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Research Article

Losslessness of Nonlinear Stochastic Discrete-Time Systems

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This paper will study stochastic losslessness theory for nonlinear stochastic discrete-time systems, which are expressed by the Itô-type difference equations. A necessary and sufficient condition is developed for a nonlinear stochastic discrete-time system to be lossless. By the stochastic lossless theory, we show that a nonlinear stochastic discrete-time system can be lossless via state feedback if and only if it has relative degree $(0, \ldots, 0)$ and lossless zero dynamics. The effectiveness of the proposed results is illustrated by a numerical example.

1. Introduction

Since Willems [1] first founded the concepts of dissipativity and passivity of nonlinear deterministic systems [1], dissipativity and passivity have been studied by many authors; see [2–9] and the references therein. The central result in [2] was a sufficient and necessary condition for an affine nonlinear system to be passive. Reference [6] dealt with the problem of a nonlinear deterministic system feedback equivalence to a passive system. Some results in [7] were derived for a nonlinear deterministic discrete-time system, which are parallel to analogous ones in [6]. For the past decade, many researchers have extended the existing methodology from deterministic systems to stochastic systems; see [10–26] and the references therein. Based on the dissipative point of view, the H_∞ control problem for nonlinear stochastic Itô systems was discussed in [12, 17]. Moreover, Zhang et al. [18] gave two sufficient conditions for H_∞ control of nonlinear stochastic Itô systems by Hamilton-Jacobi inequality. Reference [19] studies the H_∞ control problem for nonlinear Markovian jump by means of geometric control method. Li [20–22] discussed the problems of state-feedback stabilization for high-order stochastic nonlinear systems. Lin et al. [23] addressed the issues of stochastic passivity, feedback equivalence, and global stabilization for a general nonlinear stochastic system. Liu et al. [24] was devoted to the indefinite stochastic discrete-time systems with a linear equality constraint on the terminal state. Sheng et al. [25] have investigated the relationship between Nash equilibrium strategies and the finite horizon H_2/H_∞ control of stochastic Markov jump systems with multiplicative noise.

According to the above results, we have been interested in the concepts of losslessness, relative degree, and zero dynamics for stochastic discrete-time systems. This paper will study losslessness theory for nonlinear stochastic discrete-time systems, which are expressed by the Itô-type difference equations. The main contributions of this paper can be summarized as follows. A necessary and sufficient condition is developed for a nonlinear stochastic discrete-time system to be lossless, which can be viewed as a stochastic generalized version of [7]. Likewise, we show that a stochastic discrete-time lossless system can be asymptotically stabilized in probability by output feedback. Then, a necessary and sufficient condition is yielded for a nonlinear stochastic discrete-time system to be feedback equivalent to a lossless system.

The rest of this paper is organized as follows. Section 2 considers the lossless theory for a nonlinear stochastic discrete-time system paralleling that of [7]. Theorem 5 is a necessary and sufficient condition for system to be lossless, which extends Theorem 2.6 of [7] and is used in Section 3. Under some given conditions, Section 3 is concerned with the problem of feedback equivalence to a lossless system. An numerical example is presented to illustrate the effectiveness of our results. In Section 4, conclusions are drawn.

Before concluding this section, let us introduce some notations. M^T represents the transpose of a matrix M; $M > 0$ ($M \geq 0$) means that M is positive definite (positive semidefinite) symmetric matrix; $E[x]$ represents the mathematical expectation of a random variable x; R^k is the k-dimensional Euclidean space with the usual 2-norm $\|\cdot\|$; $R^{m\times n}$ is the vector space of all $m \times n$ matrices with entries in R; I is the identity matrix with appropriate dimension; C^2 is the class of functions $V(x)$ twice continuously differential with respect to x; $N = \{0, 1, 2, \ldots\}$; $N_K = \{0, 1, 2, \ldots, K\}$.

