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Research Article

Impulsive Multiorders Riemann-Liouville Fractional Differential Equations

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Impulsive multiorders fractional differential equations are studied. Existence and uniqueness results are obtained for first- and second-order impulsive initial value problems by using Banach's fixed point theorem in an appropriate weighted space. Examples illustrating the main results are presented.

1. Introduction

Fractional calculus has become very useful over the last years because of its many applications in almost all applied sciences. By now, almost all fields of research in science and engineering use fractional calculus to better describe them.

Fractional differential equations have been of great interest and are caused both by the intensive development of the theory of fractional calculus itself and by the applications of such constructions in various science such as physics, mechanics, chemistry, and engineering. For details and some recent results on the subject we refer to the papers [1–3], books [4–7], and references cited therein.

Recently in [8], Wang et al. studied existence and uniqueness results for the following impulsive multipoint fractional integral boundary value problem involving multiorders fractional derivatives and deviating argument:

$$\begin{aligned}
&{}^{c}D_{t_k^+}^{\alpha_k} u(t) = f(t, u(t), u(\theta(t))), \quad 1 < \alpha_k \le 2, \\
&\Delta u(t_k) = I_k(u(t_k)), \\
&\Delta u'(t_k) = I_k^*(u(t_k)), \\
&\qquad\qquad k = 1, 2, \ldots, p, \\
&u(0) = \sum_{k=0}^{p} \lambda_k \mathcal{J}_{t_k^+}^{\beta_k} u(\eta_k), \\
&u'(0) = 0, \\
&\qquad\qquad t_k < \eta_k < t_{k+1},
\end{aligned} \tag{1}$$

where ${}^{c}D_{t_k^+}^{\alpha_k}$ is the *Caputo fractional derivative* of order α_k, $\mathcal{J}_{t_k^+}^{\beta_k}$ is fractional Riemann-Liouville integral of order $\beta_k > 0$, $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $I_k, I_k^* \in C(\mathbb{R}, \mathbb{R})$, $\theta \in C(J, J)$, $J = [0, T]$ $(T > 0)$, $0 = t_0 < t_1 < \cdots < t_k < \cdots < t_p < t_{p+1} = T$, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, and $\Delta u'(t_k) = u'(t_k^+) - u'(t_k^-)$ where $u(t_k^+), u'(t_k^+)$ and $u(t_k^-), u'(t_k^-)$ denote the right and left hand limits of $u(t)$ and $u'(t)$ at $t = t_k$ $(k = 1, 2, \ldots, p)$.

We notice that there are some discussions on the concept of solution for impulsive fractional differential equations for both Riemann-Liouville and Caputo fractional derivatives. We refer the interested reader to some recent papers [9–11] and the references cited therein. However, we can point out the problems caused by using the definition of Caputo fractional derivatives of order α with the lower limit 0 for a function f as

$$ {}^{c}D_{0}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{0}^{t} (t-s)^{n-\alpha-1} f^{(n)}(s)\, ds. \quad (2) $$

If there are impulse points t_k such that $t_k \in (0,t)$ for some $k \in \mathbb{N}$, then the $f^{(n)}(t_k)$ does not exist, which leads to non-integrability of the right-hand side of (2). The key idea for solving this problem is to apply the definition of fractional derivative only on an interval (t_k, t_{k+1}) and combining all intervals by using impulsive conditions.

In this paper, we study impulsive multiorders Riemann-Liouville fractional differential equations. More precisely, in Section 3 we study the existence and uniqueness of solutions for the following initial value problem for impulsive multiorders Riemann-Liouville fractional differential equations of order $0 < \alpha_k \leq 1$ of the form

$$ \begin{aligned} & D_{t_k}^{\alpha_k} x(t) = f(t, x(t)), \quad t \in J,\ t \neq t_k, \\ & \widetilde{\Delta} x(t_k) = \varphi_k(x(t_k)), \quad k = 1,2,3,\ldots,m, \\ & x(0) = 0, \end{aligned} \quad (3) $$

where $D_{t_k}^{\alpha_k}$ is the *Riemann-Liouville fractional derivative* of order $0 < \alpha_k \leq 1$ on intervals J_k, $k = 0,1,2,\ldots,m$, $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$, $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and $\varphi_k \in C(\mathbb{R},\mathbb{R})$. The notation $\widetilde{\Delta} x(t_k)$ is defined by

$$ \widetilde{\Delta} x(t_k) = I_{t_k}^{1-\alpha_k} x(t_k^+) - I_{t_{k-1}}^{1-\alpha_{k-1}} x(t_k), \quad k = 1,2,3,\ldots,m, \quad (4) $$

where $I_{t_k}^{1-\alpha_k}$ is the Riemann-Liouville fractional integral of order $1-\alpha_k$ on interval J_k. It should be noticed that if $\alpha_k = 1$ in (4), then $\widetilde{\Delta} x(t_k) = \Delta x(t_k) = x(t_k^+) - x(t_k)$ for $k = 1,2,3,\ldots,m$.

In Section 4, we investigate the initial value problem of impulsive Riemann-Liouville fractional differential equations of the form

$$ \begin{aligned} & D_{t_k}^{\alpha_k} x(t) = f(t, x(t)), \quad t \in J,\ t \neq t_k, \\ & \widetilde{\Delta} x(t_k) = \varphi_k(x(t_k)), \quad k = 1,2,3,\ldots,m, \\ & \Delta^* x(t_k) = \varphi_k^*(x(t_k)), \quad k = 1,2,3,\ldots,m, \\ & x(0) = 0, \\ & D^{\alpha_0 - 1} x(0) = \beta, \end{aligned} \quad (5) $$

where $\beta \in \mathbb{R}$, $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$, $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\varphi_k, \varphi_k^* \in C(\mathbb{R},\mathbb{R})$ for $k = 1,2,\ldots,m$, and $D_{t_k}^{\alpha_k}$ is the Riemann-Liouville fractional derivative of order $1 < \alpha_k \leq 2$ on intervals J_k for $k = 0,1,2,\ldots,m$. The notation $\widetilde{\Delta} x(t_k)$ is defined by (4) and $\Delta^* x(t_k)$ is defined by

$$ \Delta^* x(t_k) = I_{t_k}^{2-\alpha_k} x(t_k^+) - I_{t_{k-1}}^{2-\alpha_{k-1}} x(t_k), \quad k = 1,2,\ldots,m, \quad (6) $$

where $I_{t_k}^{2-\alpha_k}$ is the Riemann-Liouville fractional integral of order $2-\alpha_k > 0$ on J_k. It should be noticed that if $\alpha_k = 2$ in (6), then $\widetilde{\Delta} x(t_k) = D_{t_k} x(t_k^+) - D_{t_{k-1}} x(t_k)$ and $\Delta^* x(t_k) = \Delta x(t_k) = x(t_k^+) - x(t_k)$ for $k = 1,2,\ldots,m$.

