Research Article

Bifurcation and Solitary-Like Solutions for Compound KdV-Burgers-Type Equation

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Firstly, based on the improved sub-ODE method and the bifurcation method of dynamical systems, we investigate the bifurcation of solitary waves in the compound KdV-Burgers-type equation. Secondly, numbers of solitary patterns solutions are given for each parameter condition and numerical simulations are used to display the dynamical characteristics. Finally, we obtain twelve solitary patterns solutions under some parameter conditions, such as the trigonometric function solutions and the hyperbolic function solutions.

1. Introduction and Main Results

Consider the following compound KdV-Burgers-type equation with nonlinear terms of any order:

$$u_t + (a_1 + a_2 u^p + a_3 u^{2p}) u_x + a_4 u_{xxx} = 0, \quad (1)$$

$$p > 0, \quad (a_1, a_2, a_3, a_4) \in \mathbb{R}^4,$$

and the compound KdV-type equation with nonlinear terms of any order:

$$u_t + a_2 u^p u_x + a_3 u^{2p} u_x + a_4 u_{xxx} = 0, \quad (2)$$

$$p > 0, \quad (a_2, a_3, a_4) \in \mathbb{R}^3.$$

These equations include a number of equations which have been studied by many authors [1–3]. Song et al. [4, 5] gave some solitary wave solutions and bifurcation phase portraits of (1) with $a_4 = 1$, p being odd or even. It is necessary to point out that when we take the different parameters values, the following equations can be derived from (1).

If $a_1, a_2, a_3 = \text{constants}$ and $a_4 = 1$, (1) becomes the equation

$$u_t + a_2 u^p u_x + a_3 u^{2p} u_x + u_{xxx} = 0, \quad (3)$$

$$p > 0, \quad (a_2, a_3, a_4) \in \mathbb{R}^3,$$

which was studied in [1].

If $a_1 = 0, a_2 = 6, a_3 = \pm 6$, and $p = a_4 = 1$, (1) becomes the mKdV and combined KdV equation [6]:

$$u_t + 6uu_x \pm 6u^2 u_x + u_{xxx} = 0. \quad (4)$$

If $p = a_4 = 1, a_1 = 0, a_2 = 2\alpha$, and $a_3 = -3\beta$, (1) becomes Gardner's equation [7]:

$$u_t + 2\alpha uu_x - 3\beta u^2 u_x + u_{xxx} = 0. \quad (5)$$

Biswas and Zerrad [7] also studied the soliton perturbation theory of (5).

If $a_1 \neq 0, a_2 \neq 0$ and $a_3 \neq 0$, and $p = a_4 = 1$, (1) becomes the equation [8]

$$u_t + (a_1 + a_2 u + a_3 u^2) u_x + u_{xxx} = 0. \quad (6)$$

Dey and Coffey [9, 10] considered the kink-profile solitary wave solutions for (1) with $p = 1$ and $p = 2$. The compound KdV-Burgers-type equation (2) with $p > 1$ is a model for long-wave propagation in nonlinear media with dispersion and dissipation [11]; Coffey [3] considered soliton solutions, conservation laws, Backlund transformation, and other properties for (1) with $p = 1$ and so on.

In this paper, we apply the bifurcation method of dynamical systems [12–23] and the improved sub-ODE method [24–30] with the help of symbolic computation *Mathematica* and *Maple* to study solitary wave solutions of (1).

TABLE 1: The property of singular point $O(0, 0)$.

Conditions	Property of $O(0, 0)$
$k > 2n + 1$	The origin is the saddle node.
$k < 2n + 1$, or $\phi = 0$	The origin is a cusp.

TABLE 2: The property of singular point $O(0, 0)$.

Conditions	Property of $O(0, 0)$
n even, $k > 2\lambda + 1$ or n even, $k = 2\lambda + 1, b_n^2 + 4k(\lambda + 1) \geq 0$	Stable node
n odd, $k = 2\lambda + 1, b_n^2 + 4k(\lambda + 1) \geq 0$ or n odd, $k > 2\lambda + 1$	Elliptic saddle
$k = 2\lambda + 1, b_n^2 + 4k(\lambda + 1) < 0$ or $k < 2\lambda + 1$	Focus or center

Based on the bifurcation method and the improved sub-ODE method with the help of symbolic computation *Mathematica*, the bifurcation for (1) is derived by use of a proper transformation. To our knowledge, this type of transformation obtained has not been ever seen before in the literature. Then based on this transformation, some new solutions for (1) and (2) are found.

