Research Article

Bifurcation Analysis for Nonlinear Recurrence Relations with Threshold Control and $2k$ -Periodic Coefficients

Liping Dou,¹ Chengmin Hou,¹ and Sui Sun Cheng²

¹Department of Mathematics, Yanbian University, Yanji 133002, China

²Department of Mathematics, Tsing Hua University, Hsinchu 30043, Taiwan

Correspondence should be addressed to Chengmin Hou; houchengmin@aliyun.com

Received 18 January 2015; Revised 10 March 2015; Accepted 11 March 2015

Academic Editor: Garyfalos Papashinopoulos

Copyright © 2015 Liping Dou et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

A nonlinear recurrence involving a piecewise constant McCulloch-Pitts function and $2k$ -periodic coefficient sequences is investigated. By allowing the threshold parameter to vary from 0 to ∞ , we work out a complete bifurcation analysis for the asymptotic behaviors of the corresponding solutions. Among other things, we show that each solution tends towards one of four different limits. Furthermore, the accompanying initial regions for each type of solutions can be determined. It is hoped that our analysis will provide motivation for further results for recurrent McCulloch-Pitts type neural networks.

1. Introduction

It is of great interest to see how an artificial neural network uses components which are closer to “biological” components. One particular important component can be described by means of the step (activation) function defined by

$$f_{\lambda}(u) = \begin{cases} 1, & \text{if } u \in (0, \lambda], \\ 0, & \text{if } u \in (-\infty, 0] \cup (\lambda, +\infty) \end{cases} \quad (1)$$

with a nonnegative (threshold) real parameter λ . Roughly, the activation function mimics a so called McCulloch neuron that may receive an excitatory value (indicated by 1) if the input signal has strength within limits 0 and λ , and otherwise it remains intact with an inhibitory value (indicated by 0).

If we let x_n be the state value of a neural unit during the time period n , then the recurrence relation

$$x_n = F(f_{\lambda}(x_{n-1}), x_{n-2}) \quad (2)$$

may be used to describe a one neuron McCulloch-Pitts system where the state value is updated from the two most recent state values.

Let $\mathbb{N} = \{0, 1, \dots\}$. Zhu and Huang [1] discussed the “limit cycles” of the recurrence relation

$$x_n = ax_{n-2} + bf_{\lambda}(x_{n-1}), \quad n \in \mathbb{N}, \quad (3)$$

where $a \in (0, 1)$, $b \in (0, +\infty)$, and $f_{\lambda} : \mathbb{R} \rightarrow \mathbb{R}$ is defined by (1), in which the positive threshold λ can be regarded as a bifurcation parameter. Then, Chen [2] considered the following recurrence relation:

$$x_n = a_n x_{n-2} + b_n f_{\lambda}(x_{n-1}), \quad n \in \mathbb{N}, \quad (4)$$

where $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are 2-periodic sequences with $a_0, a_1 \in (0, 1)$ and $b_0, b_1 \in (0, +\infty)$. Asymptotic behaviors of these equations reflect their differences (see [1, 2]). A good reason for studying (4) is that the constants a and b used in the physical model described by (3) may not be truly constant but exhibit fluctuating behaviors between two limits. Since in (4) we have chosen to consider the case where a and b are replaced by 2-periodic sequences, the question then arises as to what will happen if we choose general periodic sequences.

In this paper, we offer partial answers by considering the difference equation

$$x_n = a_n x_{n-2} + b_n f_{\lambda}(x_{n-1}), \quad n \in \mathbb{N}, \quad (5)$$

where $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are $2k$ -periodic sequences with $a_i \in (0, 1)$, $b_i = 1 - a_i$, and $i = 0, \dots, 2k - 1$. By studying this equation, we hope that the subsequent results will lead to much more general ones for complex systems involving similar periodic parameters and discontinuous controls.

In order to study the asymptotic behavior of (5), let us first note that it is a three-term recurrence relation so that, given x_{-2} and x_{-1} , we may calculate x_0, x_1, x_2 , and so forth in a sequential manner. The resulting sequence $\{x_n\}_{n=-2}^{\infty}$ is naturally called a solution of (5). For example, when $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are 4-periodic sequences, we may write

$$\begin{aligned} x_0 &= a_0 x_{-2} + b_0 f_{\lambda}(x_{-1}), \\ x_1 &= a_1 x_{-1} + b_1 f_{\lambda}(x_0), \\ x_2 &= a_2 x_0 + b_2 f_{\lambda}(x_1), \\ x_3 &= a_3 x_1 + b_3 f_{\lambda}(x_2), \\ x_4 &= a_4 x_2 + b_4 f_{\lambda}(x_3) = a_0 x_2 + b_0 f_{\lambda}(x_3), \\ x_5 &= a_5 x_3 + b_5 f_{\lambda}(x_4) = a_1 x_3 + b_1 f_{\lambda}(x_4), \\ &\vdots \end{aligned} \quad (6)$$

This motivates us to define a vector equation. Given a sequence $x = \{x_n\}_{n=a}^{\infty}$, its Casoratian vector sequence is $\{\langle x_i \rangle\}_{i=a}^{\infty}$, where $\langle x_i \rangle = \text{col}(x_i, x_{i+1}, \dots, x_{i+2k-1})$, $i = a, a+1, \dots$. Then (5) is equivalent to the asynchronous vector equation

$$\langle x_{2kn} \rangle = A \langle x_{2kn-2} \rangle + B f_{\lambda}(\langle x_{2kn-1} \rangle), \quad n = 0, 1, 2, \dots, \quad (7)$$

where

$$\begin{aligned} A &= \begin{pmatrix} a_0 & 0 & \dots & 0 \\ 0 & a_1 & \dots & 0 \\ & \ddots & \ddots & 0 \\ 0 & 0 & \dots & a_{2k-1} \end{pmatrix}, \\ B &= \begin{pmatrix} b_0 & 0 & \dots & 0 \\ 0 & b_1 & \dots & 0 \\ & \ddots & \ddots & 0 \\ 0 & 0 & \dots & b_{2k-1} \end{pmatrix}, \end{aligned} \quad (8)$$

$$f_{\lambda} \langle x_i \rangle = \text{col}(f_{\lambda}(x_i), f_{\lambda}(x_{i+1}), \dots, f_{\lambda}(x_{i+2k-1})).$$

Note that, given (x_{-2}, x_{-1}) , we may use (7) to generate $\langle x_0 \rangle, \langle x_{2k} \rangle, \langle x_{4k} \rangle, \dots$ which, when “lined up,” yields the same $x_0, x_1, x_2, \dots$ as described above. For this reason, the sequence $\{\langle x_i \rangle\}_{i=0}^{\infty}$ will be called the solution of (7) determined by (x_{-2}, x_{-1}) .

Therefore, to obtain complete asymptotic behaviors of (5), we need to derive the behaviors of solutions of (7) determined by vectors (x_{-2}, x_{-1}) in the entire plane.

The following result, however, can help us concentrate on solutions determined by vectors (x_{-2}, x_{-1}) in $\mathbb{R}^2 \setminus (-\infty, 0]^2$.

Theorem 1. A solution $\{\langle x_{2kn} \rangle\}_{n=0}^{\infty}$ of (7) with (x_{-2}, x_{-1}) in the nonpositive orthant $(-\infty, 0]^2$ is nonpositive and tends towards $\langle 0 \rangle$.

Proof. Let $(x_{-2}, x_{-1}) \in (-\infty, 0]^2$. Then, by (7) and by induction, for any $n \in \mathbb{N}$ and $m \in \{0, 1, \dots, 2k-1\}$, we may get $x_{2ki+m} \in (-\infty, 0]$, that is,

$$\langle x_{2kn} \rangle = A \langle x_{2k(n-1)} \rangle = \dots = A^n \langle x_0 \rangle. \quad (9)$$

Since $\langle x_0 \rangle \in (-\infty, 0]^{2k}$ and A^{2n} tends towards 0 as n tends towards $+\infty$, we see that $\langle x_{2kn} \rangle$ tends towards $\langle 0 \rangle$. The proof is complete. $\square$

Next, note that our system is autonomous (time invariant), and hence, if $\{\langle x_{2kn} \rangle\}_{n=0}^{\infty}$ is a solution of (7), then, for any $m \in \mathbb{N}$, the sequence, $\langle y_{2kn} \rangle$ defined by $\langle y_{2kn} \rangle = \langle x_{2kn+m} \rangle$ for $n = 0, 1, \dots$, is also a solution of (7).

Suppose $\{\langle x_{2kn} \rangle\}_{n=0}^{\infty}$ is a solution of (7). Then, we say that

$\{\langle x_{2kn} \rangle\}_{n=0}^{\infty}$ approaches a limit 1-cycle $\{\langle \alpha \rangle\}$ if $\lim_{i \rightarrow \infty} (x_{2ki}, x_{2ki+1}, \dots, x_{2ki+2k-1}) = (\alpha, \alpha, \dots, \alpha)$;

$\{\langle x_{2kn} \rangle\}_{n=0}^{\infty}$ approaches a limit 2-cycle $\{\langle \alpha, \beta \rangle\}$ if $\alpha \neq \beta$ and $\lim_{i \rightarrow \infty} (x_{2ki}, x_{2ki+2}, \dots, x_{2ki+2k-2}) = (\alpha, \alpha, \dots, \alpha)$ and $\lim_{i \rightarrow \infty} (x_{2ki+1}, x_{2ki+3}, \dots, x_{2ki+2k-1}) = (\beta, \beta, \dots, \beta)$.

