1}) + 2i(HS_n - a_n), \\
i(HS_n - v_n - w_n) &= 2i(HS_{n-1}) + i(HS_n - a_n) \\
\end{align*}
\]

For a random hexagon chain $HS_{n,p}$, the Merrifield-Simmons indices $i(HS_{n,p})$, $i(HS_{n,p} - a_n)$, $i(HS_{n,p} - b_n)$, and $i(HS_{n,p} - a_n - b_n)$ are also random variables, and in not confusion circumstances we also denote their expected values by $I_n = E[i(HS_{n,p})], U_n = E[i(HS_{n,p} - a_n)], V_n = E[i(HS_{n,p} - b_n)]$, and $W_n = E[i(HS_{n,p} - a_n - b_n)]$, respectively. Then we immediately have that, for $n \geq 2$,

\[
\begin{align*}
  I_n &= 3I_{n-1} + 2[U_{n-1} + V_{n-1}] + W_{n-1}, \\
  U_n &= pE[i(HS_{n,p} - t_n)] + pE[i(HS_{n,p} - v_n)] \\
  & \quad + (1 - 2p) E[i(HS_{n,p} - u_n)] \\
  &= pE[i(HS_{n-1,p}) + 2i(HS_{n-1,p} - b_{n-1})]
\end{align*}
\]
+ \rho E \left[ 2i \left( H_{n-1,p} \right) + i \left( H_{n-1,p} - a_{n-1} \right) \right]
+ 2i \left( H_{n-1,p} - b_{n-1} \right)
+i \left( H_{n-1,p} - a_{n-1} - b_{n-1} \right)]
+ (1 - 2\rho)
\cdot E \left[ 2i \left( H_{n-1,p} \right) \right]
+ 2i \left( H_{n-1,p} - a_{n-1} \right) + i \left( H_{n-1,p} - b_{n-1} \right)
+i \left( H_{n-1,p} - a_{n-1} - b_{n-1} \right)]
= (2 + \rho) E \left[ i \left( H_{n-1,p} \right) \right]
+ (2 - 3\rho) E \left[ i \left( H_{n-1,p} - a_{n-1} \right) \right]
+ (1 + 2\rho) E \left[ i \left( H_{n-1,p} - b_{n-1} \right) \right]
+ (1 - \rho) E \left[ i \left( H_{n-1,p} - a_{n-1} - b_{n-1} \right) \right]
= (2 + \rho) I_{n-1} + (2 - 3\rho) U_{n-1}
+ (1 + 2\rho) V_{n-1} + (1 - \rho) W_{n-1}.

W_n = \rho E \left[ i \left( H_{n,p} - u_n - v_n \right) \right] + \rho E \left[ i \left( H_{p,n} - v_n - w_n \right) \right]
+ (1 - 2\rho) E \left[ i \left( H_{n-1,p} - u_n - v_n \right) \right]
= \rho E \left[ 2i \left( H_{n-1,p} \right) + i \left( H_{n-1,p} - b_{n-1} \right) \right]
+ \rho \left[ 2i \left( H_{n-1,p} \right) + i \left( H_{n-1,p} - a_{n-1} \right) \right]
+ (1 - 2\rho) E \left[ i \left( H_{n-1,p} \right) + i \left( H_{n-1,p} - b_{n-1} \right) \right]
+i \left( H_{n-1,p} - a_{n-1} \right)
+i \left( H_{n-1,p} - a_{n-1} - b_{n-1} \right)]
= (1 + 2\rho) E \left[ i \left( H_{n-1,p} \right) \right]
+ (1 - \rho) E \left[ i \left( H_{n-1,p} - a_{n-1} \right) \right]
+ (1 - \rho) E \left[ i \left( H_{n-1,p} - b_{n-1} \right) \right]
+ (1 - 2\rho) E \left[ i \left( H_{n-1,p} - a_{n-1} - b_{n-1} \right) \right]
= (1 + 2\rho) I_{n-1} + (1 - \rho) U_{n-1}
+ (1 - \rho) V_{n-1} + (1 - 2\rho) W_{n-1}.

By the symmetry,

\begin{align*}
V_n &= (2 + \rho) I_{n-1} + (1 + 2\rho) U_{n-1}
+ (2 - 3\rho) V_{n-1} + (1 - \rho) W_{n-1}.
\end{align*}

Just as above, we set

\begin{align*}
I(t) &= \sum_{n \geq 1} I_n t^n, \\
U(t) &= \sum_{n \geq 1} U_n t^n, \\
V(t) &= \sum_{n \geq 1} V_n t^n, \\
W(t) &= \sum_{n \geq 1} W_n t^n.
\end{align*}