This paper is organized as follows. In Section 2, we derive the bifurcations of phase portraits of (1). In Section 3, we give the main results via the sub-ODE method. In Section 4, the main results and some exact travelling wave solutions for (1) are obtained. Conclusions are given in the last section.

2. Bifurcations of Phase Portraits of (1)

In this section, we discuss some bifurcation phase portraits of (1). To study dynamical behavior of (1), we first need to give the lemmas [21, 31] in this section.

Lemma 1. *Let $(0, 0)$ be a nilpotent singular point of the vector field $(y + P(x, y), Q(x, y))$, where P and Q are analytic functions in a neighborhood of the origin at least with quadratic terms in the variables x and y . Let $y = f(x)$ be the solution of the equation $y + P(x, y) = 0$ in a neighborhood of $(0, 0)$. Assume that the development of the function $Q(x, f(x))$ is of the form $a_k x^k + \text{Hot}$ (higher order terms) and $\phi(x) = (dP/dx + dQ/dy)(x, f(x)) = b_n x^n + \text{Hot}$ with $a_k \neq 0, k \geq 2$, and $n \geq 1$. For $k = 2n$ or $k = 2n + 1$, one gives the property of singular point $o(0, 0)$ as in Tables 1 and 2.*

Case 1. For this case, k is even.

Case 2. For this case, k is odd and $a_k < 0$.

To this end, substituting $u = \phi(\xi)$ with $\xi = x - Vt$ into (1), we have

$$-V\phi' + a_1\phi' + a_2\phi^p\phi' + a_3\phi^{2p}\phi' + a_4\phi''' = 0. \quad (7)$$

If $|\xi|$ tends to ∞ , we obtain the solitary wave solutions and kink wave solutions tend to some constants. This implies that

ϕ', ϕ'' , and ϕ''' tend to zero, when $|\xi|$ tends to ∞ . Integrating (7) once we get

$$(-V + a_1)\phi + \frac{a_2}{p+1}\phi^{p+1} + \frac{a_3}{2p+1}\phi^{2p+1} + a_4\phi'' = g, \quad (8)$$

where g is the constant of integration. Letting $y = \phi'$, we obtain a planar system:

$$\begin{aligned} \frac{d\phi}{d\xi} &= y, \\ \frac{dy}{d\xi} &= f(\phi) - g, \end{aligned} \quad (9)$$

where $f(\phi) = -(a_3/(2p+1)a_4)\phi^{2p+1} - (a_2/(p+1)a_4)\phi^{p+1} + ((V - a_1)/a_4)\phi$.

System (9) has the following first integral:

$$\begin{aligned} H(\phi, y) &= \frac{1}{2}y^2 + \frac{a_3}{(2p+1)(2p+2)a_4}\phi^{2p+2} \\ &\quad + \frac{a_2}{(p+1)(p+2)a_4}\phi^{p+2} \\ &\quad + \frac{-V + a_1}{2a_4}\phi^2 - \frac{g\phi}{a_4} = h, \end{aligned} \quad (10)$$

where h is the constant of integration. To study the distribution of singular points of system (9), we need to investigate the zero points of $f(\phi)$. If $g = 0$, p is odd, $a_3, a_4 \neq 0$, and $\Delta > 0$, then we get three zero points of $f(\phi)$ as

$$\begin{aligned} \phi_1 &= 0, \\ \phi_2 &= \left[-\frac{(2p+1)a_2 - a_4\sqrt{\Delta}}{2a_3(p+1)} \right]^{1/p}, \\ \phi_3 &= \left[-\frac{(2p+1)a_2 + a_4\sqrt{\Delta}}{2a_3(p+1)} \right]^{1/p}, \end{aligned} \quad (11)$$

where $\Delta = (a_2^2/a_4^2(p+1))^2 + (4(V - a_1)a_3/a_4^2(2p+1))$. Suppose $(\phi, 0)$ is a singular point of system (9). Let

$$M(\phi, 0) = \begin{pmatrix} 0 & 1 \\ -\frac{a_3\phi^{2p} + a_2\phi^p + (-V + a_1)}{a_4} & 0 \end{pmatrix} \quad (12)$$

be the coefficient matrix of the linearized system of (9). Then the eigenvalue of $M(\phi, 0)$ at $(\phi, 0)$ is

$$\lambda_{\pm}(\phi, 0) = \pm \sqrt{-\frac{a_3\phi^{2p} + a_2\phi^p + (-V + a_1)}{a_4}} = \pm \sqrt{f'(\phi)}. \quad (13)$$