For the sake of convenience, we also need to introduce some notations. The numbers δ and ρ are defined as

$$\delta = a_0 a_2 \dots a_{2k-2}, \quad \rho = a_1 a_3 \dots a_{2k-1}, \quad (10)$$

while the numbers

$$\begin{aligned} &A_i^{(j_0, j_1, \dots, j_m)}, B_i^{(j_0, j_1, \dots, j_m)}, C_i^{(j_0, j_1, \dots, j_m)}, \\ &D_i^{(j_0, j_1, \dots, j_m)}, E_i^{(j_0, j_1, \dots, j_m)}, G_i^{(j_0, j_1, \dots, j_m)}, \\ &L_i^{(j_0, j_1, \dots, j_m)}, M_i^{(j_0, j_1, \dots, j_m)}, H_i^{(j_0, j_1, \dots, j_m)} \end{aligned} \quad (11)$$

and their properties are listed in the Appendix. These numbers are introduced in order to break the plane into different parts such that the behavior of each solution of (7) which originates from each part may be traced.

In the sequel, we distinguish the cases (i) $\lambda > 1$, (ii) $\lambda = 1$, and (iii) $0 < \lambda < 1$, and then, for different $(x_{-2}, x_{-1}) \in \mathbb{R}^2 \setminus (-\infty, 0]^2$, we find the asymptotic behaviors of the corresponding solutions.

2. The Case $\lambda > 1$

We begin with the following.

Lemma 2. Let $\lambda > 1$. If $\{\langle x_{2kn} \rangle\}_{n=0}^{\infty}$ is a solution of (7) with $(x_{-2}, x_{-1}) \in \mathbb{R}^2 \setminus (-\infty, 0]^2$, then there exist integers $r \in \{0, 1, \dots, 2k-2\}$ and $j \in \mathbb{N}$ such that $0 < x_{2kj+r}, x_{2kj+r+1} \leq \lambda$.

Proof. From our assumption, we have $a_i \lambda + b_i < \lambda$ for $i = 0, 1, \dots, 2k-1$. Let $\{\langle x_{2kn} \rangle\}_{n=0}^{\infty}$ be a solution of (7) with $(x_{-2}, x_{-1}) \in \mathbb{R}^2 \setminus (-\infty, 0]^2$. Then, there are eight cases.

Case 1. If $(x_{-2}, x_{-1}) \in (0, \lambda]^2$, then by (7) we may easily obtain $(x_0, x_1) \in (0, \lambda]^2$, our assertion is true by taking $j = 0$ and $r = 0$.

Case 2. Suppose $(x_{-2}, x_{-1}) \in (0, \lambda] \times (\lambda, +\infty)$. Then by (7) and induction we may see that $x_{2ki+2m} \in (0, \lambda]$ for any $m \in \{1, 2, \dots, k-1\}$ and $i \in \mathbb{N}$; and hence,

$$\begin{aligned} x_{2ki+2m-1} &= a_{2m-1}x_{2ki+2m-3} + b_{2m-1} \\ &= a_1a_3 \cdots a_{2m-3}\rho^i x_{-1} + 1 \\ &\quad - a_1a_3 \cdots a_{2m-3}\rho^i, \quad m \in \{1, 2, \dots, k\}. \end{aligned} \quad (12)$$

Thus, $\lim_{i \rightarrow \infty} x_{2ki+2m-1} = 1 \in (0, \lambda]$ for any $m \in \{1, 2, \dots, k\}$. Then, there exists enough large $j \in \mathbb{N}$ such that $(x_{2kj}, x_{2kj+1}) \in (0, \lambda]^2$.

Case 3. Suppose $(x_{-2}, x_{-1}) \in (\lambda, +\infty) \times (0, \lambda]$. As in Case 2, we may show that there exists enough large $j \in \mathbb{N}$ such that $(x_{2kj}, x_{2kj+1}) \in (0, \lambda]^2$. Our assertion is thus true.

Case 4. Suppose $(x_{-2}, x_{-1}) \in (\lambda, +\infty) \times (\lambda, +\infty)$. We may first show that there is a $j \in \{0, 1, \dots\}$ such that $x_j \in (0, \lambda]$. Otherwise, for any $j \in \{0, 1, \dots\}$, we have $x_j \in (\lambda, +\infty)$. Then by (7) we have

$$\langle x_{2kn} \rangle = A \langle x_{2k(n-1)} \rangle = \cdots = A^n \langle x_0 \rangle. \quad (13)$$

Since $\langle x_0 \rangle \in (0, \infty)^{2k}$ and A^{2n} tends towards 0 as n tends towards $+\infty$, we see that $\langle x_{2kn} \rangle$ tends towards $\langle 0 \rangle$, which is a contradiction. The rest of the proof comes from the assertions in Cases 2 and 3.

Case 5. Suppose $(x_{-2}, x_{-1}) \in (-\infty, 0] \times (0, \lambda]$. Then by (7) and induction we may see that $x_{2ki+2m-1} \in (0, \lambda]$ for any $m \in \{1, 2, \dots, k\}$ and $i \in \mathbb{N}$; and hence,

$$\begin{aligned} x_{2ki+2m-2} &= a_{2m-2}x_{2ki+2m-4} + b_{2m-2}f_\lambda(x_{2ki-1}) \\ &= a_0a_2 \cdots a_{2m-2}\delta^i x_{-2} + 1 \\ &\quad - a_0a_2 \cdots a_{2m-2}\delta^i, \quad m \in \{1, 2, \dots\}. \end{aligned} \quad (14)$$

Thus, $\lim_{i \rightarrow \infty} x_{2ki+2m-2} = 1 \in (0, \lambda]$ for any $m \in \{1, 2, \dots, k\}$. Then, there exists enough large $j \in \mathbb{N}$ such that $(x_{2kj}, x_{2kj+1}) \in (0, \lambda]^2$.

Case 6. Suppose $(x_{-2}, x_{-1}) \in (0, \lambda] \times (-\infty, 0]$. As in Case 5, we may show that there exists enough large $j \in \mathbb{N}$ such that $(x_{2kj}, x_{2kj+1}) \in (0, \lambda]^2$. Our assertion is thus true.

Case 7. Suppose $(x_{-2}, x_{-1}) \in (\lambda, +\infty) \times (-\infty, 0]$. In view of (A.3), there exist $m \in \{0, 1, \dots, k-1\}$ and $i \in \mathbb{N}$ such that $(x_{-2}, x_{-1}) \in (D_i^{(0,2,\dots,2m-2)}, D_i^{(0,2,\dots,2m)}) \times (-\infty, 0]$. Next, we consider two cases.

(I) Consider $i = 0, m \geq 0$. Then by (7) we have

$$\begin{aligned} x_0 &= a_0x_{-2} + b_0f_\lambda(x_{-1}) \\ &= a_0x_{-2} \in (D_0^{(2,\dots,2m-2)}, D_0^{(2,\dots,2m)}], \\ x_1 &= a_1x_{-1} + b_1f_\lambda(x_0) = a_1x_{-1} \in (-\infty, 0], \\ &\vdots \\ x_{2m-1} &= a_{2m-1}x_{2m-2} + b_{2m-1}f_\lambda(x_{2m-1}) \\ &= a_1a_3 \cdots a_{2m-1}x_{-1} \in (-\infty, 0], \\ x_{2m} &= a_{2m}x_{2m-2} + b_{2m}f_\lambda(x_{2m-1}) \\ &= a_0a_2 \cdots x_{-2} \in (a_{2m}D_0^{(-2)}, D_0^{(-2)}) \subset (0, \lambda], \end{aligned} \quad (15)$$

that is, $(x_{2m-1}, x_{2m}) \in (-\infty, 0] \times (0, \lambda]$. Our assertion is true by Case 6.

(II) Consider $i > 0, m \geq 0$. Then by (7) we have

$$\begin{aligned} x_0 &= a_0x_{-2} + b_0f_\lambda(x_{-1}) \\ &= a_0x_{-2} \in (D_i^{(2,\dots,2m-2)}, D_i^{(2,\dots,2m)}], \\ x_1 &= ax_{-1} + b_1f_\lambda(x_0) = a_1x_{-1} \in (-\infty, 0], \\ &\vdots \\ x_{2k-2} &= a_{2k-2}x_{2k-4} + b_{2k-2}f_\lambda(x_{2k-3}) \\ &= \delta x_{-2} \in (D_{i-1}^{(0,2,\dots,2m-2)}, D_{i-1}^{(0,2,\dots,2m)}], \\ x_{2k-1} &= a_{2k-1}x_{2k-3} + b_{2k-1}f_\lambda(x_{2k-2}) \\ &= \rho x_{-1} \in (-\infty, 0] \end{aligned} \quad (16)$$

and, by induction, we have

$$\begin{aligned} x_{2ki} &= a_0x_{2ki-2} + b_0f_\lambda(x_{2ki-1}) \\ &= a_0\delta^i x_{-2} \in (D_0^{(2,\dots,2m-2)}, D_0^{(2,\dots,2m)}], \\ x_{2ki+1} &= a_1x_{2ki-1} + b_1f_\lambda(x_{2ki}) \\ &= a_1\rho^i x_{-1} \in (-\infty, 0], \\ &\vdots \\ x_{2ki+2m-1} &= a_{2m-1}x_{2ki+2m-3} + b_{2m-1}f_\lambda(x_{2ki+2m-2}) \\ &= a_1a_3 \cdots a_{2m-1}\rho^i x_{-1} \in (-\infty, 0], \\ x_{2ki+2m} &= a_{2m}x_{2ki+2m-2} + b_{2m}f_\lambda(x_{2ki+2m-1}) \\ &= a_0a_2 \cdots a_{2m}\delta^i x_{-2} \in (a_{2m}D_0^{(-2)}, D_0^{(-2)}) \subset (0, \lambda], \end{aligned} \quad (17)$$

that is, $(x_{2ki+2m-1}, x_{2ki+2m}) \in (-\infty, 0] \times (0, \lambda]$. Our assertion is true by means of Case 6.