2. Lossless Systems

Consider the following nonlinear stochastic discrete-time system governed by the Itô difference equation:

$$\begin{aligned} x(t+1) &= f_1(x(t)) + g_1(x(t))u(t) + f_2(x(t))\omega(t) \\ z(t) &= J(x(t)) + h(x(t))u(t), \end{aligned} \tag{1}$$

where f_1, g_1, f_2, h, and J are smooth mappings with appropriate dimensions and $f_1(0) = 0$ and $f_2(0) = 0$. $x \in R^n$ is called the system state, $u \in R^m$ is the control input, and $z \in R^m$ is the regulated output. Let Ω be a nonempty set, F a σ-field consisting of subsets of Ω, and P a probability measure; that is, P is a map from F to $[0, 1]$. We call the triple (Ω, F, P) a probability space. $\omega(t)$ is a sequence of second-order stationary random variables defined on the complete probability space (Ω, F, P), such that $E[\omega(t)] = 0$ and $E[\omega(t)\omega(s)] = \delta_{st}$, where δ_{st} is the Kronecker delta. x, u, f_1, g_1, and f_2 are supposed to be independent of ω.

We denote by F_t the σ-algebra generated by $\omega(t)$, $t \in N_K$; that is, $F_t = \sigma(\omega(t) : t \in N_K)$. Let $L^2(\Omega, R^k)$ represent the space of R^k-valued, square integrable random vectors and $l^2_\omega(N_K, R^k)$ consists of all finite sequences $y = \{y(t) : y(t) \in R^k\}_{t\in N_K}$, such that $y(t) \in L^2(\Omega, R^k)$ is F_{t-1} measurable for $t \in N_K$, where $F_{-1} = \{\phi, \Omega\}$; that is, $y(0)$ is constant. The l^2-norm of $l^2_\omega(N_K, R^k)$ is defined by $\|y(\cdot)\|_{l^2_\omega(N_K,R^k)} = (\sum_{t=0}^{K} E\|y(t)\|^2)^{1/2}$.

A function $W(\cdot,\cdot) : R^m \times R^m$ is called the supply rate on N_K if for any $u \in l^2_\omega(N_K, R^k)$, $x(t) \in R^n$, $z(t)$ of (1) is satisfied $\sum_{t=0}^{K} E|W(u(t), z(t))| < \infty$. A nonnegative function $V : R^n \to R^+$ with $V(0) = 0$ is called the storage function.

Definition 1. System (1) with supply rate W is said to be dissipative on N_K if there exists a storage function V, such that for all $u(t) \in R^m$ and $t \in N_K$,

$$E[V(x(t+1))] - E[V(x(t))] \leq E[W(z(t), u(t))]. \tag{2}$$

In this paper, we mainly study the dissipative systems with supply rate $W(z(t), u(t)) = z(t)^T u(t)$.

Definition 2. System (1) is said to be passive if there is a storage function V such that $\forall t \in N_K$ and $\forall u(t) \in R^m$

$$E[V(x(t+1))] - E[V(x(t))] \leq E\left[z(t)^T u(t)\right]. \tag{3}$$

For simplicity of our discussion, we give the following definitions.

Definition 3. System (1) with storage function V is said to be strictly passive if for all $t \in N_K$ and $u(t) \in R^m$

$$E[V(x(t+1))] - E[V(x(t))] < E\left[z(t)^T u(t)\right]. \tag{4}$$

It is equivalent to the following inequality:

$$\begin{aligned} &E[V(x(t+1))] - E[V(x(t))] \\ &\quad \leq E\left[z(t)^T u(t)\right] - E[S(x(t))], \end{aligned} \tag{5}$$

where $S : R^n \to R^+$.

Definition 4. System (1) with storage function V is said to be lossless if for all $t \in N_K$ and $u(t) \in R^m$

$$E[V(x(t+1))] - E[V(x(t))] = E\left[z(t)^T u(t)\right]. \tag{6}$$

It is easy to show that system (1) with storage function V is lossless if and only if

$$\begin{aligned} E[V(x(K))] - V(x(0)) &= \sum_{t=0}^{K-1} E\left[z(t)^T u(t)\right], \\ &\forall t \in N_K,\ \forall u(t),\ \forall x(0). \end{aligned} \tag{7}$$

In what follows, we give a fundamental property of lossless systems.

Theorem 5. *System (1) with a C^2-storage function V is lossless if and only if V satisfies*

$$E\left[V\left(f_1(x(t)) + f_2(x(t))\omega(t)\right)\right] = E[V(x(t))], \tag{8}$$

$$E\left[\left.\frac{\partial V(\alpha)}{\partial \alpha}\right|_{\alpha=f_1(x)+f_2(x)\omega} g_1(x)\right] = E\left[J(x)^T\right], \tag{9}$$

$$\begin{aligned} &E\left[g_1(x)^T \left.\frac{\partial V^2(\alpha)}{\partial \alpha^2}\right|_{\alpha=f_1(x)+f_2(x)\omega} g_1(x)\right] \\ &\quad = E\left[h(x) + h(x)^T\right], \end{aligned} \tag{10}$$

$$V(x(t+1)) = V\left(f_1(x) + g_1(x)u + f_2(x)\omega\right) \tag{11}$$

which is quadratic in u.