By using Banach's fixed point theorem we prove existence and uniqueness results for the problem (3) and (5) in an appropriate weighted space.

The paper is organized as follows: Section 2 contains some preliminary notations, definitions, and lemmas that we need in the sequel. In Section 3 we present the main results for problem (3), while in Section 4 we present the main results for problem (5). Examples illustrating the obtained results are also presented.

2. Preliminaries

In this section, we introduce some notations and definitions of fractional calculus and present preliminary results needed in our proofs later.

Definition 1. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (a,b) \to \mathbb{R}$ is defined by

$$ D_{a}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^{n} \int_{a}^{t} (t-s)^{n-\alpha-1} f(s)\, ds, \quad n-1 < \alpha < n,\ t \in (a,b), \quad (7) $$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of a real number α, provided the right-hand side is pointwise defined on (a,b), where Γ is the gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-s} s^{\alpha-1} ds$.

Definition 2. For at least n-times differentiable function $f : (a,b) \to \mathbb{R}$, the Caputo derivative of fractional order α is defined as

$$ {}^{c}D_{a}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{t} (t-s)^{n-\alpha-1} f^{(n)}(s)\, ds, \quad n-1 < \alpha < n,\ t \in (a,b), \quad (8) $$

where $n = [\alpha] + 1$.

Definition 3. The Riemann-Liouville fractional integral of order $\beta > 0$ of a continuous function $f : (a,b) \to \mathbb{R}$ is defined by

$$ I_{a}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{t} (t-s)^{\alpha-1} f(s)\, ds, \quad t \in (a,b) \quad (9) $$

provided the right-hand side is pointwise defined on (a,b).

Lemma 4 (see [5]). *Let $\alpha > 0$ and $x \in C(a,b) \cap L(a,b)$. Then the fractional differential equation*

$$ D_{a}^{\alpha} x(t) = 0 \quad (10) $$

has a unique solution

$$ x(t) = k_1 (t-a)^{\alpha-1} + k_2 (t-a)^{\alpha-2} + \cdots + k_n (t-a)^{\alpha-n}, \quad (11) $$

where $k_i \in \mathbb{R}$, $i = 1,2,\ldots,n$, and $n-1 < q < n$.

Lemma 5 (see [5]). *Let $\alpha > 0$. Then for $x \in C(a,b) \cap L(a,b)$ it holds*

$$I_a^\alpha D_a^\alpha x(t) = x(t) - \sum_{j=1}^{n} \frac{\left(I_a^{n-\alpha} x\right)^{(n-j)}(a)}{\Gamma(\alpha - j + 1)} (t-a)^{\alpha - j}, \quad (12)$$

where $n-1 < \alpha < n$.

3. Impulsive Riemann-Liouville Fractional Differential Equations of Orders $0 < \alpha_k \le 1$

Let $J = [0,T]$, $J_0 = [t_0, t_1]$, $J_k = (t_k, t_{k+1}]$ for $k = 1,2,3,\ldots,m$. Let $PC(J,\mathbb{R}) = \{x : J \to \mathbb{R}, x(t)$ is continuous everywhere except for some t_k at which $x(t_k^+)$ and $x(t_k^-)$ exist, and $x(t_k^-) = x(t_k)$, $k = 1,2,3,\ldots,m\}$. For $\gamma \in \mathbb{R}_+$, we introduce the space $C_{\gamma,k}(J_k,\mathbb{R}) = \{x : J_k \to \mathbb{R} : (t-t_k)^\gamma x(t) \in C(J_k,\mathbb{R})\}$ with the norm $\|x\|_{C_{\gamma,k}} = \sup_{t\in J_k} |(t-t_k)^\gamma x(t)|$ and $PC_\gamma = \{x : J \to \mathbb{R} :$ for each $t \in J_k$ and $(t-t_k)^\gamma x(t) \in C(J_k,\mathbb{R})$, $k = 0,1,2,\ldots,m\}$ with the norm $\|x\|_{PC_\gamma} = \max\{\sup_{t\in J_k} |(t-t_k)^\gamma x(t)| : k = 0,1,2,\ldots,m\}$. Clearly PC_γ is a Banach space.

In this section we study problem (3).

Lemma 6. *If $x \in PC(J,\mathbb{R})$ is a solution of (3), then, for any $t \in J_k$, $k = 0,1,2,\ldots,m$,*

$$x(t) = \frac{(t-t_k)^{\alpha_k - 1}}{\Gamma(\alpha_k)} \left[\sum_{0<t_k<t} \left(I_{t_{k-1}}^1 f(t_k, x(t_k)) + \varphi_k(x(t_k)) \right) \right] + I_{t_k}^{\alpha_k} f(t, x(t)), \quad (13)$$

with $\sum_{0<0}(\cdot) = 0$. The converse is also true.