Case 8. Suppose $(x_{-2}, x_{-1}) \in (-\infty, 0] \times (\lambda, +\infty)$. Then, there exist $m \in \{0, 1, \dots, k-1\}$ and $i \in \mathbb{N}$ such that $(x_{-2}, x_{-1}) \in (-\infty, 0] \times (E_i^{(1,3,\dots,2m-1)}, E_i^{(1,3,\dots,2m+1)})$. As in Case 7, we may show that $(x_{2ki+2m}, x_{2ki+2m+1}) \in (-\infty, 0] \times (0, \lambda]$. Our assertion is true by means of Case 5. $\square$

Theorem 3. Let $\lambda > 1$. If $\{\langle x_{2kn} \rangle\}_{n=0}^\infty$ is a solution of (7) with $(x_{-2}, x_{-1}) \in \mathbb{R}^2 \setminus (-\infty, 0]^2$, then it approaches the limit 1-cycle $\langle 1 \rangle$.

Proof. By Lemma 2, we may assume without loss of generality that $(x_{-2}, x_{-1}) \in (0, \lambda]^2$. Then, by (7) and induction, $x_{2ki+j} = a_j x_{2ki+j-2} + b_j \in (0, \lambda]$ for $i \in \mathbb{N}$ and $j \in \{0, 1, \dots, 2k-1\}$; that is,

$$\begin{aligned} \langle x_{2kn} \rangle &= A \langle x_{2k(n-1)} \rangle + B = \dots \\ &= A^n \langle x_0 \rangle + A^{n-1}B + A^{n-2}B + \dots + B \quad (18) \\ &= A^n \langle x_0 \rangle + E - A^i. \end{aligned}$$

Since $\langle x_0 \rangle \in (-\infty, 0]^{2k}$ and A^{2n} tends towards 0 as n tends towards $+\infty$, we see that $\{\langle x_{2kn} \rangle\}$ tends towards the limit 1-cycle $\langle 1 \rangle$. The proof is complete. $\square$

3. The Case $\lambda = 1$

In this section, we assume $\lambda = 1$. Set

$$\begin{aligned} \Psi &= (0, \lambda]^2 \cup (-\infty, 0] \times (0, \lambda] \cup (0, \lambda] \times (-\infty, 0] \\ &\quad \cup (-\infty, 0] \times (\lambda, \infty) \cup (\lambda, \infty) \times (-\infty, 0], \\ \Omega &= (0, \lambda] \times (\lambda, \infty) \cup (\lambda, \infty) \times (0, \lambda] \cup (\lambda, \infty) \times (\lambda, \infty). \end{aligned} \quad (19)$$

Note that a simple plot of the regions Ψ and Ω in the plane shows that they are disjoint and form the complement of the region $(-\infty, \lambda]^2$.

Lemma 4. Let $\lambda = 1$. If $\{\langle x_{2kn} \rangle\}_{n=0}^\infty$ is a solution of (7) with $(x_{-2}, x_{-1}) \in \Psi$, then there exist integer $r \in \{0, 1, \dots, 2k-2\}$ and $j \in \mathbb{N}$ such that $0 < x_{2kj+r}, x_{2kj+r+1} \leq \lambda$.

The proof is similar to those discussed in Cases 5 through 8 in the proof of Lemma 2 and, hence, is skipped.

Theorem 5. Let $\lambda = 1$. If $\{\langle x_{2kn} \rangle\}_{n=0}^\infty$ is a solution of (7) with $(x_{-2}, x_{-1}) \in \Psi$, then it approaches the limit 1-cycle $\langle 1 \rangle$.

Proof. By Lemma 4, we may assume without loss of generality that $(x_{-2}, x_{-1}) \in (0, \lambda]^2$. Then, similar to the proof of Theorem 3, we have $x_{2ki+j} = a_j x_{2ki+j-2} + b_j \in (0, \lambda]$ for $i \in \mathbb{N}$ and $j \in \{0, 1, \dots, 2k-1\}$; that is,

$$\begin{aligned} \langle x_{2kn} \rangle &= A \langle x_{2k(n-1)} \rangle + B = \dots \\ &= A^n \langle x_0 \rangle + A^{n-1}B + A^{n-2}B + \dots + B \quad (20) \\ &= A^n \langle x_0 \rangle + E - A^i. \end{aligned}$$

Since $\langle x_0 \rangle \in (-\infty, 0]^{2k}$ and A^{2n} tends towards 0 as n tends towards $+\infty$, we see that $\{\langle x_{2kn} \rangle\}$ tends towards the limit 1-cycle $\langle 1 \rangle$. The proof is complete. $\square$

Theorem 6. A solution $\{\langle x_{2kn} \rangle\}_{n=0}^\infty$ of (7) with $(x_{-2}, x_{-1}) \in (0, \lambda] \times (\lambda, +\infty)$ tends towards the limit 2-cycle $\langle 0, 1 \rangle$.

Proof. By (7), we have

$$\begin{aligned} x_0 &= a_0 x_{-2} + b_0 f_\lambda(x_{-1}) = a_0 x_{-2} \in (0, \lambda], \\ x_1 &= a_1 x_{-1} + b_1 f_\lambda(x_0) = a_1 x_{-1} + b_1 > \lambda, \\ &\vdots \\ x_{2k-2} &= a_{2k-2} x_{2k-4} + b_{2k-2} f_\lambda(x_{2k-3}) \quad (21) \\ &= \delta x_{-2} \in (0, \lambda], \\ x_{2k-1} &= a_{2k-1} x_{2k-3} + b_{2k-1} f_\lambda(x_{2k-2}) \\ &= \rho x_{-1} + 1 - \rho > \lambda, \end{aligned}$$

and by induction,

$$\begin{aligned} x_{2kn} &= a_0 x_{2kn-2} + b_0 f_\lambda(x_{2kn-1}) \\ &= a_0 \delta^n x_{-2} \in (0, \lambda], \\ x_{2kn+1} &= a_1 x_{2kn-1} + b_1 f_\lambda(x_{2kn}) \\ &= a_1 \rho^n x_{-1} + 1 - a_1 \rho^n > \lambda, \quad (22) \\ &\vdots \\ x_{2kn+2k-1} &= a_{2k-1} x_{2kn+2k-3} + b_{2k-1} f_\lambda(x_{2kn+2k-2}) \\ &= \rho^{n+1} x_{-1} + 1 - \rho^{n+1} > \lambda. \end{aligned}$$

Therefore, $\{\langle x_{2kn} \rangle\}$ tends towards the limit 2-cycle $\langle 0, 1 \rangle$. $\square$

Theorem 7. A solution $\{\langle x_{2kn} \rangle\}_{n=0}^\infty$ of (7) with $(x_{-2}, x_{-1}) \in (\lambda, +\infty) \times (0, \lambda]$ tends towards the limit 2-cycle $\langle 1, 0 \rangle$.

The proof is similar to Theorem 6 and is skipped.

Theorem 8. Suppose that $\{\langle x_{2kn} \rangle\}_{n=0}^\infty$ is a solution of (7) with $(x_{-2}, x_{-1}) \in (D_i^{(0,2,\dots,2m-2)}, D_i^{(0,2,\dots,2m)}) \times (E_j^{(1,3,\dots,2l-1)}, E_j^{(1,3,\dots,2l+1)})$, where $i, j \in \{0, 1, \dots\}$ and $m, l \in \{0, 1, 2, \dots, k-1\}$. Then,

- (i) $\lim_{n \rightarrow \infty} \langle x_{2kn} \rangle = \langle 0, 1 \rangle$ for $i = j, m \leq l$;
- (ii) $\lim_{n \rightarrow \infty} \langle x_{2kn} \rangle = \langle 1, 0 \rangle$ for $i = j, m > l$;
- (iii) $\lim_{n \rightarrow \infty} \langle x_{2kn} \rangle = \langle 1, 0 \rangle$ for $i < j$;
- (iv) $\lim_{n \rightarrow \infty} \langle x_{2kn} \rangle = \langle 0, 1 \rangle$ for $i > j$.

Proof. Suppose that $(x_{-2}, x_{-1}) \in (D_i^{(0,2,\dots,2m-2)}, D_i^{(0,2,\dots,2m)}) \times (E_j^{(1,3,\dots,2l-1)}, E_j^{(1,3,\dots,2l+1)})$.

(i) We distinguish two different cases.