Proof. If system (1) is lossless, by Definition 4, there exists a storage function V such that

$$\begin{aligned} E[V(x(t+1))] &= E[V(x(t))] + E\left[z(t)^T u(t)\right] \\ &= E[V(x(t))] + E\left[J(x)^T u\right] \\ &\quad + E\left[u^T \frac{h(x) + h(x)^T}{2} u\right]. \end{aligned} \tag{12}$$

It is clear that $V(x(t+1)) = V(f_1(x) + g_1(x)u + f_2(x)\omega)$ is quadratic in u. Let $u = 0$, then $E[V(f_1(x(t)) + f_2(x(t))\omega(t))] = E[V(x(t))]$.

Hence, we can obtain that

$$\begin{aligned} E\left[\frac{\partial V(x(t+1))}{\partial u}\right] &= E\left[\left.\frac{\partial V(\alpha)}{\partial \alpha}\right|_{\alpha=f_1(x)+g_1(x)u+f_2(x)\omega}(g_1(x))\right] \\ &= E\left[\left.\frac{\partial V(\alpha)}{\partial \alpha}\right|_{\alpha=f_1(x)+g_1(x)u+f_2(x)\omega} g_1(x)\right] \\ &= E\left[J(x)^T\right] + E\left[u^T\left(h(x)+h(x)^T\right)\right], \end{aligned} \tag{13}$$

$$\begin{aligned} E\left[\frac{\partial^2 V(x(t+1))}{\partial u^2}\right] &= E\left[g_1(x)^T \left.\frac{\partial^2 V(\alpha)}{\partial \alpha^2}\right|_{\alpha=f_1(x)+g_1(x)u+f_2(x)\omega} g_1(x)\right] \\ &= E\left[h(x)+h(x)^T\right]. \end{aligned}$$

Taking $u = 0$, (9) and (10) are given.

Conversely, since (11) is quadratic in u, we use Taylor's expansion formula and have

$$\begin{aligned} V(x(t+1)) &= V\left(f_1(x)+g_1(x)u+f_2(x)\omega\right) \\ &= A(x)+B(x)u+\frac{1}{2}u^T C(x)u. \end{aligned} \tag{14}$$

Obviously,

$$\begin{aligned} A(x) &= V\left(f_1(x(t))+f_2(x(t))\omega(t)\right), \\ B(x) &= \left.\frac{\partial V\left(f_1(x)+g_1(x)u+f_2(x)\omega\right)}{\partial u}\right|_{u=0} g_1(x) \\ &= \left.\frac{\partial V(\alpha)}{\partial \alpha}\right|_{\alpha=f_1(x)+f_2(x)\omega} g_1(x), \\ C(x) &= g_1(x)^T \left.\frac{\partial^2 V\left(f_1(x)+g_1(x)u+f_2(x)\omega\right)}{\partial u^2}\right|_{u=0} \\ &\quad \cdot g_1(x) = g_1(x)^T \left.\frac{\partial^2 V(\alpha)}{\partial \alpha^2}\right|_{\alpha=f_1(x)+f_2(x)\omega} g_1(x). \end{aligned} \tag{15}$$

Together with (8)–(10) and (1), we conclude that

$$\begin{aligned} E[V(x(t+1))] - E[V(x(t))] &= E\left[z(t)^T u(t)\right], \\ &\forall t \in N_K,\ \forall u(t) \in R^m. \end{aligned} \tag{16}$$

This implies that system (1) is lossless with $V \in C^2$. □

Corollary 6. *Consider a discrete-time linear system*

$$\begin{aligned} x(t+1) &= A_1 x(t) + B_1 u(t) + A_2 x(t)\omega(t), \\ z(t) &= Cx(t) + Du(t). \end{aligned} \tag{17}$$

By Theorem 5, system (17) with $V(x) = x^T Px$ is lossless if and only if there exists a matrix $P \geq 0$ such that

$$\begin{aligned} E\left[x^T\left(A_1^T P A_1 + A_2^T P A_2\right)x\right] &= E\left[x^T P x\right], \\ E\left[B_1^T\left(PA_1+PA_2\right)x\right] &= E[Cx], \\ B_1^T P B_1 &= D + D^T. \end{aligned} \tag{18}$$

In the sequel, we give the following definitions about stochastic stability and zero-state observability, which are useful in treating the stabilization problem for lossless systems.