Proof. For $t \in J_0$, taking the Riemann-Liouville fractional integral of order α_0 to the first equation of (3) and using Lemma 5, we have

$$x(t) = I_{t_0}^{\alpha_0} f(t, x(t)) + c_0 \frac{t^{\alpha_0 - 1}}{\Gamma(\alpha_0)}, \quad (14)$$

where $c_0 = I_{t_0}^{1-\alpha_0} x(0)$. The initial condition $x(0) = 0$ implies $c_0 = 0$. Then for $x \in J_0$, we get

$$x(t) = I_{t_0}^{\alpha_0} f(t, x(t)). \quad (15)$$

Applying the Riemann-Liouville fractional integral of order $1-\alpha_0$ from 0 to t_1, we get

$$I_{t_0}^{1-\alpha_0} x(t_1) = I_{t_0}^{1-\alpha_0} I_{t_0}^{\alpha_0} f(t_1, x(t_1)) = I_{t_0}^1 f(t_1, x(t_1)). \quad (16)$$

For $t \in J_1$, taking the Riemann-Liouville fractional integral of order α_1 to the first equation of (3) and using Lemma 5, we have

$$x(t) = \frac{(t-t_1)^{\alpha_1 - 1}}{\Gamma(\alpha_1)} I_{t_1}^{1-\alpha_1} x(t_1^+) + I_{t_1}^{\alpha_1} f(t, x(t)). \quad (17)$$

Since $I_{t_1}^{1-\alpha_1} x(t_1^+) = I_{t_0}^{1-\alpha_0} x(t_1) + \varphi_1(x(t_1))$, it follows for $t \in J_1$ that

$$x(t) = \frac{(t-t_1)^{\alpha_1 - 1}}{\Gamma(\alpha_1)} \left[I_{t_0}^1 f(t_1, x(t_1)) + \varphi_1(x(t_1)) \right] + I_{t_1}^{\alpha_1} f(t, x(t)). \quad (18)$$

Applying the Riemann-Liouville fractional integral of order $1-\alpha_1$ to the above equation and substituting $t = t_1$, one has

$$I_{t_1}^{1-\alpha_1} x(t) = I_{t_0}^1 f(t_1, x(t_1)) + \varphi_1(x(t_1)) + I_{t_1}^1 f(t, x(t)). \quad (19)$$

For $t \in J_2$, using the Riemann-Liouville fractional integral of order α_2 for (3), we have

$$\begin{aligned} x(t) &= \frac{(t-t_2)^{\alpha_2 - 1}}{\Gamma(\alpha_2)} I_{t_2}^{1-\alpha_2} x(t_2^+) + I_{t_2}^{\alpha_2} f(t, x(t)) \\ &= \frac{(t-t_2)^{\alpha_2 - 1}}{\Gamma(\alpha_2)} \left[I_{t_1}^{1-\alpha_1} x(t_2) + \varphi_2(x(t_2)) \right] \\ &\quad + I_{t_2}^{\alpha_2} f(t, x(t)) = \frac{(t-t_2)^{\alpha_2 - 1}}{\Gamma(\alpha_2)} \left[I_{t_0}^1 f(t_1, x(t_1)) \right. \\ &\quad \left. + \varphi_1(x(t_1)) + I_{t_1}^1 f(t_2, x(t_2)) + \varphi_2(x(t_2)) \right] \\ &\quad + I_{t_2}^{\alpha_2} f(t, x(t)). \end{aligned} \quad (20)$$

Repeating the above procession for each J_k, we obtain (13).

On the other hand, assume that x is a solution of (3). Applying the Riemann-Liouville fractional derivative of order α_k for (13) on J_k, $k = 0,1,2,\ldots,m$, and using $\Gamma(0) = \infty$, it follows that

$$D_{t_k}^{\alpha_k} x(t) = f(t, x(t)). \quad (21)$$

It is easy to verify that $\widetilde{\Delta} x(t_k) = \varphi_k(x(t_k))$, $k = 1,2,3,\ldots,m$, and $x(0) = 0$. The proof is complete. □

Next we will prove that problem (3) has a unique solution by using Banach's fixed point theorem.

Theorem 7. *Assume that*

(H_1) *$f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function and satisfies*

$$|f(t,x) - f(t,y)| \le L_1 |x-y|, \quad L_1 > 0, \ \forall t \in J, \ x, y \in \mathbb{R}; \quad (22)$$

(H_2) *$\varphi_k : \mathbb{R} \to \mathbb{R}$, $k = 1,2,3,\ldots,m$, are continuous functions and satisfy*

$$|\varphi_k(x) - \varphi_k(y)| \le L_2 |x-y|, \quad L_2 > 0, \ \forall x, y \in \mathbb{R}. \quad (23)$$

If

$$\Lambda_1 := \frac{T^*}{\Gamma^*} (L_1 + L_1 T + L_2 m) < 1, \quad (24)$$

where $T^* = \max\{T^{\gamma+\alpha_k-1}, T^{\gamma+\alpha_k}\}$, $\Gamma^* = \min\{\Gamma(\alpha_k), \Gamma(\alpha_k+1)\}$, and $\gamma + \alpha_k > 1$ for $k = 0,1,2,\ldots,m$, then the initial value problem (3) has a unique solution on $[0,T]$.

Proof. In view of Lemma 6, we define the operator $\mathscr{K} : PC \to PC$ as

$$\mathscr{K}x(t) = \frac{(t-t_k)^{\alpha_k-1}}{\Gamma(\alpha_k)}\left[\sum_{0<t_k<t}\left(I^1_{t_{k-1}}f(t_k,x(t_k)) + \varphi_k(x(t_k))\right)\right] + I^{\alpha_k}_{t_k}f(t,x(t)). \tag{25}$$

In addition, we define a ball $B_r = \{x \in PC_\gamma(J,\mathbb{R}), \|x\|_{PC_\gamma} \le r\}$. To show that $\mathscr{K}x \in PC_\gamma$, we suppose $\tau \in J_k$ and then