TABLE 3: The local bifurcation phase portraits of system (9).

a_i ($i = 1, \dots, 4$)	$\Gamma_1 < 0$	$\Gamma_1 = 0$	Γ_1, Γ_2 $0 < \Gamma_1 < \Gamma_2$	$\Gamma_2 = 0$	$\Gamma_2 > 0$
$a_2 > 0, a_3 < 0,$ $(v - a_1)/a_4 > 0, p$ is odd.					
$a_2 < 0, a_3 < 0,$ $(v - a_1)/a_4 > 0, p$ is odd.					
$a_2 > 0, a_3 > 0,$ $(v - a_1)/a_4 > 0.$					
$a_2 < 0, a_3 > 0,$ $(v - a_1)/a_4 > 0.$					
$a_2 > 0, a_3 < 0,$ $(v - a_1)/a_4 > 0, p$ is even.					
$a_2 < 0, a_3 < 0,$ $(v - a_1)/a_4 > 0, p$ is even.					

According to Lemma 1 and the theory of dynamical systems [4, 5, 12–16], we obtain the following conclusions.

- (i) If $f'(\phi_i) > 0$, $(\phi_i, 0)$ is a saddle point.
- (ii) If $f'(\phi_i) = 0$, $(\phi_i, 0)$ is a degenerate saddle point.
- (iii) If $f'(\phi_i) < 0$, $(\phi_i, 0)$ is a center point. Solving equations $f(\phi_i) = 0$ and $H(\phi_i, 0) = 0$, respectively, we get two bifurcation curves Γ_1 and Γ_2 as follows:

$$\begin{aligned}\Gamma_1 : a_3 &= -\frac{2p+1}{4(V-a_1)(p+1)^2}a_2^2, \\ \Gamma_2 : a_3 &= -\frac{2p+1}{(V-a_1)(p+1)(p+2)^2}a_2^2.\end{aligned}\quad (14)$$

From the above analysis, we obtain the bifurcation phase portraits of system (9) as in Table 3.

3. The Improved Method and Statement of the Main Results

In this section, we will give some exact parametric representations of travelling wave solutions of (1). To study solitary wave solutions and the qualitative behavior of system (1), we first need to introduce the improved method based on [25] as follows. Suppose a nonlinear PDE,

$$p(u, u_t, u_x, u_{xt}, u_{tt}, u_{xx}, \dots) = 0, \quad (15)$$

where P is a polynomial in its arguments.

Step 1 (reduce NPDE to nonlinear ODE). By taking the transformation $u(x, t) = u(\xi)$, $\xi = x - Vt$, where V is arbitrary nonzero constants, and (15) transform it to the ordinary differential equation reduces to be

$$G(u', u'', u''', \dots) = 0. \quad (16)$$

Step 2 (determine the parameters). Determine the highest order nonlinear term and the linear term of the highest order term in (15) or (16). Then, in the resulting terms, balance the highest order nonlinear term and the linear term of the highest order term; we get a balance constant k (k is usually a positive integer). If k is a negative integer or a fraction, we make the following transformation:

$$u(\xi) = v^k(\xi). \quad (17)$$

We obtain the solutions of (15) to be as the following forms.

Case 1. If $R \neq 0$ in (19)–(20),

$$u(\xi) = A_0 + \sum_{i=1}^m \sigma^{i-1} [A_i \sigma(\xi) + B_i \tau(\xi)], \quad (18)$$

where $\sigma(\xi)$ and $\tau(\xi)$ satisfy the following equations:

$$\begin{aligned}\sigma'(\xi) &= \epsilon \sigma(\xi) \tau(\xi), & \tau'(\xi) &= R + \epsilon \tau^2(\xi) - \mu \sigma(\xi), \\ \epsilon &= \pm 1, & R, \mu &= \text{constant},\end{aligned}\quad (19)$$

which admits the first integral with $R \neq 0$,

$$\tau^2(\xi) = -\epsilon \left[R - 2\mu\sigma(\xi) + \frac{\mu^2 - 1}{R}\sigma^2(\xi) \right], \quad (R \neq 0), \quad (20)$$

where $l = d/\xi$. When $\epsilon = -1$, $R = 1$, and $\mu \rightarrow \mu/K$, (20) becomes a projective Riccati equation [25].