Definition 7 (see [26]). Consider the following stochastic system:

$$\begin{aligned} x(t+1) &= f(x(t)) + g(x(t))\omega(t), \\ x(0) &= x_0 \in R^n, \\ f(0) &= 0, \\ g(0) &= 0. \end{aligned} \tag{19}$$

(a) $x = 0$ of (19) is called stable in probability if for any $\epsilon > 0$,

$$\lim_{x_0 \to 0} P\left(\sup \|x(t)\| > \epsilon,\ t \geq 0\right) = 0. \tag{20}$$

(b) $x = 0$ of (19) is called locally asymptotically stable in probability if (20) holds and

$$\lim_{x_0 \to 0} P\left(\lim_{t\to\infty} x(t) = 0\right) = 1. \tag{21}$$

(c) $x = 0$ of (19) is called globally asymptotically stable in probability if (20) holds and

$$P\left(\lim_{t\to\infty} x(t) = 0\right) = 1. \tag{22}$$

Definition 8. System (1) is called locally (resp., globally) zero-state observable if there is a neighborhood U of $x = 0$ such that, for all $x(0) = x_0 \in U$ (resp., R^n),

$$E\left[z(t)|_{u(t)=0}\right] = E[J(x(t))] = 0 \quad \text{implies } x_0 = 0. \tag{23}$$

Remark 9. By Definition 7, it is clear that system (1) is zero-state observable iff

$$\left\{x_0 \in R^n \mid E[J(x(t))] = 0,\ \forall t \in N_K\right\} = \{0\}. \tag{24}$$

Below, we point out the zero-state observability condition for lossless system.

Theorem 10. *Assume that system (1) with $V \in C^2$ is lossless. Let*

$$\begin{aligned} S = \Bigg\{x_0 \in R^n \mid E\left[\left.\frac{\partial V(\alpha)}{\partial \alpha}\right|_{\alpha=f_1(x)+f_2(x)\omega} g_1(x)\right] \\ = 0,\ \forall t \in N_K\Bigg\}; \end{aligned} \tag{25}$$

then, system (1) is zero-state observable iff $S = \{0\}$.

Proof. Because system (1) is lossless, by Theorem 5 it follows that

$$E\left[J(x(t))^T\right] = E\left[\left.\frac{\partial V(\alpha)}{\partial \alpha}\right|_{\alpha=f_1(x)+f_2(x)\omega} g_1(x)\right], \quad \forall t \in N_K. \tag{26}$$

We can show that $S = \{0\}$ iff

$$\{x_0 \in R^n \mid E[J(x(t))] = 0,\ \forall t \in N_K\} = \{0\}. \tag{27}$$

Together with Definition 7 and Remark 9, Theorem 10 is easily obtained. □

Based on the above, we study the problem of stabilization for system (1).

Theorem 11. *If system (1) is lossless, ψ is a smooth mapping with $\psi(0) = 0$, and $z^T\psi(z) > 0$ for any $z \neq 0$, then the close-loop system*

$$x(t+1) = f_1(x(t)) - g_1(x(t))\psi(z(t)) + f_2(x(t))\omega(t) \tag{28}$$

is locally asymptotically stable in probability if and only if system (1) is locally zero-state observable. If V is proper (i.e., for any $a > 0$, $V^{-1}([0,a])$ is compact), then it is globally asymptotically stable in probability if and only if system (1) is globally zero-state observable.

Proof (sufficiency). By Definition 4 and (28), we have

$$\begin{aligned} &E[V(x(t+1))] - E[V(x(t))] \\ &\quad = -E\left[z(t)^T\psi(z(t))\right] \le 0. \end{aligned} \tag{29}$$

According to Theorem 2.1 in [26], system (1) is stable in probability. Define

$$\begin{aligned} \gamma = \{&x \in R^n : E[\Delta V(x(t))] = E[V(x(t+1))] \\ &- E[V(x(t))] = 0\}. \end{aligned} \tag{30}$$