$$\begin{aligned}
&\left|(t-t_k)^\gamma \mathscr{K}x(t) - (\tau-t_k)^\gamma \mathscr{K}x(\tau)\right| \\
&\le \left|\frac{(t-t_k)^{\gamma+\alpha_k-1}}{\Gamma(\alpha_k)}\left[\sum_{j=1}^{k}\left(I^1_{t_{j-1}}f(t_j,x(t_j)) + \varphi_j(x(t_j))\right)\right] + (t-t_k)^\gamma I^{\alpha_k}_{t_k}f(t,x(t)) - \frac{(\tau-t_k)^{\gamma+\alpha_k-1}}{\Gamma(\alpha_k)}\left[\sum_{j=1}^{k}\left(I^1_{t_{j-1}}f(t_j,x(t_j)) + \varphi_j(x(t_j))\right)\right] - (\tau-t_k)^\gamma I^{\alpha_k}_{t_k}f(\tau,x(\tau))\right| \\
&\le \left|\frac{(t-t_k)^{\gamma+\alpha_k-1} - (\tau-t_k)^{\gamma+\alpha_k-1}}{\Gamma(\alpha_k)}\right|\left|\sum_{j=1}^{k}\left(I^1_{t_{j-1}}f(t_j,x(t_j)) + \varphi_j(x(t_j))\right)\right| + \left|(t-t_k)^\gamma I^{\alpha_k}_{t_k}f(t,x(t)) - (\tau-t_k)^\gamma \cdot I^{\alpha_k}_{t_k}f(\tau,x(\tau))\right|.
\end{aligned} \tag{26}$$

As $t \to \tau$, we get $|(t-t_k)^\gamma \mathscr{K}x(t) - (\tau-t_k)^\gamma \mathscr{K}x(\tau)| \to 0$ for each $k = 0,1,2,\ldots,m$. Therefore, we have $\mathscr{K}x(t) \in PC_\gamma$. Next we will show that $\mathscr{K}B_r \subset B_r$. Suppose that $\sup_{t\in J}|f(t,0)| = M$, $\max\{|I_k(0)|,\ k = 1,2,3,\ldots,m\} = N$. Setting

$$\Lambda_2 = \frac{T^*}{\Gamma^*}(M + MT + Nm) \tag{27}$$

we choose r such that

$$r \ge \frac{\Lambda_2}{1-\Lambda_1}. \tag{28}$$

For any $x \in B_r$ and for each $t \in J_k$, we have

$$\begin{aligned}
|(\mathscr{K}x)(t)| &\le \frac{(t-t_k)^{\alpha_k-1}}{\Gamma(\alpha_k)}\left[\sum_{j=1}^{k}\left(I^1_{t_{j-1}}|f(s,x(s))|(t_j) + |\varphi_j(x(t_j))|\right)\right] + I^{\alpha_k}_{t_k}|f(s,x(s))|(t) \\
&\le \frac{(t-t_k)^{\alpha_k-1}}{\Gamma(\alpha_k)}\left[\sum_{j=1}^{k}\left(I^1_{t_{j-1}}(|f(s,x(s)) - f(s,0)| + |f(s,0)|) \cdot (t_j) + |\varphi_j(x(t_j)) - \varphi_j(0)| + |\varphi_j(0)|\right)\right] + I^{\alpha_k}_{t_k}(|f(s,x(s)) - f(s,0)| - |f(s,0)|)(t) \le \frac{(t-t_k)^{\alpha_k-1}}{\Gamma(\alpha_k)}[(L_1r+M)t_k + (L_2r+N)k] + \frac{(t-t_k)^{\alpha_k}}{\Gamma(\alpha_k+1)}(L_1r+M).
\end{aligned} \tag{29}$$

Multiplying both sides of the above inequality by $(t-t_k)^\gamma$ for each $t \in J_k$, we obtain

$$\begin{aligned}
&(t-t_k)^\gamma |(\mathscr{K}x)(t)| \\
&\le \frac{(t-t_k)^{\gamma+\alpha_k-1}}{\Gamma(\alpha_k)}[(L_1r+M)t_k + (L_2r+N)k] + \frac{(t-t_k)^{\gamma+\alpha_k}}{\Gamma(\alpha_k+1)}(L_1r+M) \\
&\le r\left(\frac{T^*}{\Gamma^*}(L_1 + L_1T + L_2m)\right) + \left(\frac{T^*}{\Gamma^*}(M + MT + Nm)\right) = r\Lambda_1 + \Lambda_2 \le r.
\end{aligned} \tag{30}$$

This implies that $\mathscr{K}B_r \subset B_r$.

Finally we will show that $\mathscr{K}$ is a contraction mapping on B_r. For $x, y \in PC_\gamma(J,\mathbb{R})$ and for each $t \in J$ we have

$$\begin{aligned}
&|(\mathscr{K}x)(t) - (\mathscr{K}y)(t)| \\
&\le \frac{(t-t_k)^{\alpha_k-1}}{\Gamma(\alpha_k)}\left[\sum_{j=1}^{k}\left(I^1_{t_{j-1}}|f(s,x(s)) - f(s,y(s))| \cdot (t_j) + |\varphi_j(x(t_j)) - \varphi_j(y(t_j))|\right)\right] + I^{\alpha_k}_{t_k}|f(s,x(s)) - f(s,y(s))|(t) \\
&\le \frac{(t-t_k)^{\alpha_k-1}}{\Gamma(\alpha_k)}\left[\sum_{j=1}^{k}\left(L_1(t_j - t_{j-1})\|x-y\| + L_2\|x-y\|\right)\right] + \frac{(t-t_k)^{\alpha_k}}{\Gamma(\alpha_k+1)}L_1\|x-y\|.
\end{aligned} \tag{31}$$

Multiplying both sides of the above inequality by $(t-t_k)^\gamma$ for each $t \in J_k$, we have

$$\begin{aligned}
&\left|(t-t_k)^\gamma(\mathscr{K}x)(t) - (t-t_k)^\gamma(\mathscr{K}y)(t)\right| \\
&\le \frac{(t-t_k)^{\gamma+\alpha_k-1}}{\Gamma(\alpha_k)}(L_1t_k\|x-y\| + L_2k\|x-y\|)
\end{aligned}$$

$$+\frac{(t-t_k)^{\gamma+\alpha_k}}{\Gamma(\alpha_k+1)}L_1\|x-y\| \leq \frac{T^*}{\Gamma^*}(L_1+L_1T+L_2m)\|x-y\|. \quad (32)$$

It follows that

$$\|\mathscr{K}x-\mathscr{K}y\| \leq \Lambda_1\|x-y\|. \quad (33)$$

Since $\Lambda_1 < 1$, $\mathscr{K}$ is a contraction mapping on B_r. Therefore (3) has a unique solution on $[0,T]$. □