Case 2. If $R = \mu = 0$ in (19)-(20),

$$u(\xi) = \sum_{i=0}^k A_i \tau^i(\xi), \quad (21)$$

where $\tau'(\xi) = \tau^2(\xi)$.

Step 3. (I) If $R \neq 0$, substitute (18) along with conditions (19)-(20) into (15).

(II) If $R = \mu = 0$, substituting (21) along with $\tau'(\xi) = \tau^2(\xi)$ into (15) yields a set of algebraic equations for $\sigma^j(\xi)\tau^l(\xi)$, ($j = 0, 1, \dots; i = 0, 1$) ($\tau^l(\xi)$, $l = 0, 1, \dots$). Setting the coefficients of these terms $\sigma^j\tau^l$ (or $\tau^l(\xi)$) to zero yields a set of overdetermined algebraic equations in V , A_i , B_i , R , and μ .

Step 4. According to the theory of dynamical systems with the aid of *Mathematica*, solving the above set of equations yields the values of A_i , B_i , R , and λ .

Step 5. We know that (15) admits the following solutions.

Case 1. If $\epsilon = -1$, $R \neq 0$,

$$\begin{aligned} \sigma_1(\xi) &= \frac{R \operatorname{sech}(\sqrt{R}\xi)}{\mu \operatorname{sech}(\sqrt{R}\xi) + 1}, & \tau_1(\xi) &= \frac{\sqrt{R} \tanh(\sqrt{R}\xi)}{\mu \operatorname{sech}(\sqrt{R}\xi) + 1}, \\ \sigma_2(\xi) &= \frac{R \operatorname{csch}(\sqrt{R}\xi)}{\mu \operatorname{csch}(\sqrt{R}\xi) + 1}, & \tau_2(\xi) &= \frac{\sqrt{R} \coth(\sqrt{R}\xi)}{\mu \operatorname{csch}(\sqrt{R}\xi) + 1}. \end{aligned} \quad (22)$$

Case 2. If $\epsilon = 1$, $R \neq 0$,

$$\begin{aligned} \sigma_3(\xi) &= \frac{R \sec(\sqrt{R}\xi)}{\mu \sec(\sqrt{R}\xi) + 1}, & \tau_3(\xi) &= \frac{\sqrt{R} \tan(\sqrt{R}\xi)}{\mu \sec(\sqrt{R}\xi) + 1}, \\ \sigma_4(\xi) &= \frac{R \csc(\sqrt{R}\xi)}{\mu \csc(\sqrt{R}\xi) + 1}, & \tau_4(\xi) &= -\frac{\sqrt{R} \cot(\sqrt{R}\xi)}{\mu \csc(\sqrt{R}\xi) + 1}. \end{aligned} \quad (23)$$

Case 3. If $R = \mu = 0$,

$$\sigma_5(\xi) = \frac{c}{\xi} = c\epsilon\tau_5(\xi), \quad \tau_5(\xi) = \frac{1}{\epsilon\xi}, \quad (24)$$

where c is a constant. According to (18), (21), (22)–(24), and the conclusions in Step 4, we can obtain many solutions for (15).

Under the above method and some parameter conditions, exact solitary wave solutions are obtained. Our results are as follows.

Proposition 2. (1) If $V = a_1 - (a_4/a_3)((2p+1)a_2^2/(2+p)^2)$, (1) has two solitary wave trigonometric function solutions:

$$\begin{aligned} u_1(x, t) &= \left(\pm \sqrt{R}B_1 + B_1 \sqrt{R} \tanh(\sqrt{R}(x - Vt)) \right)^{1/p}, \\ u_2(x, t) &= \left(\pm \sqrt{R}B_1 + B_1 \sqrt{R} \coth(\sqrt{R}(x - Vt)) \right)^{1/p}, \end{aligned} \quad (25)$$

where $R = -(2p+1)p^2a_2^2/4a_3(2+p)^2$ and $B_1 = \pm \sqrt{a_4(p+1)(2p+1)/-p^2a_3}$.