Case 1. Consider $0 = i = j$, $0 \leq m \leq l$. Then,

$$\begin{aligned}
 x_0 &= a_0 x_{-2} + b_0 f_\lambda(x_{-1}) \\
 &= a_0 x_{-2} \in (D_0^{(2, \dots, 2m-2)}, D_0^{(2, \dots, 2m)}], \\
 x_1 &= a_1 x_{-1} + b_1 f_\lambda(x_0) \\
 &= a_1 x_{-1} \in (E_0^{(3, \dots, 2l-1)}, E_0^{(3, \dots, 2l+1)}], \\
 &\vdots \\
 x_{2m} &= a_{2m} x_{2m-2} + b_{2m} f_\lambda(x_{2m-1}) \\
 &= a_1 a_3 \cdots a_{2m} \in (a_{2m} D_0^{(-2)}, D_0^{(-2)}) \subset (0, \lambda], \\
 x_{2m+1} &= a_{2m+1} x_{2m-2} + b_{2m+1} f_\lambda(x_{2m-1}) \\
 &> a_{2m+1} \lambda + b_{2m+1} > \lambda.
 \end{aligned} \tag{23}$$

Case 2. Consider $0 < i = j$, $0 \leq m \leq l$. Then,

$$\begin{aligned}
 x_0 &= a_0 x_{-2} + b_0 f_\lambda(x_{-1}) \\
 &= a_0 x_{-2} \in (D_i^{(2, \dots, 2m-2)}, D_i^{(2, \dots, 2m)}], \\
 x_1 &= a_1 x_{-1} + b_1 f_\lambda(x_0) \\
 &= a_1 x_{-1} \in (E_j^{(3, \dots, 2l-1)}, E_j^{(3, \dots, 2l+1)}], \\
 &\vdots \\
 x_{2k-2} &= a_{2k-2} x_{2k-4} + b_{2k-2} f_\lambda(x_{2k-3}) \\
 &= a_0 a_2 \cdots a_{2k-2} x_{-2} \in (D_{i-1}^{(0, 2, \dots, 2m-2)}, D_{i-1}^{(0, 2, \dots, 2m)}], \\
 x_{2k-1} &= a_{2k-1} x_{2k-3} + b_{2k-1} f_\lambda(x_{2k-2}) \\
 &= \rho x_{-1} \in (E_{j-1}^{(1, 3, \dots, 2l-1)}, E_{j-1}^{(1, 3, \dots, 2l+1)}],
 \end{aligned} \tag{24}$$

and by induction, we have

$$\begin{aligned}
 x_{2ki} &= a_0 x_{2ki-2} + b_0 f_\lambda(x_{2ki-1}) \\
 &= a_0 x_{2ki-2} \in (D_0^{(2, \dots, 2m-2)}, D_0^{(2, \dots, 2m)}], \\
 x_{2ki+1} &= a_1 x_{2ki-1} + b_1 f_\lambda(x_{2ki}) \\
 &= a_1 x_{2ki-1} \in (E_0^{(3, \dots, 2l-1)}, E_0^{(3, \dots, 2l+1)}], \\
 &\vdots \\
 x_{2ki+2m-1} &= a_{2m-1} x_{2ki+2m-3} + b_{2m-1} f_\lambda(x_{2ki+2m-2}) \\
 &= a_1 a_3 \cdots a_{2m-1} \rho^i x_{-1} \\
 &\in (E_{j-i}^{(2m+1, \dots, 2l+1)}, E_{j-i}^{(2m+1, \dots, 2l+1)}], \\
 x_{2ki+2m} &= a_{2m} x_{2ki+2m-2} + b_{2m} f_\lambda(x_{2ki+2m-1}) \\
 &= a_1 a_3 \cdots a_{2m} \delta^i \in (a_{2m} D_0^{(-2)}, D_0^{(-2)}) \subset (0, \lambda].
 \end{aligned} \tag{25}$$

(ii) Similar to (i), by distinguishing two different cases and by induction, we have the following.

Case 1. Consider $(x_{2l}, x_{2l+1}) \in (\lambda, +\infty) \times (0, \lambda]$, $0 = i = j$, $0 \leq l \leq m$.

Case 2. Consider $(x_{2ki+2l+1}, x_{2ki+2l+2}) \in (0, \lambda] \times (\lambda, +\infty)$, $0 < i = j$, $0 \leq l < m$.

(iii) We distinguish four different cases.

Case 1. Consider $j > i = 0$, $m \geq l \geq 0$. By (7),

$$\begin{aligned}
 x_0 &= a_0 x_{-2} + b_0 f_\lambda(x_{-1}) \\
 &= a_0 x_{-2} \in (D_0^{(2, \dots, 2m-2)}, D_0^{(2, \dots, 2m)}], \\
 x_1 &= a_1 x_{-1} + b_1 f_\lambda(x_0) \\
 &= a_1 x_{-1} \in (E_j^{(3, \dots, 2l-1)}, E_j^{(3, \dots, 2l+1)}], \\
 &\vdots
 \end{aligned} \tag{26}$$

$$\begin{aligned}
 x_{2m} &= a_{2m} x_{2m-2} + b_{2m} f_\lambda(x_{2m-1}) \\
 &= a_{2m} x_{2m-2} \in (a_{2m} D_0^{(-2)}, D_0^{(-2)}) \subset (0, \lambda],
 \end{aligned}$$

$$\begin{aligned}
 x_{2m+1} &= a_{2m+1} x_{2m-1} + b_{2m+1} f_\lambda(x_{2m}) \\
 &= a_{2m+1} x_{2m-1} + b_{2m+1} > \lambda.
 \end{aligned}$$

Case 2. Consider $j > i = 0$, $l \geq m \geq 0$. Then, by (7),

$$\begin{aligned}
 x_0 &= a_0 x_{-2} + b_0 f_\lambda(x_{-1}) \\
 &= a_0 x_{-2} \in (D_0^{(2, \dots, 2m-2)}, D_0^{(2, \dots, 2m)}], \\
 x_1 &= a_1 x_{-1} + b_1 f_\lambda(x_0) \\
 &= a_1 x_{-1} \in (E_j^{(3, \dots, 2l-1)}, E_j^{(3, \dots, 2l+1)}], \\
 &\vdots
 \end{aligned} \tag{27}$$

$$\begin{aligned}
 x_{2m} &= a_{2m} x_{2m-2} + b_{2m} f_\lambda(x_{2m-1}) \\
 &= a_{2m} x_{2m-2} \in (a_{2m} D_0^{(-2)}, D_0^{(-2)}) \subset (0, \lambda],
 \end{aligned}$$

$$\begin{aligned}
 x_{2m+1} &= a_{2m+1} x_{2m-1} + b_{2m+1} f_\lambda(x_{2m}) \\
 &= a_{2m+1} x_{2m-1} + b_{2m+1} > \lambda.
 \end{aligned}$$

Case 3. Consider $j > i > 0$, $m \geq l \geq 0$. Then,

$$\begin{aligned}
 x_0 &= a_0 x_{-2} + b_0 f_\lambda(x_{-1}) \\
 &= a_0 x_{-2} \in (D_i^{(2, \dots, 2m-2)}, D_i^{(2, \dots, 2m)}], \\
 x_1 &= a_1 x_{-1} + b_1 f_\lambda(x_0) \\
 &= a_1 x_{-1} \in (E_j^{(3, \dots, 2l-1)}, E_j^{(3, \dots, 2l+1)}], \\
 &\vdots
 \end{aligned} \tag{28}$$

$$\begin{aligned}
 x_{2k-2} &= a_{2k-2} x_{2k-4} + b_{2k-2} f_\lambda(x_{2k-3}) \\
 &= a_0 a_2 \cdots a_{2k-2} x_{-2} \in (D_{i-1}^{(0, 2, \dots, 2m-2)}, D_{i-1}^{(0, 2, \dots, 2m)}], \\
 x_{2k-1} &= a_{2k-1} x_{2k-3} + b_{2k-1} f_\lambda(x_{2k-2}) \\
 &= a_1 a_3 \cdots a_{2k-1} x_{-1} \in (E_{j-1}^{(1, 3, \dots, 2l-1)}, E_{j-1}^{(1, 3, \dots, 2l+1)}],
 \end{aligned}$$

and, by induction, we have

$$\begin{aligned}
 x_{2ki} &= a_0 x_{2ki-2} + b_0 f_\lambda(x_{2ki-1}) \\
 &= a_0 x_{2ki-2} \in (D_0^{(2, \dots, 2m-2)}, D_0^{(2, \dots, 2m)}], \\
 x_{2ki+1} &= a_1 x_{2ki-1} + b_1 f_\lambda(x_{2ki}) \\
 &= a_1 x_{2ki-1} \in (E_{j-i}^{(3, \dots, 2l-1)}, E_{j-i}^{(3, \dots, 2l+1)}], \\
 &\vdots \\
 x_{2ki+2m-1} &= a_{2m-1} x_{2ki+2m-3} \\
 &\quad + b_{2m-1} f_\lambda(x_{2ki+2m-2}) \\
 &= a_1 a_3 \cdots a_{2m-1} \rho^i x_{-1} \\
 &\quad \in (E_{j-i}^{(1, 3, \dots, 2l-1, 2m+1, \dots, 2k-1)}, E_{j-i}^{(1, 3, \dots, 2l+1, 2m+1, \dots, 2k-1)}], \\
 x_{2ki+2m} &= a_{2m} x_{2ki+2m-2} + b_{2m} f_\lambda(x_{2ki+2m-1}) \\
 &= a_0 a_2 \cdots a_{2m} \delta^i x_{-2} \in (a_{2m} D_0^{(-2)}, D_0^{(-2)}) \subset (0, \lambda].
 \end{aligned} \tag{29}$$