Example 8. Consider the following impulsive multiorders Riemann-Liouville fractional initial value problem:

$$\begin{gathered} D_{t_k}^{(k+1)/(k+2)}x = \frac{|x(t)|\ln(t+1)^2}{(4t+3)^3(1+2|x(t)|)} + \frac{e^t\cos t}{5-2t}, \quad t\in\left[0,\frac{11}{10}\right],\ t\neq t_k, \\ \widetilde{\Delta}x(t_k) = \frac{|x(t_k)|}{(4k)!+|x(t_k)|}, \quad k=1,2,\ldots,10,\ t_k=\frac{k}{10}, \\ x(0)=0. \end{gathered} \quad (34)$$

Here $\alpha_k = (k+1)/(k+2)$, $k=0,1,2,\ldots,10$, $m=10$, $T=11/10$, $f(t,x)=(|x|\ln(t+1)^2)/((4t+3)^3(1+2|x|)) + (e^t\cos t)/(5-2t)$, and $\varphi_k(x)=|x|/((4k)!+|x|)$. Since $|f(t,x)-f(t,y)| \leq (2/27)|x-y|$ and $|\varphi_k(x)-\varphi_k(y)| \leq (1/24)|x-y|$, for $k=1,2,\ldots,10$, then (H_1) and (H_2) are satisfied with $L_1 = 2/27$, $L_2 = 1/24$.

By choosing $\gamma = 1$, we find that $T^* = 1.200428$, $\Gamma^* = 0.886227$, and

$$\Lambda_1 = \frac{T^*}{\Gamma^*}(L_1+L_1T+L_2m) = 0.775096 < 1. \quad (35)$$

Hence, by Theorem 7, the initial value problem (34) has a unique solution on $[0,11/10]$.

4. Impulsive Riemann-Liouville Fractional Differential Equations of Orders $1 < \alpha_k \leq 2$

Problem (5) is studied in this section.

Lemma 9. *The unique solution of problem (5) is given by*

$$\begin{aligned} x(t) &= \frac{(t-t_k)^{\alpha_k-2}}{\Gamma(\alpha_k-1)}\Bigg[\beta t_k + \sum_{j=1}^{k-1}(t_k-t_j)\left(I_{t_{j-1}}^1 f(t_j,x(t_j)) + \varphi_j(x(t_j))\right) \\ &\quad + \sum_{j=1}^{k}\left(I_{t_{j-1}}^2 f(t_j,x(t_j)) + \varphi_j^*(x(t_j))\right)\Bigg] \\ &\quad + \frac{(t-t_k)^{\alpha_k-1}}{\Gamma(\alpha_k)}\Bigg[\beta + \sum_{j=1}^{k}\left(I_{t_{j-1}}^1 f(t_j,x(t_j)) + \varphi_j x(t_j)\right)\Bigg] + I_{t_k}^{\alpha_k} f(t, x(t)), \end{aligned} \quad (36)$$

for $t\in J_k$, $k=0,1,2,\ldots,m$ with $\sum_{j=a}^{b}=0$ for $b<a$.

Proof. For $t \in J_0$, taking the Riemann-Liouville fractional integral of order α_0 for the first equation of (5) and applying Lemma 5, we obtain

$$x(t) = I_{t_0}^{\alpha_0} f(t,x(t)) + c_0\frac{t^{\alpha_0-1}}{\Gamma(\alpha_0)} + c_1\frac{t^{\alpha_0-2}}{\Gamma(\alpha_0-1)}, \quad (37)$$

where $c_0 = I_{t_0}^{1-\alpha_0}x(0)$ and $c_1 = I_{t_0}^{2-\alpha_0}x(0)$. The initial condition $x(0)=0$ implies $c_1=0$ which leads to

$$x(t) = I_{t_0}^{\alpha_0} f(t,x(t)) + c_0\frac{t^{\alpha_0-1}}{\Gamma(\alpha_0)}. \quad (38)$$

Using the Riemann-Liouville fractional derivative of order α_0-1 for (38) on J_0, we get

$$D^{\alpha_0-1}x(t) = I_{t_0}^1 f(t,x(t)) + c_0. \quad (39)$$

From the second initial condition of (5), we get

$$x(t) = I_{t_0}^{\alpha_0} f(t,x(t)) + \beta\frac{t^{\alpha_0-1}}{\Gamma(\alpha_0)}. \quad (40)$$

Taking the Riemann-Liouville fractional integral of order $1-\alpha_0$ and $2-\alpha_0$ for (40) and substituting $t=t_1$, we have

$$\begin{aligned} I_{t_0}^{1-\alpha_0}x(t_1) &= I_{t_0}^1 f(t_1,x(t_1)) + \beta, \\ I_{t_0}^{2-\alpha_0}x(t_1) &= I_{t_0}^2 f(t_1,x(t_1)) + \beta t_1. \end{aligned} \quad (41)$$

For $t \in J_1$, taking the Riemann-Liouville fractional integral of order α_1 for (5), we have

$$x(t) = \frac{(t-t_1)^{\alpha_1-2}}{\Gamma(\alpha_1-1)}I_{t_1}^{2-\alpha_1}x(t_1^+) + \frac{(t-t_1)^{\alpha_1-1}}{\Gamma(\alpha_1)}I_{t_1}^{1-\alpha_1}x(t_1^+) + I_{t_1}^{\alpha_1}f(t,x(t)). \quad (42)$$

Since $I_{t_1}^{1-\alpha_1}x(t_1^+) = I_{t_0}^{1-\alpha_0}x(t_1) + \varphi_1(x(t_1))$ and $I_{t_1}^{2-\alpha_1}x(t_1^+) = I_{t_0}^{2-\alpha_0}x(t_1) + \varphi_1^*(x(t_1))$, it follows that, for $t \in J_1$,

$$\begin{aligned} x(t) &= \frac{(t-t_1)^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\left[\beta t_1 + I_{t_0}^2 f(t_1, x(t_1)) + \varphi_1^*(x(t_1))\right] \\ &+ \frac{(t-t_1)^{\alpha_1-1}}{\Gamma(\alpha_1)}\left[\beta + I_{t_0}^1 f(t_1, x(t_1)) + \varphi_1(x(t_1))\right] \\ &+ I_{t_1}^{\alpha_1} f(t, x(t)). \end{aligned} \quad (43)$$