For any $x_0 = x \in \gamma$, it follows that

$$\begin{aligned} 0 &= E[V(x(t+1))] - E[V(x(t))] \\ &= -E\left[z(t)^T\psi(z(t))\right] \le 0. \end{aligned} \tag{31}$$

It means that $z(t)^T\psi(z(t)) = 0$; otherwise, it contradicts $z^T\psi(z) > 0, \forall z \neq 0$. Then,

$$\begin{aligned} &z(t) = 0, \\ &u(t) = -\psi(z(t)) = 0, \\ &\forall t \in N. \end{aligned} \tag{32}$$

By Definition 8, it yields $x = 0$. By Theorem 2.3 in [26], $x = 0$ is locally asymptotically stable in probability. If V is proper, then $x = 0$ is globally asymptotically stable in probability.

The necessity is similar to the proof given in [27] and is omitted. □

3. Feedback Equivalence to a Lossless System

In this section, we solve the problem of feedback equivalence to a lossless system via state feedback. To this end, some preliminary definitions are needed, such as relative degree and zero dynamics. These concepts play crucial roles in this paper. It will be shown that the losslessness of system (1) can be achieved by means of state feedback if and only if system (1) have relative degree $(0,\ldots,0)$ and lossless zero dynamics.

Definition 12. System (1) is said to have relative degree (resp., uniform relative degree) $(0,\ldots,0)$ at $x = 0$ if $h(0)$ (resp., $h(x)$, $\forall x \in R^n$) is nonsingular.

By Definition 12, if system (1) has relative degree $(0,\ldots,0)$, there is a neighborhood $U \subset R^n$ of $x = 0$ such that $h^{-1}(x)$ exists. Set $u^*(t) = -h^{-1}(x(t))J(x(t))$, then $E[z(t)] \equiv 0$ for any $x(t) \in U$ and

$$\begin{aligned} x(t+1) = f_1(x(t)) + g_1(x(t))u^*(t) \\ + f_2(x(t))\omega(t). \end{aligned} \tag{33}$$

Let $Z^* = \{x \in R^n \mid E[z(t)] = 0\}$, it is obvious that $Z^* = U$, given

$$u^*(t) = -h^{-1}(x(t))J(x(t)), \quad \forall x \in U = Z^*. \tag{34}$$

Definition 13. Systems (33)-(34) are called the zero dynamics of system (1).

If system (1) has uniform relative degree $(0,\ldots,0)$, then $Z^* = U \equiv R^n$ and the global zero dynamics (33)-(34) exist.

In the following, we give definition of system (1) having lossless zero dynamics.

Definition 14. Assume that $h(0)$ is nonsingular. System (1) is said to have locally lossless zero dynamics if there exists a function $V \in C^2$, which is positive definite and is locally defined on the neighborhood U of $x = 0$ such that the following conditions are satisfied:

$$\begin{aligned} &E\left[V\left(f_1(x) + g_1(x)u^* + f_2(x)\omega\right)\right] = E[V(x)], \\ &V\left(f_1(x) + g_1(x)u^* + f_2(x)\omega + g_1(x)u\right) \end{aligned} \tag{35}$$

which is quadratic in u.

System (1) has globally lossless zero dynamics if there exists a positive definite function $V \in C^2$ satisfying (35) for all $x \in R^n$.

Now, we analyze a lossless system which has relative degree $(0,\ldots,0)$ at $x = 0$ and present the following theorem.

Theorem 15. *Assume that system (1) is lossless with $V \in C^2$ and V is nondegenerate at $x = 0$, then*

(1) $\mathrm{Rank}\{g_1(0)\} = m$ *if and only if* $E[h(0)^T + h(0)] > 0$;

(2) *System (1) has relative degree* $(0,\ldots,0)$ *at* $x = 0$.

Proof. (1) By Theorem 5 and (10), we have

$$E\left[g_1(0)^T \left.\frac{\partial V^2(\alpha)}{\partial \alpha^2}\right|_{\alpha=f_1(x)+f_2(x)\omega} g_1(0)\right] = E\left[h(0)+h(0)^T\right]. \tag{36}$$

It is obvious that for all $x \in R^m$

$$E\left[(g_1(0)x)^T \left.\frac{\partial V^2(\alpha)}{\partial \alpha^2}\right|_{\alpha=f_1(x)+f_2(x)\omega} (g_1(0)x)\right] = E\left[x^T(h(0)+h(0))x\right]. \tag{37}$$

It is easy to show that $\mathrm{Rank}\{g_1(0)\} = m$ implies $E[h(0)^T + h(0)] > 0$.