Case 4. Consider $j > i > 0, l \geq m \geq 0$. Then,

$$\begin{aligned}
 x_0 &= a_0 x_{-2} + b_0 f_\lambda(x_{-1}) \\
 &= a_0 x_{-2} \in (D_i^{(2, \dots, 2m-2)}, D_i^{(2, \dots, 2m)}], \\
 x_1 &= a_1 x_{-1} + b_1 f_\lambda(x_0) \\
 &= a_1 x_{-1} \in (E_j^{(3, \dots, 2l-1)}, E_j^{(3, \dots, 2l+1)}], \\
 &\vdots \\
 x_{2k-2} &= a_{2k-2} x_{2k-4} + b_{2k-2} f_\lambda(x_{2k-3}) \\
 &= a_0 a_2 \cdots a_{2k-2} x_{-2} \in (D_{i-1}^{(0, 2, \dots, 2m-2)}, D_{i-1}^{(0, 2, \dots, 2m)}], \\
 x_{2k-1} &= a_{2k-1} x_{2k-3} + b_{2k-1} f_\lambda(x_{2k-2}) \\
 &= a_1 a_3 \cdots a_{2k-1} x_{-1} \in (E_{j-1}^{(1, 3, \dots, 2l-1)}, E_{j-1}^{(1, 3, \dots, 2l+1)}]
 \end{aligned} \tag{30}$$

and, by induction, we have

$$\begin{aligned}
 x_{2ki} &= a_0 x_{2ki-2} + b_0 f_\lambda(x_{2ki-1}) \\
 &= a_0 x_{2ki-2} \in (D_0^{(2, \dots, 2m-2)}, D_0^{(2, \dots, 2m)}], \\
 x_{2ki+1} &= a_1 x_{2ki-1} + b_1 f_\lambda(x_{2ki}) \\
 &= a_1 x_{2ki-1} \in (E_{j-i}^{(3, \dots, 2l-1)}, E_{j-i}^{(3, \dots, 2l+1)}], \\
 &\vdots \\
 x_{2ki+2m-1} &= a_{2m-1} x_{2ki+2m-3} + b_{2m-1} f_\lambda(x_{2ki+2m-2}) \\
 &= a_1 a_3 \cdots a_{2m-1} \rho^i x_{-1} \\
 &\quad \in (E_{j-i}^{(2m+1, \dots, 2l-1)}, E_{j-i}^{(2m+1, \dots, 2l+1)}], \\
 x_{2ki+2m} &= a_{2m} x_{2ki+2m-2} + b_{2m} f_\lambda(x_{2ki+2m-1}) \\
 &= a_1 a_3 \cdots a_{2m} \delta^i \in (a_{2m} D_0^{(-2)}, D_0^{(-2)}) \subset (0, \lambda].
 \end{aligned} \tag{31}$$

(iv) Similar to (iii), by distinguishing four different cases and by induction, we have the following.

Case 1. Consider $(x_{2l}, x_{2l+1}) \in (\lambda, +\infty) \times (0, \lambda], i > j = 0, l \geq m \geq 0$.

Case 2. Consider $(x_{2l+1}, x_{2l+2}) \in (0, \lambda] \times (\lambda, +\infty), i > j = 0, m > l \geq 0$.

Case 3. Consider $(x_{2kj+2l}, x_{2kj+2l+1}) \in (\lambda, +\infty) \times (0, \lambda], i > j > 0, m \geq l \geq 0$.

Case 4. Consider $(x_{2kj+2l+1}, x_{2kj+2l+2}) \in (0, \lambda] \times (\lambda, +\infty), i > j > 0, m \geq l \geq 0$.

Note that, in view of (A.3) and (A.4), the above result handles all solutions of (7) originated from $(\lambda, +\infty)^2$. Together with Theorems 6 and 7, we have taken care of all the cases where $(x_{-2}, x_{-1}) \in \Omega$. Since $\Psi \cup \Omega \cup (-\infty, 0]^2$ is the complement of the $\mathbb{R}^2$, Theorems 1 and 5–7 can be used to take care of all possible solutions of (7). $\square$

4. The Case $0 < \lambda < 1$

In this section, we assume that $0 < \lambda < 1$ and we set $(p, q] = \emptyset$ if $p \geq q$. We continue to use those notations in the case where $\lambda = 1$.

Lemma 9. Let $0 < \lambda < 1$. If $\{x_{2kn}\}_{n=0}^\infty$ is a solution of (7) with $(x_{-2}, x_{-1}) \in \mathbb{R}^2 \setminus (-\infty, 0]^2$, then there exist integers $i \in \mathbb{N}$ and $m \in \{0, 1, \dots, 2k-2\}$ such that $(x_{2ki+m}, x_{2ki+m+1}) \in (0, \lambda] \times (\lambda, +\infty) \cup (\lambda, +\infty) \times (0, \lambda]$.

Proof. There are seven cases.

(i) If $(x_{-2}, x_{-1}) \in (0, \lambda] \times (\lambda, +\infty)$ or $(x_{-2}, x_{-1}) \in (\lambda, +\infty) \times (0, \lambda]$, then, by (7), we may easily see that $(x_0, x_1) \in (0, \lambda] \times (\lambda, +\infty)$ or $(x_0, x_1) \in (\lambda, +\infty) \times (0, \lambda]$.

(ii) Suppose $(x_{-2}, x_{-1}) \in (0, \lambda]^2$. If $x_{2ki+m} \in (0, \lambda]$ for all $i \in \mathbb{N}, m \in \{0, 1, \dots, 2k-1\}$, then $x_{2ki+m} = a_m x_{2ki+m-2} + b_m$ for all $i \in \mathbb{N}, m \in \{0, 1, \dots, 2k-1\}$. Thus, $\lim_{i \rightarrow \infty} x_{2ki+m} = 1 > \lambda$, which is a contradiction. Therefore, there exists $j \in \mathbb{N}$ such that

$$\begin{aligned}
 x_{2kj} &\in (\lambda, +\infty), x_0, x_1, \dots, x_{2k-1}, x_{2k}, x_{2k+1}, \dots, \\
 x_{4k-1}, \dots, x_{2kj-2}, x_{2kj-1} &\in (0, \lambda].
 \end{aligned} \tag{32}$$

(iii) Suppose $(x_{-2}, x_{-1}) \in (\lambda, +\infty) \times (\lambda, +\infty)$. If $x_{2ki+m} \in (\lambda, +\infty)$ for all $i \in \mathbb{N}, m \in \{0, 1, \dots, 2k-1\}$, then $x_{2ki+m} = a_m x_{2ki+m-2}$ for all $i \in \mathbb{N}, m \in \{0, 1, \dots, 2k-1\}$. Thus $\lim_{j \rightarrow \infty} x_{2kj+m} = 0$, which is a contradiction. Therefore, there exists $j \in \mathbb{N}$ such that

$$\begin{aligned}
 x_{2kj} &\in (0, \lambda], x_0, x_1, \dots, x_{2k-1}, x_{2k}, x_{2k+1}, \dots, \\
 x_{4k-1}, \dots, x_{2kj-2}, x_{2kj-1} &> \lambda.
 \end{aligned} \tag{33}$$

(iv) If $(x_{-2}, x_{-1}) \in (-\infty, 0] \times (0, \lambda]$, then, in view of (A.15), there exist $i \in \mathbb{N}, r \in \{0, 1, \dots, k-1\}$ such that $x_{-2} \in (G_i^{(0, 2, \dots, 2r)}, L_i^{(0, 2, \dots, 2r)}) \cup (L_i^{(0, 2, \dots, 2r)}, G_i^{(0, 2, \dots, 2r-2)})$. Furthermore, if $x_{-2} \in (G_i^{(0, 2, \dots, 2r)}, L_i^{(0, 2, \dots, 2r)})$, we distinguish two cases.

Case 1. Consider $i = 0, r \geq 0$. Then,

$$\begin{aligned}
 x_0 &= a_0 x_{-2} + b_0 f_\lambda(x_{-1}) \\
 &= a_0 x_{-2} + b_0 \in (G_0^{(2, \dots, 2r)}, L_0^{(2, \dots, 2r)}], \\
 x_1 &= a_1 x_{-1} \in (0, \lambda], \\
 &\vdots \\
 x_{2r} &= a_{2r} x_{2r-2} + b_{2r} \in (G_0^{(-2)}, L_0^{(-2)}) = (0, \lambda], \\
 x_{2r+1} &= a_1 \cdots a_{2r+1} x_{2r-1} + b_{2r+1} \in (0, +\infty).
 \end{aligned} \tag{34}$$

Case 2. Consider $i > 0, r \geq 0$. Then,

$$\begin{aligned}
 x_0 &= a_0 x_{-2} + b_0 f_\lambda(x_{-1}) \\
 &= a_0 x_{-2} + b_0 \in (G_i^{(2, \dots, 2r)}, L_i^{(2, \dots, 2r)}], \\
 x_1 &= a_1 x_{-1} \in (0, \lambda], \\
 &\vdots \\
 x_{2ki+2r} &= a_{2r} x_{2ki+2r-2} + b_{2r} \in (G_0^{(-2)}, L_0^{(-2)}) = (0, \lambda], \\
 x_{2ki+2r+1} &= a_{2r+1} x_{2ki+2r-1} + b_{2r+1} \in (0, +\infty).
 \end{aligned} \tag{35}$$

Next, suppose $x_{-2} \in (L_i^{(0, 2, \dots, 2r)}, G_i^{(0, 2, \dots, 2r-2)}) \neq \emptyset$.