The Riemann-Liouville integrating of the above equation of order $1-\alpha_1$ and $2-\alpha_1$ for $t = t_2$ leads to

$$\begin{aligned} I_{t_1}^{2-\alpha_1}x(t_2) &= \beta t_1 + I_{t_0}^2 f(t_1, x(t_1)) + \varphi_1^*(x(t_1)) \\ &+ (t_2 - t_1)(\beta + I_{t_0}^1 f(t_1, x(t_1)) \\ &+ \varphi_1(x(t_1)) + I_{t_1}^2 f(t_2, x(t_2)), \\ I_{t_1}^{1-\alpha_1}x(t_2) &= \beta + I_{t_0}^1 f(t_1, x(t_1)) + \varphi_1(x(t_1)) \\ &+ I_{t_1}^1 f(t_2, x(t_2)). \end{aligned} \quad (44)$$

For $t \in J_2$, applying the Riemann-Liouville fractional integral of order α_2 for (5) and substituting values $I_{t_2}^{1-\alpha_2}x(t_2^+) = I_{t_1}^{1-\alpha_1}x(t_2) + \varphi_2(x(t_2))$ and $I_{t_2}^{2-\alpha_2}x(t_2^+) = I_{t_1}^{2-\alpha_1}x(t_2) + \varphi_2^*(x(t_2))$, we get

$$\begin{aligned} x(t) &= \frac{(t-t_2)^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\left[\beta t_2 + (t_2-t_1)\left(I_{t_0}^1 f(t_1, x(t_1))\right.\right. \\ &+ \left.\varphi_1(x(t_1))\right) + \left(I_{t_0}^2 f(t_1, x(t_1)) + \varphi_1^*(x(t_1))\right. \\ &+ \left.\left.I_{t_1}^2 f(t_2, x(t_2)) + \varphi_2^*(x(t_2))\right)\right] + \frac{(t-t_2)^{\alpha_2-1}}{\Gamma(\alpha_2)}\left[\beta\right. \\ &+ I_{t_0}^1 f(t_1, x(t_1)) + \varphi_1(x(t_1)) + I_{t_1} f(t_2, x(t_2)) \\ &+ \left.\varphi_2(x(t_2))\right] + I_{t_2}^{\alpha_2} f(t, x(t)). \end{aligned} \quad (45)$$

Repeating the above process, for $t \in J$, we obtain (36) as requested. □

Next, we will prove the existence and uniqueness of a solution to the initial value problem (5) by using Banach's fixed point theorem.

Theorem 10. *Assume that (H_1) and (H_2) hold. In addition we suppose that*

(H_3) $\varphi_k^* : \mathbb{R} \to \mathbb{R}$, $k = 1, 2, 3, \ldots, m$, are continuous functions and satisfy

$$\left|\varphi_k^*(x) - \varphi_k^*(y)\right| \le L_3 |x - y|, \quad L_3 > 0, \ \forall x, y \in \mathbb{R}. \quad (46)$$

If

$$\Omega_1 := \frac{T_2^*}{\Gamma_2^*}\left(L_1(\Phi + T + 1) + L_2(2m-1) + L_3 m\right) < 1, \quad (47)$$

where

$$\Phi = \frac{1}{2}\sum_{j=1}^{m}\left(t_j - t_{j-1}\right)\left(2t_m - t_j - t_{j-1}\right), \quad (48)$$

$T_2^* = \max\{T^{\gamma+\alpha_k-2}, T^{\gamma+\alpha_k-1}, T^{\gamma+\alpha_k}\}$, $\Gamma_2^* = \min\{\Gamma(\alpha_k - 1), \Gamma(\alpha_k)$, and $\Gamma(\alpha_k + 1)\}$, $\gamma + \alpha_k > 2$ for $k = 0, 1, 2, \ldots, m$, then problem (5) has a unique solution on $[0, T]$.

Proof. We define the operator $\mathscr{A} : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ as follows:

$$\begin{aligned} \mathscr{A}x(t) &= \frac{(t-t_k)^{\alpha_k-2}}{\Gamma(\alpha_k-1)}\Bigg[\beta t_k \\ &+ \sum_{j=1}^{k-1}\left(\left(t_k - t_j\right)\left(I_{t_{j-1}}^1 f\left(t_j, x\left(t_j\right)\right) + \varphi_j\left(x\left(t_j\right)\right)\right)\right) \\ &+ \sum_{j=1}^{k}\left(I_{t_{j-1}}^2 f\left(t_j, x\left(t_j\right)\right) + \varphi_j^*\left(x\left(t_j\right)\right)\right)\Bigg] \\ &+ \frac{(t-t_k)^{\alpha_k-1}}{\Gamma(\alpha_k)}\Bigg[\beta \\ &+ \sum_{j=1}^{k}\left(I_{t_{j-1}}^1 f\left(t_j, x\left(t_j\right)\right) + \varphi_j\left(x\left(t_j\right)\right)\right)\Bigg] \\ &+ I_{t_k}^{\alpha_k} f(t, x(t)), \end{aligned} \quad (49)$$

for $t \in J_k$, $k = 0, 1, 2, \ldots, m$ with $\sum_{j=a}^{b} = 0$ for $b < a$.