(2) If $V = (R^2A_1^2a_3 + 2a_1p^2 + 3a_1p + a_1)/(p+1)(2p+1)$, (1) has two solitary wave solutions:

$$\begin{aligned} u_3(x, t) &= \left(A_1R \operatorname{sech}(\sqrt{R}(x - Vt)) \right)^{1/p}, \\ u_4(x, t) &= \left(A_1R \operatorname{csch}(\sqrt{R}(x - Vt)) \right)^{1/p}, \end{aligned} \quad (26)$$

where $R = a_4(2p+1)(p+1)/a_3A_1^2p^2$ and A_1 is arbitrary constant.

Proposition 3. (1) If $V = -(-2a_1p^2 - 3a_1p - a_1 + a_1(p+1)(2p+1)R/p^2a_3)(p+1)^{-1}(2p+1)^{-1}$, (1) has two solitary wave solutions:

$$\begin{aligned} u_5(x, t) &= \left(\mp \sqrt{R}B_1 + B_1 \frac{\sqrt{R} \tanh(\sqrt{R}(x - Vt))}{\operatorname{sech}(\sqrt{R}(x - Vt)) + 1} \right)^{1/p}, \\ u_6(x, t) &= \left(\mp \sqrt{R}B_1 + B_1 \frac{\sqrt{R} \coth(\sqrt{R}(x - Vt))}{\operatorname{csch}(\sqrt{R}(x - Vt)) + 1} \right)^{1/p}, \end{aligned} \quad (27)$$

where $R = (1/4)(a_2^2(2p+1)^2/a_3^2B_1^2(2+p)^2)$, $\mu = 1$, and $B_1 = \pm(1/2)(\sqrt{-a_3a_4(2p^2+3p+1)/a_3p})$.

(2) If $V = (Ra_2A_1 + 3a_1p + a_1p^2 + 2a_1)/(2+p)(p+1)$, (1) has two solitary wave solutions:

$$\begin{aligned} u_7(x, t) &= \left(A_1R \frac{\sec(\sqrt{R}(x - Vt))}{\sec(\sqrt{R}(x - Vt)) + 1} \right)^{1/p}, \\ u_8(x, t) &= \left(A_1R \frac{\operatorname{sech}(\sqrt{R}(x - Vt))}{\operatorname{csch}(\sqrt{R}(x - Vt)) + 1} \right)^{1/p}, \end{aligned} \quad (28)$$

where $A_1 = -a_4(p+1)(2+p)/p^2a_2$, $\mu = 1$, and R is arbitrary constant.

Proposition 4. (1) If $V = a_1$, $a_4(p+1)(2p+1) = 0$, (1) has two solitary wave hyperbolic function solutions:

$$u_9(x, t) = \left(\mp \sqrt{R} B_1 \pm \frac{B_1}{\sqrt{R}} \frac{R \operatorname{sech}(\sqrt{R}(x - Vt))}{\operatorname{sech}(\sqrt{R}(x - Vt)) + 1} + B_1 \frac{\sqrt{R} \tanh(\sqrt{R}(x - Vt))}{\operatorname{sech}(\sqrt{R}(x - Vt))} \right)^{1/p},$$

$$u_{10}(x, t) = \left(\mp \sqrt{R} B_1 \pm \frac{B_1}{\sqrt{R}} \frac{R \operatorname{csch}(\sqrt{R}(x - Vt))}{\operatorname{csch}(\sqrt{R}(x - Vt)) + 1} + B_1 \frac{\sqrt{R} \coth(\sqrt{R}(x - Vt))}{\operatorname{csch}(\sqrt{R}(x - Vt))} \right)^{1/p},$$
(29)

where $\mu = -1$ and B_1 and R are arbitrary constants.

(2) If $V = -(R^2 A_1^2 a_3 - 2a_1 p^2 \mu^2 + 2a_1 p^2 - 3a_1 p \mu^2 + 3a_1 p - a_1 \mu^2 + a_1)/(\mu^2 - 1)(p+1)(2p+1)$, (1) has two solitary wave solutions:

$$u_{11}(x, t) = \left(A_1 R \frac{\sec(\sqrt{R}(x - Vt))}{\mu \sec(\sqrt{R}(x - Vt)) + 1} \right)^{1/p},$$

$$u_{12}(x, t) = \left(A_1 R \frac{\csc(\sqrt{R}(x - Vt))}{\mu \csc(\sqrt{R}(x - Vt)) + 1} \right)^{1/p},$$
(30)

where $R = -a_4(p+1)(2p+1)(\mu^2 - 1)/p^2 A_1^2 a_3$, $A_1 = -a_2(2p\mu^2 - 2p + \mu^2 - 1)/(2+p)R\mu a_3$ and μ is arbitrary constant.