Case 1. Consider $i = 0, r \geq 0$. From $L_0^{(0, 2, \dots, 2r)} \leq G_0^{(0, 2, \dots, 2r-2)}$, we may get $0 < \lambda < 1 - a_{2r} = b_{2r}$. Then,

$$\begin{aligned}
 x_0 &= a_0 x_{-2} + b_0 f_\lambda(x_{-1}) \\
 &= a_0 x_{-2} + b_0 \in (L_0^{(2, \dots, 2r)}, G_0^{(2, \dots, 2r-2)}], \\
 x_1 &= a_1 x_{-1} \in (0, \lambda], \\
 &\vdots \\
 x_{2r} &= a_{2r} x_{2r-2} + b_{2r} \in (L_0^{(-2)}, a_{2r} G_0^{(-2)} + b_{2r}) = (\lambda, b_{2r}], \\
 x_{2r+1} &= a_{2r+1} x_{2r-1} \in (0, \lambda].
 \end{aligned} \tag{36}$$

Case 2. Consider $i > 0, r \geq 0$. From $L_i^{(0, 2, \dots, 2r)} \leq G_i^{(0, 2, \dots, 2r-2)}$, we may get $0 < \lambda < 1 - a_{2r} = b_{2r}$. Then,

$$\begin{aligned}
 x_0 &= a_0 x_{-2} + b_0 f_\lambda(x_{-1}) \\
 &= a_0 x_{-2} + b_0 \in (L_i^{(2, \dots, 2r)}, G_i^{(2, \dots, 2r-2)}], \\
 x_1 &= a_1 x_{-1} \in (0, \lambda], \\
 &\vdots \\
 x_{2ki+2r} &= a_{2r} x_{2ki+2r-2} + b_{2r} \\
 &\in (L_0^{(-2)}, a_{2r} G_0^{(-2)} + b_{2r}) = (\lambda, b_{2r}], \\
 x_{2ki+2r+1} &= a_{2r+1} x_{2ki+2r-1} \in (0, \lambda].
 \end{aligned} \tag{37}$$

Theorem 10. Let $0 < \lambda < 1$. Then, any solution $\{\langle x_{2kn} \rangle\}_{n=0}^\infty$ of (7) tends asymptotically to the 2-cycle $\langle 0, 1 \rangle$ or $\langle 1, 0 \rangle$.

Proof. In view of Lemma 9, we may assume without loss of generality that $0 < x_{-2} < \lambda, x_{-1} > \lambda$. Then, by (7),

$$\begin{aligned}
 x_0 &= a_0 x_{-2} + b_0 f_\lambda(x_{-1}) = a_0 x_{-2} \in (0, a_0 \lambda], \\
 x_1 &= a_1 x_{-1} + b_1 f_\lambda(x_0) = a_1 x_{-1} + b_1 > \lambda, \\
 &\vdots
 \end{aligned}$$

$$\begin{aligned}
 x_{2k-2} &= a_{2k-2} x_{2k-4} + b_{2k-2} f_\lambda(x_{2k-3}) \\
 &= \delta x_{-2} + 1 - \delta \in (0, \delta \lambda],
 \end{aligned}$$

$$\begin{aligned}
 x_{2k-1} &= a_{2k-1} x_{2k-3} + b_{2k-1} f_\lambda(x_{2k-2}) \\
 &= \delta x_{-1} + 1 - \delta > \lambda,
 \end{aligned}$$

$$x_{2k} = a_0 x_{2k-2} + b_0 f_\lambda(x_{2k-1}) = a_0 \delta x_{-2} \in (0, a_0 \delta \lambda],$$

$$x_{2k+1} = a_1 x_{2k-1} + b_1 f_\lambda(x_{2k}) = a_1 \rho x_{-1} + 1 - a_1 \rho > \lambda,$$

$\vdots$

$$\begin{aligned}
 x_{4k-2} &= a_{2k-2} x_{4k-4} + b_{2k-2} f_\lambda(x_{4k-3}) \\
 &= \delta^2 x_{-2} \in (0, \delta^2 \lambda],
 \end{aligned}$$

$$\begin{aligned}
 x_{4k-1} &= a_{2k-1} x_{4k-3} + b_{2k-1} f_\lambda(x_{4k-2}) \\
 &= \rho^2 x_{-1} + 1 - \rho^2 > \lambda,
 \end{aligned} \tag{38}$$

and by induction, we have $(x_{2kn+2m}, x_{2kn+2m+1}) \in (0, \lambda] \times (\lambda, +\infty)$ for $n \in \mathbb{N}, m \in \{0, 1, \dots, k-1\}$. Therefore, $\{\langle x_{2kn} \rangle\}_{n=0}^\infty$ tends towards the limit 2-cycle $\langle 0, 1 \rangle$. The proof is complete. $\square$

By reviewing Theorems 1, 3, 5–8 and 10 carefully, we may see that all solutions of (7) tend towards the limit 1-cycles $\langle 0 \rangle$ or $\langle 1 \rangle$ or towards the limit 2-cycles $\langle 1, 0 \rangle$ or $\langle 0, 1 \rangle$. In Theorems 1, 3, 5–8, we are precise about the limit cycle for each solution from $\mathbb{R}^2 \setminus (-\infty, 0]^2$, but not in Theorem 10. We can also be precise in the last case. There are more technical details, however. Hence, we refer the readers who are interested in the precise “initial vector-limit vector” relations to the Appendix.

5. Discussions

The results in the previous sections are stated in terms of the $2k$ -dimensional asynchronous dynamical system (7). Note that (7) is asynchronous in the sense that, given x_{-2} and x_{-1} , the subsequent $x_0, x_1, \dots$ are calculated one by one from the first, second, third, $\dots$ equation of (7), respectively. $\square$

We may also regard the original equation (5) as the following 2-dimensional asynchronous system:

$$\begin{aligned}x_n &= a_n x_{n-2} + b_n f_\lambda(x_{n-1}), \\x_{n+1} &= a_{n+1} x_{n-1} + b_{n+1} f_\lambda(x_n), \\n &= 0, 1, 2, \dots\end{aligned}\quad (39)$$

Let us say that a solution $\{(x_n, x_{n+1})\}_{n=-2}^\infty$ of (39) approaches a limit 1-cycle (α, α) if $\lim_n(x_n, x_{n+1}) = (\alpha, \alpha)$ and a limit 2-cycle (α, β) if $\alpha \neq \beta$ and $\lim_n(x_{2n}, x_{2n+1}) = (\alpha, \beta)$. Let us further say that a solution $\{(x_n, x_{n+1})\}_{n=-2}^\infty$ of (39) eventually falls into a region Y if $(x_n, x_{n+1}) \in Y$ for all large n and eventually alternates between two regions Y_1 and Y_2 if $(x_{2n}, x_{2n+1}) \in Y_1$ and $(x_{2n+1}, x_{2n+2}) \in Y_2$ for all large n .

Then, we may restate the previous theorems as follows.

- (i) The vectors $(0, 0)$, $(1, 1)$, $(0, 1)$, and $(1, 0)$ form the corners of a square in the plane.
- (ii) A solution $\{(x_n, x_{n+1})\}_{n=-2}^\infty$ of (39) with (x_{-2}, x_{-1}) in the nonpositive orthant $(-\infty, 0]^2$ is nonpositive and tends towards the limit 1-cycle $(0, 0)$.
- (iii) Suppose $\lambda > 1$. Then, a solution $\{(x_n, x_{n+1})\}_{n=-2}^\infty$ of (39) with $(x_{-2}, x_{-1}) \in \mathbb{R}^2 \setminus (-\infty, 0]^2$ will tend towards the limit 1-cycle $(1, 1)$.
- (iv) Suppose $0 < \lambda < 1$. Then, a solution $\{(x_n, x_{n+1})\}_{n=-2}^\infty$ of (39) with $(x_{-2}, x_{-1}) \in \mathbb{R}^2 \setminus (-\infty, 0]^2$ will tend towards the limit 2-cycle $(0, 1)$ or $(1, 0)$.
- (v) Suppose $\lambda = 1$. Then, a solution $\{(x_n, x_{n+1})\}_{n=-2}^\infty$ of (39) with $(x_{-2}, x_{-1}) \in \Psi$ will eventually fall into $(0, \lambda] \times (0, \lambda]$ and approach the limit 1-cycle $(1, 1)$.
- (vi) Suppose $\lambda = 1$. Then, a solution $\{(x_n, x_{n+1})\}_{n=-2}^\infty$ of (39) with $(x_{-2}, x_{-1}) \in \Omega$ will eventually alternate between $(0, \lambda] \times (\lambda, +\infty)$ and $(\lambda, +\infty) \times (0, \lambda]$ and approach the limit 2-cycle $(0, 1)$ or $(1, 0)$.