Then we have that

\begin{align*}
I(t) - 18t &= 3t I(t) + 2t [U(t) + V(t)] + tW(t), \\
U(t) - 13t &= (2 + \rho) tI(t) + (2 - 3\rho) tU(t) \\
&\quad + (1 + 2\rho) tV(t) + (1 - \rho) tW(t), \\
V(t) - 13t &= (2 + \rho) tI(t) + (2 - 3\rho) tU(t) \\
&\quad + (1 + 2\rho) tV(t) + (1 - \rho) tW(t), \\
W(t) - 8t &= (1 + 2\rho) tI(t) + (1 - \rho) tU(t) \\
&\quad + (1 - 2\rho) tV(t) + (1 - 2\rho) tW(t).
\end{align*}

Recall that \( I_1 = 18, U_1 = V_1 = 13, \) and \( W_1 = 8. \)

Solving the equations, we have that

\begin{align*}
I(t) &= \frac{6t (3 - 2t + 9\rho t)}{1 - 7t + 6\rho t + 4t^2 - 36\rho t^2} \\
&= \frac{At}{1 - \left( \left( 7 - 3\rho + \sqrt{33 + 30\rho + 9\rho^2} \right)/2 \right) t}
\quad + \frac{Bt}{1 - \left( \left( 7 - 3\rho - \sqrt{33 + 30\rho + 9\rho^2} \right)/2 \right) t},
\end{align*}

where

\begin{align*}
A &= 9 + \frac{(59 - 9\rho) \sqrt{33 + 30\rho + 9\rho^2}}{33 + 30\rho + 9\rho^2}, \\
B &= 9 - \frac{(59 - 9\rho) \sqrt{33 + 30\rho + 9\rho^2}}{33 + 30\rho + 9\rho^2}.
\end{align*}

So we have the following result.

**Theorem 7.** If \( 0 \leq \rho \leq 1/2, \) then for \( n \geq 2 \) one has that

\begin{align*}
E \left[ i \left( H_{n,p} \right) \right] &= A \left( \frac{7 - 3\rho + \sqrt{33 + 30\rho + 9\rho^2}}{2} \right)^{n-1} \\
&\quad + B \left( \frac{7 - 3\rho - \sqrt{33 + 30\rho + 9\rho^2}}{2} \right)^{n-1},
\end{align*}

(24)
where
\[
A = 9 + \frac{(59 - 9p) \sqrt{33 + 30p + 9p^2}}{33 + 30p + 9p^2},
\]
\[
B = 9 - \frac{(59 - 9p) \sqrt{33 + 30p + 9p^2}}{33 + 30p + 9p^2}.
\]

Specifically, we have that
\[
E[i(HS_{n,1/2})] = \left(9 + \frac{109\sqrt{201}}{201}\right)\left(11 + \frac{\sqrt{201}}{4}\right)^{n-1} + \left(9 - \frac{109\sqrt{201}}{201}\right)\left(11 - \frac{\sqrt{201}}{4}\right)^{n-1},
\]
\[
E[i(HS_{n,1/3})] = \left(9 + \frac{28\sqrt{11}}{11}\right)\left(3 + \sqrt{11}\right)^{n-1} + \left(9 - \frac{28\sqrt{11}}{11}\right)\left(3 - \sqrt{11}\right)^{n-1},
\]
\[
E[i(HS_{n,0})] = \left(9 + \frac{59\sqrt{33}}{33}\right)\left(7 + \frac{\sqrt{33}}{2}\right)^{n-1} + \left(9 - \frac{59\sqrt{33}}{33}\right)\left(7 - \frac{\sqrt{33}}{2}\right)^{n-1}.
\]

The following corollary is easily obtained from Theorem 7 which gives the limit of \(\log E[i(HS_{n,p})]/|V(HS_{n,p})|\) as \(n \to +\infty\), where \(V(HS_{n,p})\) is the vertex set of \(HS_{n,p}\).

**Corollary 8.** If \(0 \leq p \leq 1/2\), then for \(n \geq 2\), one has
\[
\lim_{n \to +\infty} \frac{\log E[i(HS_{n,p})]}{|V(HS_{n,p})|} = \lim_{n \to +\infty} \frac{\log E[i(HS_{n,p})]}{4n + 2}
\]
\[
= \frac{7 - 3p + \sqrt{33 + 30p + 9p^2}}{8}.
\]

It is easy to check that \(g(p) = (7 - 3p + \sqrt{33 + 30p + 9p^2})/8\) is a monotonic decreasing function on \(p\), so the limit of \(\log E[i(HS_{n,p})]/|V(HS_{n,p})|\) has the maximum value \((7 + \sqrt{33})/2 = 1.062\) at \(p = 0\), and the minimum value \((11 + \sqrt{201})/4 \approx 1.049\) at \(p = 1/2\). That is say, for different \(p \in [0, 1/2]\), the limit of \(\log E[i(HS_{n,p})]/|V(HS_{n,p})|\) has little difference.

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**References**


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