It is straightforward to show that $\mathscr{A}x \in PC_\gamma(J, \mathbb{R})$; see Theorem 7. Next we will show that $\mathscr{A}B_r \subset B_r$, where a ball B_r is defined by $B_r = \{x \in PC_\gamma(J, \mathbb{R}), \|x\|_{PC_\gamma} \le r\}$. Assume that $\sup_{t\in J}|f(t, 0)| = M$, $\max\{|\varphi_k(0)| : k = 1, 2, 3, \ldots, m\} = N$ and $\max\{|\varphi_k^*(0)| : k = 1, 2, 3, \ldots, m\} = P$. Setting

$$\begin{aligned} \Omega_2 &= \frac{T_2^*}{\Gamma_2^*}\left(M(\Phi + T + 1) + N(2m-1) + Pm\right. \\ &+ \left.|\beta|(T+1)\right), \end{aligned} \quad (50)$$

we choose a constant r such that

$$r \ge \frac{\Omega_2}{1 - \Omega_1}. \quad (51)$$

Let $x \in B_r$. For any $t \in J_k$, $k = 0, 1, 2, \ldots, m$, we have

$$
\begin{aligned}
|(\mathscr{A}x)(t)| \le{} & \frac{(t-t_k)^{\alpha_k-2}}{\Gamma(\alpha_k-1)}\left[|\beta| t_k + \sum_{j=1}^{k-1}\left(\left(t_k-t_j\right)\left(I_{t_{j-1}}^{1}|f(s,x(s))|\left(t_j\right)+\left|\varphi_j\left(x\left(t_j\right)\right)\right|\right)\right)\right.\\
&\left.+\sum_{j=1}^{k}\left(I_{t_{j-1}}^{2}|f(s,x(s))|\left(t_j\right)+\left|\varphi_j^{*}\left(x\left(t_j\right)\right)\right|\right)\right] + \frac{(t-t_k)^{\alpha_k-1}}{\Gamma(\alpha_k)}\left[|\beta| + \sum_{j=1}^{k}\left(I_{t_{j-1}}^{1}|f(s,x(s))|\left(t_j\right)+\left|\varphi_j\left(x\left(t_j\right)\right)\right|\right)\right]\\
&+I_{t_k}^{\alpha_k}|f(s,x(s))|(t) \le \frac{(t-t_k)^{\alpha_k-2}}{\Gamma(\alpha_k-1)}\left[|\beta| t_k\right.\\
&+\sum_{j=1}^{k-1}\left(\left(t_k-t_j\right)\left(I_{t_{j-1}}^{1}\left(|f(s,x(s))-f(s,0)|+|f(s,0)|\right)\left(t_j\right)+\left|\varphi_j\left(x\left(t_j\right)\right)-\varphi_j(0)\right|+\left|\varphi_j(x(0))\right|\right)\right)\\
&\left.+\sum_{j=1}^{k}\left(I_{t_{j-1}}^{2}\left(|f(s,x(s))-f(s,0)|+|f(s,0)|\right)\left(t_j\right)+\left|\varphi_j^{*}\left(x\left(t_j\right)\right)-\varphi_j^{*}(0)\right|+\left|\varphi_j^{*}(0)\right|\right)\right] + \frac{(t-t_k)^{\alpha_k-1}}{\Gamma(\alpha_k)}\left[|\beta|\right.\\
&\left.+\sum_{j=1}^{k}\left(I_{t_{j-1}}^{1}\left(|f(s,x(s))-f(s,0)|+|f(s,0)|\right)\left(t_j\right)+\left|\varphi_j\left(x\left(t_j\right)\right)-\varphi_j(0)\right|+\left|\varphi_j(0)\right|\right)\right] + I_{t_k}^{\alpha_k}\left(|f(s,x(s))-f(s,0)|\right.\\
&\left.+|f(s,0)|\right)(t) \le \frac{(t-t_k)^{\alpha_k-2}}{\Gamma(\alpha_k-1)}\left[|\beta| t_k + (L_1r+M)\left(\sum_{j=1}^{k-1}\left(t_k-t_j\right)\left(t_j-t_{j-1}\right)\right) + (L_2r+N)(k-1)\right.\\
&\left.+(L_1r+M)\left(\sum_{j=1}^{k}\frac{\left(t_j-t_{j-1}\right)^2}{2}\right)+(L_3r+P)k\right] + \frac{(t-t_k)^{\alpha_k-1}}{\Gamma(\alpha_k)}\left[|\beta| + (L_1r+M)\left(\sum_{j=1}^{k}\left(t_j-t_{j-1}\right)\right)+(L_2r+N)k\right]\\
&+(L_1r+M)\frac{(t-t_k)^{\alpha_k}}{\Gamma(\alpha_k+1)}.
\end{aligned}
\tag{52}
$$

Multiplying both sides of the above inequality by $(t-t_k)^{\gamma}$ for $t \in J_k$, we have

$$
\begin{aligned}
(t-t_k)^{\gamma}|(\mathscr{A}x)(t)| \le{} & \frac{(t-t_k)^{\gamma+\alpha_k-2}}{\Gamma(\alpha_k-1)}\left[|\beta| t_k\right.\\
&+(L_1r+M)\left(\sum_{j=1}^{k-1}\left(t_k-t_j\right)\left(t_j-t_{j-1}\right)\right)\\
&+(L_2r+N)(k-1)\\
&\left.+(L_1r+M)\left(\sum_{j=1}^{k}\frac{\left(t_j-t_{j-1}\right)^2}{2}\right)+(L_3r+P)k\right]\\
&+\frac{(t-t_k)^{\gamma+\alpha_k-1}}{\Gamma(\alpha_k)}\left[|\beta|\right.\\
&\left.+(L_1r+M)\left(\sum_{j=1}^{k}\left(t_j-t_{j-1}\right)\right)+(L_2r+N)k\right]\\
&+(L_1r+M)\frac{(t-t_k)^{\gamma+\alpha_k}}{\Gamma(\alpha_k+1)} \le \frac{T_2^{*}}{\Gamma_2^{*}}\left[|\beta| T\right.\\
&+(L_1r+M)\left(\sum_{j=1}^{m-1}\left(t_m-t_j\right)\left(t_j-t_{j-1}\right)\right)\\
&+(L_2r+N)(m-1)\\
&+(L_1r+M)\left(\sum_{j=1}^{m}\frac{\left(t_j-t_{j-1}\right)^2}{2}\right)+(L_3r+P)+|\beta|\\
&+(L_1r+M)\left(\sum_{j=1}^{m}\left(t_j-t_{j-1}\right)\right)+(L_2r+N)m
\end{aligned}
$$

$$+\left(L_1 r+M\right)\Bigg]$$

$$=r\left[\frac{T_2^*}{\Gamma_2^*}\left(L_1(\Phi+T+1)+L_2(2m-1)+L_3 m\right)\right]$$

$$+\frac{T_2^*}{\Gamma_2^*}\left(M(\Phi+T+1)+N(2m-1)+Pm\right.$$
$$\left.+|\beta|(T+1)\right)=\Omega_1 r+\Omega_2. \tag{53}$$

This implies that $\mathscr{A}B_r \subset B_r$.