Since we have obtained a complete set of asymptotic criteria, we may deduce (bifurcation) results such as the following. If it $0 < \lambda < 1$, then all solutions $\{(x_n, x_{n+1})\}_{n=-2}^\infty$ which originated from the positive orthant approach a limit 2-cycle; if $\lambda > 1$, then all solutions that originated from the positive orthant tend towards the limit 1-cycles $(1, 1)$; if $\lambda = 1$, then all solutions originated from the positive orthant tend towards the limit 1-cycle $(1, 1)$ or 2-cycles $(1, 0)$ or $(0, 1)$. Roughly, the above statements show that when the threshold parameter λ is a relatively small positive parameter, all solutions from the positive orthant tend towards a limit 2-cycle; when the threshold parameter reaches the critical value 1, some of these solutions drift away and tend towards a limit 1-cycle, and when λ drifts beyond the critical value, all solutions tend towards the limit 1-cycle $(1, 1)$. Such an observation seems to appear in many natural processes and hence our model may be used to explain such phenomena. It is also expected that when a group of neural units interact with each other in a network where each unit is governed by evolutionary laws of the form (39), complex but manageable analytical results can be obtained. These are left for further studies.

We remark that the precise region from which each type of solution originates can also be given (by implementing a simple computer program).

Our proofs also indicate that more general multiple neuron recurrent McCulloch-Pitts type neural networks possess similar behavior. However, the derivations may involve more delicate combinatorial arguments and are left for further studies.

Appendix

We let

$$\begin{aligned}D_i^{(-2)} &= \frac{\lambda}{\delta^i}, & E_i^{(-1)} &= \frac{\lambda}{\rho^i}, \\D_i^{(j_0, j_1, \dots, j_m)} &= \frac{\lambda}{a_{j_0} a_{j_1} \dots a_{j_m} \delta^i}, \\E_i^{(j_0, j_1, \dots, j_m)} &= \frac{\lambda}{a_{j_0} a_{j_1} \dots a_{j_m} \rho^i}.\end{aligned}\quad (A.1)$$

Then,

$$\begin{aligned}D_0^{(-2)} &= E_0^{(-1)} = \lambda, \\E_i^{(1, 3, \dots, 2k-1)} &= E_{i+1}^{(-1)}, \\D_i^{(0, 2, \dots, 2k-2)} &= D_{i+1}^{(-2)}, \\E_i^{(j_0, j_1, \dots, j_m)} &< E_i^{(j_0, j_1, \dots, j_{m+1})}, \\D_i^{(j_0, j_1, \dots, j_m)} &< D_i^{(j_0, j_1, \dots, j_{m+1})},\end{aligned}\quad (A.2)$$

$$\lim_{i \rightarrow \infty} D_i^{(j_0, j_1, \dots, j_m)} = \lim_{i \rightarrow \infty} E_i^{(j_0, j_1, \dots, j_m)} = +\infty.$$

Thus,

$$(\lambda, +\infty) = \bigcup_{i=0}^{\infty} \bigcup_{m=0}^{k-1} (D_i^{(0, 2, \dots, 2m-2)}, D_i^{(0, 2, \dots, 2m)}], \quad (A.3)$$

$$(\lambda, +\infty) = \bigcup_{i=0}^{\infty} \bigcup_{m=0}^{k-1} (E_i^{(1, 3, \dots, 2m-1)}, E_i^{(1, 3, \dots, 2m+1)}]. \quad (A.4)$$

We assume $\lambda > 1$. Set

$$\begin{aligned}A_i^{(-1)} &= \frac{\lambda - (1 - \rho^i)}{\rho^i}, \\B_i^{(-2)} &= \frac{\lambda - (1 - \delta^i)}{\delta^i}, \\A_i^{(j_0, j_1, \dots, j_m)} &= \frac{\lambda - (1 - a_{j_0} a_{j_1} \dots a_{j_m} \rho^i)}{j_0 j_1 \dots j_m \rho^i}, \\B_i^{(j_0, j_1, \dots, j_m)} &= \frac{\lambda - (1 - a_{j_0} a_{j_1} \dots a_{j_m} \delta^i)}{j_0 j_1 \dots j_m \delta^i}.\end{aligned}\quad (A.5)$$

Then,

$$\begin{aligned}
 A_0^{(-1)} &= B_0^{(-2)} = \lambda, \\
 A_i^{(1,3,\dots,2k-1)} &= A_{i+1}^{(-1)}, \\
 B_i^{(0,2,\dots,2k-2)} &= B_{i+1}^{(-2)}, \\
 A_i^{(j_0,j_1,\dots,j_m)} &< A_i^{(j_0,j_1,\dots,j_{m+1})}, \\
 B_i^{(j_0,j_1,\dots,j_m)} &< B_i^{(j_0,j_1,\dots,j_{m+1})}, \\
 \lim_{i \rightarrow \infty} A_i^{(j_0,j_1,\dots,j_m)} &= \lim_{i \rightarrow \infty} B_i^{(j_0,j_1,\dots,j_m)} = +\infty.
 \end{aligned} \tag{A.6}$$

Thus,

$$\begin{aligned}
 (\lambda, +\infty) &= \bigcup_{i=0}^{\infty} \bigcup_{m=0}^{k-1} \left(A_i^{(1,3,\dots,2m-1)}, A_i^{(1,3,\dots,2m+1)} \right], \\
 (\lambda, +\infty) &= \bigcup_{i=0}^{\infty} \bigcup_{m=0}^{k-1} \left(B_i^{(0,2,\dots,2m-2)}, B_i^{(0,2,\dots,2m)} \right].
 \end{aligned} \tag{A.7}$$

We assume $0 < \lambda < 1$. Set

$$\begin{aligned}
 G_i^{(-2)} &= -\frac{1-\delta^i}{\delta^i}, & H_i^{(-1)} &= -\frac{1-\rho^i}{\rho^i}, \\
 L_i^{(-2)} &= \frac{\lambda - (1-\delta^i)}{\delta^i}, & M_i^{(-1)} &= \frac{\lambda - (1-\rho^i)}{\rho^i}, \\
 G_i^{(j_0,j_1,\dots,j_m)} &= -\frac{1 - a_{j_0} a_{j_1} \cdots a_{j_m} \delta^i}{a_{j_0} a_{j_1} \cdots a_{j_m} \delta^i}, \\
 L_i^{(j_0,j_1,\dots,j_m)} &= \frac{\lambda - (1 - a_{j_0} a_{j_1} \cdots a_{j_m} \delta^i)}{a_{j_0} a_{j_1} \cdots a_{j_m} \delta^i}, \\
 j_0, j_1, \dots, j_m &\in \{0, 2, \dots, 2k-2\}, \\
 H_i^{(j_0,j_1,\dots,j_m)} &= -\frac{1 - a_{j_0} a_{j_1} \cdots a_{j_m} \rho^i}{a_{j_0} a_{j_1} \cdots a_{j_m} \rho^i}, \\
 M_i^{(j_0,j_1,\dots,j_m)} &= \frac{\lambda - (1 - a_{j_0} a_{j_1} \cdots a_{j_m} \rho^i)}{a_{j_0} a_{j_1} \cdots a_{j_m} \rho^i}, \\
 j_0, j_1, \dots, j_m &\in \{1, 3, \dots, 2k-1\}.
 \end{aligned} \tag{A.8}$$

Then,

$$\begin{aligned}
 G_0^{(-2)} &= H_0^{(-1)} = 0, & L_0^{(-2)} &= M_0^{(-1)} = \lambda, \\
 G_i^{(0,2,\dots,2k-2)} &= G_{i+1}^{(-2)}, & H_i^{(1,3,\dots,2k-1)} &= H_{i+1}^{(-1)}, \\
 L_i^{(0,2,\dots,2k-2)} &= L_{i+1}^{(-2)}, & M_i^{(1,3,\dots,2k-1)} &= M_{i+1}^{(-1)}, \\
 G_i^{(j_0,j_1,\dots,j_m)} &< L_i^{(j_0,j_1,\dots,j_m)}, \\
 H_i^{(j_0,j_1,\dots,j_m)} &< M_i^{(j_0,j_1,\dots,j_m)},
 \end{aligned}$$

$$\lim_{i \rightarrow \infty} G_i^{(j_0,j_1,\dots,j_m)} = \lim_{i \rightarrow \infty} H_i^{(j_0,j_1,\dots,j_m)} = -\infty,$$

$$\lim_{i \rightarrow \infty} L_i^{(j_0,j_1,\dots,j_m)} = \lim_{i \rightarrow \infty} M_i^{(j_0,j_1,\dots,j_m)} = -\infty.$$

(A.9)

Since

$$\begin{aligned}
 \lambda &= L_0^{(-2)} > L_0^{(0)} > L_0^{(0,2)} > \dots > L_1^{(-2)} > \dots \\
 &> L_m^{(0,2,\dots,2s)} > L_m^{(0,2,\dots,2s+2)} \longrightarrow -\infty \quad (\text{as } m \longrightarrow \infty),
 \end{aligned} \tag{A.10}$$

there exist integers $p \in \mathbb{N}$ and $t \in \{0, 1, \dots, k-1\}$ such that $L_p^{(0,2,\dots,2t)} \leq 0$ and $L_p^{(0,2,\dots,2t-2)} > 0$.