On the other hand, by $E[h(0)^T + h(0)] > 0$, it yields that

$$E\left[(g_1(0)x)^T \left.\frac{\partial V^2(\alpha)}{\partial \alpha^2}\right|_{\alpha=f_1(x)+f_2(x)\omega} (g_1(0)x)\right] > 0 \tag{38}$$

and $g_1(0)x \neq 0$, $\forall x \neq 0$. This means that $g_1(0)x = 0$ has a unique solution $x = 0$. Then, $\mathrm{Rank}\{g_1(0)\} = m$.

(2) From (1), $J(0)$ is nonsingular. By Definition 12, system (1) has relative degree $(0,\ldots,0)$ at $x = 0$. □

Remark 16. If linear system (17) is lossless with $V(x) = x^T Px/2 > 0$, then $\partial V^2(x)/\partial x^2 = P > 0$, so V is nondegenerate at $x = 0$.

The following theorem can be viewed as the global version of Theorem 15.

Theorem 17. *If system (1) with $V \in C^2$ is lossless and satisfies one of the following conditions:*

$$\left.\frac{\partial V^2(\alpha)}{\partial \alpha^2}\right|_{\alpha=f_1(x)+f_2(x)\omega} > 0, \quad \text{or } V \text{ is a strictly convex.} \tag{39}$$

Then, for all $x \in R^n$,

(1) $\mathrm{Rank}\{g_1(x)\} = m$ *if and only if* $E[h(x)^T + h(x)] > 0$;

(2) *System (1) has uniform relative degree* $(0,\ldots,0)$.

Proof. The proof of Theorem 17 is the same as that of Theorem 15. □

From the above, we give a necessary condition under which a lossless system has lossless zero dynamics at $x = 0$.

Theorem 18. *If* $\mathrm{Rank}\{g_1(0)\} = m$ *and the conditions of Theorem 15 hold, then the zero dynamics of system (1) at $x = 0$ are lossless.*

Proof. By Theorem 15, the zero dynamics (33)-(34) of system (1) locally exist at $x = 0$.

Moreover, by the losslessness of system (1), we can obtain that

$$E\left[V\left(f_1(x)+g_1(x)u+f_2(x)\omega\right)\right] - E\left[V(x(t))\right] = E\left[z(t)^T u(t)\right]. \tag{40}$$

Take $u(t) = u^*(t)$, we have

$$E\left[V\left(f_1(x)+g_1(x)u^*(t)+f_2(x)\omega\right)\right] = E\left[V(x)\right], \quad \forall x \in Z^*. \tag{41}$$

By (40), we can show that $V(f_1(x)+g_1(x)u^*(t)+f_2(x)\omega+g_1(x)\tilde{u})$ is quadratic in $\tilde{u}$ if $u = u^* + \tilde{u}$. So, we conclude that the zero dynamics of system (1) are lossless.

Now, we attempt to consider a state feedback as follows:

$$u(t) = \alpha(x(t))v(t) + \beta(x(t)), \qquad \beta(0) = 0, \tag{42}$$

where $\alpha(x)$ and $\beta(x)$ are smooth functions defined near $x = 0$ locally or globally and $\alpha(x)$ is nonsingular. We solve the problem of feedback equivalence to a lossless system. □

Theorem 19. *If* $\mathrm{Rank}\{g_1(0)\} = m$ *and V is nondegenerate at $x = 0$, then the following conditions are equivalent.*

(1) *System (1) is locally feedback equivalent to a lossless system with $V \in C^2$;*

(2) *System (1) has relative degree $(0,\ldots,0)$ at $x = 0$ and has lossless zero dynamics.*

Proof. If there exists a state feedback (42) such that system (1) is lossless. It can be known that system

$$\begin{aligned} x(t+1) &= f_1(x(t)) + g_1(x(t))\beta(x(t)) \\ &\quad + f_2(x(t))\omega(t) \\ &\quad + g_1(x(t))\alpha(x(t))v(t), \\ z(t) &= J(x(t)) + h(x(t))\beta(x(t)) \\ &\quad + h(x(t))\alpha(x(t))v(t) \end{aligned} \tag{43}$$

is lossless with $V \in C^2$.