Finally we will show that $\mathscr{A}$ is a contraction mapping on B_r. For $x, y \in PC_\gamma(J,\mathbb{R})$ and for each $t \in J$ we have

$$\begin{aligned}|(\mathscr{A}x)(t)-(\mathscr{A}y)(t)| &\leq \frac{(t-t_k)^{\alpha_k-2}}{\Gamma(\alpha_k-1)}\left[\sum_{j=1}^{k-1}\left((t_k-t_j)\left(I_{t_{j-1}}^1|f(s,x(s))-f(s,y(s))|(t_j)+\left|\varphi_j(x(t_j))-\varphi_j(y(t_j))\right|\right)\right)\right.\\
&\left.+\sum_{j=1}^{k}\left(I_{t_{j-1}}^2|f(s,x(s))-f(s,y(s))|(t_j)+\left|\varphi_j^*(x(t_j))-\varphi_j^*(y(t_j))\right|\right)\right]\\
&+\frac{(t-t_k)^{\alpha_k-1}}{\Gamma(\alpha_k)}\left[\sum_{j=1}^{k}\left(I_{t_{j-1}}^1|f(s,x(s))-f(s,y(s))|(t_j)+\left|\varphi_j(x(t_j))-\varphi_j(y(t_j))\right|\right)\right]+I_{t_k}^{\alpha_k}|f(s,x(s))-f(s,y(s))|\\
&\cdot(t).\end{aligned} \tag{54}$$

Multiplying both sides of the above inequality by $(t-t_k)^\gamma$, we have

$$\begin{aligned}&\left|(t-t_k)^\gamma(\mathscr{A}x)(t)-(t-t_k)^\gamma(\mathscr{A}y)(t)\right|\\
&\leq\frac{(t-t_k)^{\gamma+\alpha_k-2}}{\Gamma(\alpha_k-1)}\left[L_1\|x-y\|\right.\\
&\cdot\left(\sum_{j=1}^{k-1}(t_k-t_j)(t_j-t_{j-1})\right)+L_2\|x-y\|(k-1)\\
&\left.+L_1\|x-y\|\left(\sum_{j=1}^{k}\frac{(t_j-t_{j-1})^2}{2}\right)+L_3\|x-y\|k\right]\\
&+\frac{(t-t_k)^{\gamma+\alpha_k-1}}{\Gamma(\alpha_k)}\left[L_1\|x-y\|\left(\sum_{j=1}^{k}(t_j-t_{j-1})\right)\right.\\
&\left.+L_2\|x-y\|k\right]+L_1\|x-y\|\frac{(t-t_k)^{\gamma+\alpha_k}}{\Gamma(\alpha_k+1)}\\
&\leq\Omega_1\|x-y\|.\end{aligned} \tag{55}$$

It follows that

$$\|\mathscr{A}x-\mathscr{A}y\|\leq\Omega_1\|x-y\|. \tag{56}$$

Since $\Omega_1 < 1$, $\mathscr{A}$ is a contraction mapping on B_r. Therefore (5) has a unique solution on $[0,T]$. □

Example 11. Consider the following impulsive multiorders Riemann-Liouville fractional initial value problem:

$$\begin{aligned}&D_{t_k}^{(2k^2+3k+5)/(k^2+2k+3)}x=\frac{2^t|x(t)|\cos t}{(7+|x(t)|)^2+|x(t)|\sin^2 t}\\
&\qquad+\frac{1}{2},\\
&\qquad t\in\left[0,\tan^{-1}\left(\frac{11}{3\pi}\right)\right],\ t\neq t_k,\\
&\widetilde{\Delta}x(t_k)=\frac{|x(t_k)|}{9(k^2+3k+5)+|x(t_k)|},\\
&\qquad k=1,2,\ldots,10,\ t_k=\tan^{-1}\left(\frac{k}{3\pi}\right),\\
&\Delta^*x(t_k)=\frac{|x(t_k)|}{37(1-2\cos k\pi)+|x(t_k)|},\\
&\qquad k=1,2,\ldots,10,\ t_k=\tan^{-1}\left(\frac{k}{3\pi}\right),\\
&x(0)=0\\
&D_{t_0}^{2/3}x(0)=e.\end{aligned} \tag{57}$$

Here $\alpha_k=(2k^2+3k+5)/(k^2+2k+3)$, $k=0,1,2,\ldots,10$, $m=10$, $T=\tan^{-1}(11/(3\pi))$, $f(t,x)=(2^t|x|\cos t)/((7+|x|)^2+|x|\sin^2 t)+(1/2)$, $\varphi_k(x)=|x|/(9(k^2+3k+5)+|x|)$, and $\varphi_k^*(x)=|x|/(37(1-2\cos k\pi)+|x|)$. Since $|f(t,x)-f(t,y)|\leq(2/49)|x-y|$, $|\varphi_k(x)-\varphi_k(y)|\leq(1/81)|x-y|$, and $|\varphi_k^*(x)-\varphi_k^*(y)|\leq(3/37)|x-y|$, then (H_1), (H_2), and (H_3) are satisfied with $L_1=2/49$, $L_2=1/81$, and $L_3=3/37$, respectively. By choosing

$\gamma = 2$, we can find that $T_2^* = 0.781307$, $\Gamma_2^* = 0.902745$, $\Phi = 0.332114$, and

$$\Omega_1 = \frac{T_2^*}{\Gamma_2^*}\left(L_1\left(\phi + T + 1\right) + L_2\left(2m - 1\right) + L_3 m\right) = 0.982275 < 1. \tag{58}$$

Hence, by Theorem 10, the initial value problem (57) has a unique solution on $[0, \tan^{-1} 11/(3\pi)]$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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