Proposition 5. If $(V - a_1)/a_4 > 0$, $a_3 > 0$, (1) has two solitary wave solutions:

$$u_{13}(x, t) = \left(-\frac{(p+1)(p+2)(V - a_1)}{a_4(A_1 \cosh(R(x - Vt))) - a_2} \right)^{1/p},$$
(31)

$$u_{14}(x, t) = \left(\frac{(p+1)(p+2)(V - a_1)}{a_4(A_1 \cosh(R(x - Vt))) + a_2} \right)^{1/p},$$
(32)

where

$$A_1 = -\sqrt{\frac{(2p+1)a_2^2 + (p+1)(p+2)^2(V - a_1)a_3}{(2p+1)a_4}},$$

$$R = p\sqrt{\frac{c - a_1}{a_4}}.$$
(33)

4. The Derivations of Theorem

4.1. The Derivations of Proposition Using the Improved Sub-ODE Method. In this section, we make the travelling transformation to (1) $u(x, t) = \phi(\xi)$, $\xi = x - Vt$, where V is a constant to be determined later, and thus (1) becomes (8). Integrating (8) once with regard to ξ , we obtain

$$(-V + a_1)\phi + \frac{a_2}{p+1}\phi^{p+1} + \frac{a_3}{2p+1}\phi^{2p+1} + a_4\phi'' = 0, \quad (34)$$

with the integration constants taken to be zero.

According to Step 1 in Section 3, if $a_1 \neq 0$, $a_2 \neq 0$, $a_3 \neq 0$, $a_4 \neq 0$, and $p \neq 0$, by balancing ϕ^{2p+1} and ϕ'' in (34), we get $k = 1/p$. Therefore, we make the following transformation:

$$\phi(\xi) = \varphi^{1/p}(\xi); \quad (35)$$

then substituting (34) into (35) yields

$$p^2 b_0 \varphi^2 + a_2 p^2 (2p+1) \varphi^3 + a_3 (p+1) p^2 \varphi^4 + b_1 (1-p) (\varphi')^2 + b_1 p \varphi'' \varphi = 0, \quad (36)$$

where

$$b_0 = (p+1)(2p+1)(-V + a_1),$$

$$b_1 = a_4(p+1)(2p+1). \quad (37)$$

According to Step 1 in Section 3, by balancing $\varphi(\xi)\varphi''(\xi)$ (or $\varphi'^2(\xi)$) and $\varphi^4(\xi)$ in (36), we get $k = 1$. Therefore, we suppose that (37) has the following formal solutions:

$$\varphi(\xi) = A_0 + A_1 \sigma(\xi) + B_1 \tau(\xi), \quad (38)$$

where $\sigma(\xi)$, $\tau(\xi)$ satisfy (19)-(20), where A_0, A_1, B_1 are constants to be determined later.

With the aid of *Mathematica* and *Maple*, substituting (38) along with (18)-(19) into (36) yields a set of algebraic equations for $\sigma^j(\xi)\tau^i(\xi)$ ($j = 0, 1, \dots; i = 0, 1$). Setting the coefficients of these terms $\sigma^j(\xi)\tau^i(\xi)$ to zero yields a set of overdetermined algebraic equations with respect to A_0, A_1, B_1, R , and V .

Case 1. Consider

$$\mu = A_1 = 0, \quad b_0 = 4B_1^2 Ra_3,$$

$$b_1 = -p^2 B_1^2 a_3, \quad A_0 = \pm \sqrt{R} B_1,$$

$$a_2 = \pm 2 \frac{(2+p)a_3 \sqrt{R} B_1}{2p+1}. \quad (39)$$

Case 2. Consider

$$\mu = 0, \quad b_1 = -4p^2 B_1^2 a_3, \quad b_0 = 4B_1^2 Ra_3,$$

$$A_1 = \pm \frac{B_1}{\sqrt{-R}}, \quad a_2 = -2 \frac{(2+p)a_3 \sqrt{R} B_1}{2p+1},$$