Since $\alpha(0)$ is invertible, it yields $\mathrm{Rank}\{\tilde{g}(0)\} = m$. According to Theorem 15, it is clear that $h(0)\alpha(0)$ is nonsingular and so is $h(0)$. Therefore, system (1) has relative degree $(0,\ldots,0)$ at $x = 0$.

In addition, we know that the zero dynamics of system (43) are

$$\begin{aligned} x(t+1) &= \tilde{f}^*(x(t)) \\ &= f_1(x(t)) - g_1(x(t))h^{-1}(x(t))J(x(t)) \\ &\quad + f_2(x(t))\omega(t)x(t), \quad \forall x \in \tilde{Z}^*. \end{aligned} \tag{44}$$

Because system (43) is lossless, proceed along the same lines of Theorem 18, we have $E[V(\tilde{f}^*(x))] = E[V(x)]$, and $V(\tilde{f}^*(x) + \tilde{g}(x)v)$ is quadratic in v.

Then, $E[V(f^*(x))] = E[V(x)]$ and $V(f^*(x) + g_1(x)u)$ is quadratic in u. This means that zero dynamics of system (1) are lossless.

On the other hand, when system (1) has relative degree $(0, \ldots, 0)$ at $x = 0$, we define

$$u(t) = u^*(t) + h^{-1}(x(t))\,\xi(t). \tag{45}$$

We apply (45) to system (1), system (1) can be rewritten as

$$\begin{aligned} x(t+1) &= f_1(x(t)) + g_1(x(t))\,u^*(x(t)) \\ &\quad + f_2(x(t))\,\omega(t) + g_1(x(t))\,\xi(t), \\ z(t) &= J(x(t)) + h(x(t))\,u^*(t) + \xi(t) = \xi(t). \end{aligned} \tag{46}$$

Set $z(t) = \xi(t) = \tilde{J}(x(t)) + \tilde{h}(x(t))\nu(t)$, then system (46) is replaced by

$$\begin{aligned} x(t+1) &= f_1(x(t)) + g_1(x(t))\,u^*(x(t)) \\ &\quad + f_2(x(t))\,\omega(t) \\ &\quad + g_1(x(t))\left[\tilde{J}(x(t)) + \tilde{h}(x(t))\,\nu(t)\right], \\ z(t) &= \tilde{J}(x(t)) + \tilde{h}(x(t))\,\nu(t). \end{aligned} \tag{47}$$

Since $\mathrm{Rank}\{g_1(0)\} = m$ and V is nondegenerate at $x = 0$, we can see that there exists a neighborhood U of $x = 0$ satisfying $g_1(x)^T(\partial V^2(\alpha)/\partial\alpha^2)|_{\alpha=f^*(x)}g_1(x) > 0$.

Define $\tilde{h}(x)$ and $\tilde{J}(x)$ as

$$\begin{aligned} \tilde{h}(x) &= \left(\frac{1}{2}g_1(x)^T \left.\frac{\partial V^2(\alpha)}{\partial\alpha^2}\right|_{\alpha=f^*(x)} g_1(x)\right)^{-1}, \\ \tilde{J}(x) &= -\tilde{h}(x)\left(\left.\frac{\partial V(\alpha)}{\partial\alpha}\right|_{\alpha=f^*(x)} g_1(x)\right)^T. \end{aligned} \tag{48}$$

Moreover, by the fact that $V(f^*(x) + g_1(x)u)$ is quadratic in u, then

$$\begin{aligned} &V\left(f^*(x) + g_1(x)\,\tilde{J}(x)\right) \\ &\quad = V\left(f^*(x)\right) + \left.\frac{\partial V(\alpha)}{\partial\alpha}\right|_{\alpha=f^*(x)} g_1(x)\,\tilde{J}(x) \\ &\qquad + \frac{1}{2}\tilde{J}(x)^T g_1{}^T(x) \left.\frac{\partial^2 V(\alpha)}{\partial\alpha^2}\right|_{\alpha=f^*(x)} g_1(x)\,\tilde{J}(x). \end{aligned} \tag{49}$$