$$A_0 = \sqrt{R} B_1. \quad (40)$$

Case 3. Consider

$$\mu = 0, \quad A_1 = \pm \frac{B_1}{\sqrt{-R}}, \quad b_0 = 4B_1^2 Ra_3,$$

$$b_1 = -4p^2 B_1^2 a_3, \quad a_2 = 2 \frac{(2+p)a_3 \sqrt{R} B_1}{2p+1},$$

$$A_0 = -\sqrt{R} B_1. \quad (41)$$

Case 4. Consider

$$\begin{aligned}\mu &= A_0 = a_2 = B_1 = 0, & b_1 &= a_3 A_1^2 p^2 R, \\ b_0 &= -R^2 A_1^2 a_3,\end{aligned}\quad (42)$$

where A_1 is an arbitrary constant.

Case 5. Consider

$$\begin{aligned}\mu &= \pm 1, & A_1 &= 0, & b_1 &= \pm 4p^2 B_1^2 a_3, \\ b_0 &= 4B_1^2 Ra_3, & a_2 &= \pm 2 \frac{(2+p)a_3 \sqrt{RB_1}}{2p+1}, \\ A_0 &= \mp \sqrt{RB_1}.\end{aligned}\quad (43)$$

Case 6. Consider

$$\begin{aligned}\mu &= \pm 1, & b_0 &= b_1 = a_2 = a_3 = 0, \\ A_1 &= \pm \frac{B_1}{\sqrt{R}}, & A_0 &= \pm \sqrt{RB_1}.\end{aligned}\quad (44)$$

Case 7. Consider

$$\begin{aligned}\mu &= \pm 1, & A_0 &= B_1 = a_3 = 0, \\ b_1 &= \pm \frac{A_1 a_2 p^2 (2p+1)}{2+p}, & b_0 &= \mp \frac{A_1 R a_2 (2p+1)}{2+p}.\end{aligned}\quad (45)$$

Case 8. Consider

$$\begin{aligned}A_0 &= B_1 = 0, & a_2 &= -\frac{(2+p)R\mu a_3 A_1}{2p\mu^2 - 2p + \mu^2 - 1}, \\ b_0 &= \frac{R^2 A_1^2 a_3}{\mu^2 - 1}, & b_1 &= -\frac{a_3 A_1^2 p^2 R}{\mu^2 - 1},\end{aligned}\quad (46)$$

where $\mu \neq 1$ and A_1 are arbitrary constants and b_0 and b_1 are the same as (37).

Solving the overdetermined algebraic equations, we get Propositions 2–5.

4.2. The Derivations of Proposition Using the Integral Method. In the following, we study the derivations of theorem using the integral method [4, 5, 13–16].

If $g = 0$, (9) reduces to the following system:

$$\begin{aligned}\frac{d\varphi}{d\xi} &= y, \\ \frac{dy}{d\xi} &= -((p+1)(2p+1)(-V+a_1)\varphi \\ &\quad + a_3(p+1)\varphi^{2p+1} + a_2(2p+1)\varphi^{p+1}) \\ &\quad \cdot (a_4(p+1)(2p+1))^{-1}.\end{aligned}\quad (47)$$

From Table 3, we obtain two homoclinic orbits Γ_i ($i = 1, 2$) of (9).

In $\varphi - y$ plane, L_1 is described as

$$\begin{aligned}y^2 &= \varphi^2 \left(\frac{V-a_1}{a_4} - \frac{2a_2}{(p+1)(p+2)a_4} \varphi^p \right. \\ &\quad \left. - \frac{a_3}{(2p+1)(p+1)a_4} \varphi^{2p} \right), \\ \varphi &\in [\varphi_1, \varphi_2],\end{aligned}\quad (48)$$

where

$$\begin{aligned}\varphi_1 &= \left[-\frac{(2p+1)a_2 - a_4 \sqrt{\Delta}}{2a_3(p+1)} \right]^{1/p}, \\ \varphi_2 &= \left[-\frac{(2p+1)a_2 + a_4 \sqrt{\Delta}}{2a_3(p+1)} \right]^{1/p}, \\ \Delta &= \frac{a_2^2}{a_4^2(p+1)^2} + \frac{4(V-a_1)a_3}{a_4^2(2p+1)}.\end{aligned}\quad (49)$$

We obtain the results by substituting (48) into $d\varphi/d\xi = y$:

$$\begin{aligned}\pm \left(1 \cdot \left(\varphi^2 \left(\frac{V-a_1}{a_4} - \frac{2a_2}{(p+1)(p+2)a_4} \varphi^p \right. \right. \right. \\ \left. \left. \left. - \frac{a_3}{(2p+1)(p+1)a_4} \varphi^{2p} \right) \right) \right)^{-1/2} d\varphi = d\xi.\end{aligned}\quad (50)$$

If $\varphi(0) = \varphi_1$, then, integrating (50) along homoclinic orbit L_1 , we have

$$\begin{aligned}\int_{\varphi_1}^{\varphi} \left(dw \cdot \left(w^2 \left(\frac{V-a_1}{a_4} - \frac{2a_2}{(p+1)(p+2)a_4} w^p \right. \right. \right. \\ \left. \left. \left. - \frac{a_3}{(2p+1)(p+1)a_4} w^{2p} \right) \right) \right)^{-1} \\ = \int_0^{\xi} dw \quad \xi < 0, \\ - \int_{\varphi}^{\varphi_1} \left(dw \cdot \left(w^2 \left(\frac{V-a_1}{a_4} - \frac{2a_2}{(p+1)(p+2)a_4} w^p \right. \right. \right. \\ \left. \left. \left. - \frac{a_3}{(2p+1)(p+1)a_4} w^{2p} \right) \right) \right)^{-1} \\ = \int_0^{\xi} dw \quad \xi > 0.\end{aligned}\quad (51)$$

We obtain by completing the above two integrals

$$\varphi_4(x, t) = \left(-\frac{(p+1)(p+2)(V-a_1)}{a_4(A_1 \cosh(R(x-Vt)) - a_2)} \right)^{1/p}, \quad (52)$$

where

$$A_1 = -\sqrt{\frac{(2p+1)a_2^2 + (p+1)(p+2)^2(V-a_1)a_3}{(2p+1)a_4}}, \quad (53)$$

$$R = p\sqrt{\frac{c-a_1}{a_4}}.$$

We obtain $u_{13}(x, t)$ as formula (31) via (53).

If $\varphi(0) = \varphi_2$, integrating (50) along homoclinic orbit L_2 , we have

$$\int_{\varphi_2}^{\varphi} \left(dw \cdot \left(w^2 \left(\frac{V-a_1}{a_4} - \frac{2a_2}{(p+1)(p+2)a_4} w^p - \frac{a_3}{(2p+1)(p+1)a_4} w^{2p} \right) \right)^{-1} \right)$$

$$= \int_0^{\xi} dw \quad \xi > 0, \quad (54)$$

$$- \int_{\varphi}^{\varphi_2} \left(dw \cdot \left(w^2 \left(\frac{V-a_1}{a_4} - \frac{2a_2}{(p+1)(p+2)a_4} w^p - \frac{a_3}{(2p+1)(p+1)a_4} w^{2p} \right) \right)^{-1} \right)$$

$$= \int_0^{\xi} dw \quad \xi < 0.$$

We obtain by completing the above two integrals

$$\varphi_5(x, t) = \left(\frac{(p+1)(p+2)(V-a_1)}{a_4(A_1 \cosh(R(x-Vt))) + a_2} \right)^{1/p}, \quad (55)$$

where

$$A_1 = -\sqrt{\frac{(2p+1)a_2^2 + (p+1)(p+2)^2(V-a_1)a_3}{(2p+1)a_4}}, \quad (56)$$

$$R = p\sqrt{\frac{c-a_1}{a_4}}.$$

Our results (52) and (55) can be compared with the result of Song et al. [5] by assigning special values to the arbitrary parameters involved.

From (52), (55), and traveling wave transformation (34), we get solitary wave solutions $u_{13}(x, y, t)$, $u_{14}(x, y, t)$ just as expressed in (31) and (32).

Similarly, we can get other solutions. Here we complete the derivation of Propositions 2–4.

5. Conclusion

In this paper, we were devoted to investigate the dynamical survey and the phase orbits of the compound KdV-Burgers-type equation via the improved sub-ODE method and

the integral method. Some solitary wave solutions are given for each parameter condition. At the same time, the method can be widely applied to other nonlinear equations. Not only do we obtain the explicit solitary wave solutions but also we can get other kinds of solutions such as the trigonometric function solutions, the hyperbolic function solutions, and the rational solutions.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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