We can show that

$$\begin{aligned} &E\left[\left.\frac{\partial V(\alpha)}{\partial\alpha}\right|_{\alpha=f^*(x)+g_1(x)\tilde{J}(x)} g_1(x)\,\tilde{h}(x)\right] \\ &\quad = E\left[\left.\frac{\partial V(\alpha)}{\partial\alpha}\right|_{\alpha=f^*(x)} g_1(x)\,\tilde{h}(x)\right] \\ &\qquad + E\left[\tilde{J}(x)^T g_1(x)^T \left.\frac{\partial^2 V(\alpha)}{\partial\alpha^2}\right|_{\alpha=f^*(x)} g_1(x)\,\tilde{h}(x)\right] \\ &\quad = E\left[\tilde{J}(x)^T\right], \\ &E\left[g_1(x)^T \left.\frac{\partial V^2(\alpha)}{\partial\alpha^2}\right|_{\alpha=f^*(x)+g_1(x)\tilde{J}(x)} g_1(x)\,\tilde{h}(x)\right] \\ &\quad = E\left[g_1(x)^T \left.\frac{\partial^2 V(\alpha)}{\partial\alpha^2}\right|_{\alpha=f^*(x)} g_1(x)\,\tilde{h}(x)\right] \\ &\quad = E\left[\tilde{h}(x)\right] + E\left[\tilde{h}(x)^T\right]. \end{aligned} \tag{50}$$

Taking expectation on both sides of (49) and using (50), it is easy to show that

$$E\left[V\left(f^*(x) + g_1(x)\,\tilde{J}(x)\right)\right] = E\left[V\left(f^*(x)\right)\right]. \tag{51}$$

By Definition 14, we have

$$E\left[V\left(f^*(x) + g_1(x)\,\tilde{J}(x)\right)\right] = E\left[V(x)\right] \tag{52}$$

and $V(f^*(x) + g_1(x)\tilde{J}(x) + g_1(x)\tilde{h}(x)\nu)$ is quadratic in ν.

Finally, the losslessness of system (47) is achieved by Theorem 5. This means system (1) is locally feedback equivalent to a lossless system (47). □

Similar to the proof of Theorem 19, we present the global version of Theorem 19.

Theorem 20. *If* $\mathrm{Rank}\{g_1(x)\} = m$ *and* V *satisfies one of the conditions,* $(\partial V^2(\alpha)/\partial\alpha^2)|_{\alpha=f_1(x)+f_2(x)\omega} > 0$ *or* V *is a strictly convex Lyapunov function. Then, the following are equivalent:*

(1) *System (1) is globally feedback equivalent to a lossless system with* $V \in C^2$*;*

(2) *System (1) has uniform relative degree* $(0, \ldots, 0)$ *at* $x = 0$ *and has globally lossless zero dynamics.*

In the following, an example is presented to illustrate the effectiveness of our results.

Example 21. Consider the following discrete-time nonlinear stochastic system $\sum$:

$$\begin{aligned} x(t+1) &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &\quad + \begin{pmatrix} 1 - x_1 \\ \frac{x_1}{1+x_1} \end{pmatrix} u(t) \\ &\quad + \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \omega(t), \\ z(t) &= (x_1 + x_2)\left(1 + \sqrt{2}x_1 - x_1^2\right) \\ &\quad + \left[(1-x_1)^2(1+x_1) + \frac{2x_1}{1+x_1}\right] u(t). \end{aligned} \tag{53}$$

Let us choose the storage function as $V(x) = x^T P x$, $P = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)$.

We consider a static feedback of the form

$$u(t) = (1 + x_1)\, v(t), \tag{54}$$

by Theorem 5, system $\widetilde{\sum}$

$$\begin{aligned} x(t+1) &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &\quad + \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \omega(t) \\ &\quad + \begin{pmatrix} 1 - x_1^2 \\ x_1 \end{pmatrix} v(t), \\ z(t) &= (x_1 + x_2)\left(1 + \sqrt{2}x_1 - x_1^2\right) \\ &\quad + \left[\left(1 - x_1^2\right)^2 + 2x_1\right] u(t) \end{aligned} \tag{55}$$

is lossless.

System $\sum$ is locally feedback equivalent to system $\widetilde{\sum}$; by Theorem 19, system $\sum$ have relative degree $(0, \ldots, 0)$ at $x = 0$ and lossless zero dynamics.

4. Conclusions

This paper has investigated the problem of losslessness and feedback equivalence for nonlinear stochastic discrete-time systems. A necessary and sufficient condition is developed for a nonlinear stochastic discrete-time system to be lossless. Under some conditions, it has been shown that a nonlinear stochastic discrete-time system can be lossless via state feedback if and only if the system have relative degree $(0, \ldots, 0)$ and lossless zero dynamics.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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