Set

$$\begin{aligned}
 \bar{L}_p^{(0,2,\dots,2t)} &= \bar{L}_p^{(0,2,\dots,2t+2)} = \dots = \bar{L}_{p+1}^{(-2)} = 0, \\
 \bar{L}_i^{(0,2,\dots,2s)} &= L_i^{(0,2,\dots,2s)}, \\
 \bar{L}_p^{(0,2,\dots,2l)} &= L_p^{(0,2,\dots,2l)}
 \end{aligned} \tag{A.11}$$

for $i \in \{0, 1, \dots, p-1\}$, $s \in \{0, 1, \dots, k-1\}$, and $l \in \{0, 1, \dots, t-1\}$. Similarly, since

$$\begin{aligned}
 \lambda &= M_0^{(-1)} > M_0^{(1)} > M_0^{(1,3)} > \dots > L_1^{(-1)} > \dots \\
 &> M_m^{(1,1,\dots,2s-1)} > M_m^{(1,3,\dots,2s+1)} \longrightarrow -\infty \quad (\text{as } m \longrightarrow \infty),
 \end{aligned} \tag{A.12}$$

there exist integers $q \in \mathbb{N}$ and $s \in \{0, 1, \dots, k-1\}$ such that $M_q^{(1,3,\dots,2s-1)} \leq 0$ and $M_q^{(0,2,\dots,2s+1)} > 0$.

Set $\bar{M}_q^{(1,3,\dots,2s-1)} = \bar{M}_q^{(1,3,\dots,2s+1)} = \dots = \bar{M}_{q+1}^{(-2)} = 0$, $\bar{M}_i^{(1,3,\dots,2s+1)} = M_i^{(1,3,\dots,2s+1)}$, and $\bar{M}_q^{(1,3,\dots,2l+1)} = M_q^{(1,3,\dots,2l+1)}$ for $i \in \{0, 1, \dots, q-1\}$, $s \in \{0, 1, \dots, k-1\}$, and $l \in \{0, 1, \dots, s-1\}$. Then,

$$(0, \lambda] = \bigcup_{i=0}^p \bigcup_{m=0}^{k-1} \left(\bar{L}_i^{(0,2,\dots,2m)}, \bar{L}_i^{(0,2,\dots,2m-2)} \right], \tag{A.13}$$

$$(0, \lambda] = \bigcup_{i=0}^q \bigcup_{r=0}^{k-1} \left(\bar{M}_i^{(1,3,\dots,2r+1)}, \bar{M}_i^{(1,3,\dots,2r-1)} \right], \tag{A.14}$$

$$\begin{aligned}
 (-\infty, 0] &= \bigcup_{i=0}^{\infty} \bigcup_{r=0}^{k-1} \left(G_i^{(0,2,\dots,2r)}, L_i^{(0,2,\dots,2r)} \right] \\
 &\cup \left(L_i^{(0,2,\dots,2r)}, G_i^{(0,2,\dots,2r-2)} \right],
 \end{aligned} \tag{A.15}$$

$$\begin{aligned}
 (-\infty, 0] &= \bigcup_{i=0}^{\infty} \bigcup_{r=0}^{k-1} \left(H_i^{(1,3,\dots,2r+1)}, M_i^{(1,3,\dots,2r+1)} \right] \\
 &\cup \left(M_i^{(1,3,\dots,2r+1)}, H_i^{(1,3,\dots,2r-1)} \right].
 \end{aligned} \tag{A.16}$$

Furthermore, since

$$\begin{aligned}
 G_i^{(0,2,\dots,2r)} &< \frac{\bar{L}_p^{(0,2,\dots,2t-2)} - (1 - a_0 a_2 \cdots a_{2r} \delta^i)}{a_0 a_2 \cdots a_{2r} \delta^i} < \cdots \\
 &< \frac{\bar{L}_p^{(-2)} - (1 - a_0 a_2 \cdots a_{2r} \delta^i)}{a_0 a_2 \cdots a_{2r} \delta^i} < \cdots \\
 &< \frac{\bar{L}_0^{(-2)} - (1 - a_0 a_2 \cdots a_{2r} \delta^i)}{a_0 a_2 \cdots a_{2r} \delta^i} \\
 &= \bar{L}_i^{(0,2,\dots,2r)}, \\
 H_i^{(1,3,\dots,2r+1)} &< \frac{\bar{M}_q^{(1,3,\dots,2r-1)} - (1 - a_1 a_3 \cdots a_{2r+1} \rho^i)}{a_1 a_3 \cdots a_{2r+1} \rho^i} < \cdots \\
 &< \frac{\bar{M}_{q-1}^{(-1)} - (1 - a_1 a_3 \cdots a_{2r+1} \rho^i)}{a_1 a_3 \cdots a_{2r+1} \rho^i} < \cdots \\
 &< \frac{\bar{M}_0^{(-1)} - (1 - a_1 a_3 \cdots a_{2r+1} \rho^i)}{a_1 a_3 \cdots a_{2r+1} \rho^i} \\
 &= \bar{M}_i^{(1,3,\dots,2r+1)},
 \end{aligned} \tag{A.17}$$

then if we let

$$\Theta = \{0, 1, \dots, p-1\} \times \{0, 1, \dots, k-1\}$$

$$\cup \{p\} \times \{0, 1, \dots, t-1\},$$

$$\Xi = \{0, 1, \dots, q-1\} \times \{0, 1, \dots, k-1\}$$

$$\cup \{q\} \times \{0, 1, \dots, s-1\},$$

$$Q_{i,j}^{(0,2,\dots,2r;0,2,\dots,2m-2)} = \frac{\bar{L}_j^{(0,2,\dots,2m)} - (1 - a_0 a_2 \cdots a_{2r} \delta^i)}{a_0 a_2 \cdots a_{2r} \delta^i},$$

$$(j, m) \in \Theta,$$

$$R_{i,j}^{(1,3,\dots,2r+1;1,3,\dots,2m-1)}$$

$$= \frac{\bar{M}_j^{(1,3,\dots,2m-1)} - (1 - a_1 a_3 \cdots a_{2r+1} \rho^i)}{a_1 a_3 \cdots a_{2r+1} \rho^i},$$

$$(j, m) \in \Xi,$$

$$\tag{A.18}$$

then

$$\begin{aligned}
 &[G_i^{(0,2,\dots,2r)}, L_i^{(0,2,\dots,2r)}] \\
 &= \bigcup_{(j,m) \in \Theta} (Q_{i,j}^{(0,2,\dots,2r;0,2,\dots,2m)}, Q_{i,j}^{(0,2,\dots,2r;0,2,\dots,2m-2)}],
 \end{aligned}$$

$$\begin{aligned}
 &[H_i^{(1,3,\dots,2r+1)}, M_i^{(1,3,\dots,2r+1)}] \\
 &= \bigcup_{(j,m) \in \Xi} (R_{i,j}^{(1,3,\dots,2r+1;1,3,\dots,2m+1)}, R_{i,j}^{(1,3,\dots,2r+1;1,3,\dots,2m-1)}], \\
 &(-\infty, 0] \\
 &= \bigcup_{i=0}^{\infty} \bigcup_{r=0}^{k-1} \left\{ \bigcup_{(j,m) \in \Theta} (Q_{i,j}^{(0,2,\dots,2r;0,2,\dots,2m)}, Q_{i,j}^{(0,2,\dots,2r;0,2,\dots,2m-2)}) \right. \\
 &\quad \left. \cup (L_i^{(0,2,\dots,2r)}, G_i^{(0,2,\dots,2r-2)}) \right\}, \\
 &(-\infty, 0] \\
 &= \bigcup_{i=0}^{\infty} \bigcup_{r=0}^{k-1} \left\{ \bigcup_{(j,m) \in \Xi} (R_{i,j}^{(1,3,\dots,2r+1;1,3,\dots,2m+1)}, R_{i,j}^{(1,3,\dots,2r+1;1,3,\dots,2m-1)}) \right. \\
 &\quad \left. \cup (M_i^{(1,3,\dots,2r+1)}, H_i^{(1,3,\dots,2r-1)}) \right\}.
 \end{aligned} \tag{A.19}$$

Furthermore,

$$(0, \lambda] = \left(\bigcup_{(i,m) \in \Theta} \left(\frac{\bar{L}_i^{(0,\dots,2m)}}{a_0 \cdots a_{2s} \delta^j}, \frac{\bar{L}_i^{(0,\dots,2m-2)}}{a_0 \cdots a_{2s} \delta^j} \right) \right) \cap (0, \lambda],$$

$$j \in \mathbb{N}, \quad s \in \{0, 1, \dots, k-1\},$$

$$(0, \lambda]$$

$$= \left(\bigcup_{i,m \in \Xi} \left(\frac{\bar{M}_i^{(1,\dots,2m+1)}}{a_1 a_3 \cdots a_{2r+1} \rho^j}, \frac{\bar{M}_i^{(1,\dots,2m-1)}}{a_1 \cdots a_{2r+1} \rho^j} \right) \right) \cap (0, \lambda],$$

$$j \in \mathbb{N}, \quad r \in \{0, 1, \dots, k-1\},$$

$$(-\infty, 0]$$

$$\begin{aligned}
 &= \bigcup_{i=0}^{\infty} \bigcup_{r=0}^{k-1} \left\{ \bigcup_{(j,m) \in \Theta} \left(\left(\frac{Q_{i,j}^{(0,\dots,2r;0,2,\dots,2m)}}{a_0 \cdots a_{2\xi} \delta^n}, \right. \right. \right. \\
 &\quad \left. \left. \frac{Q_{i,j}^{(0,\dots,2r;0,\dots,2m-2)}}{a_0 \cdots a_{2\xi} a_0 \cdots a_{2\eta} \delta^n} \right) \right. \\
 &\quad \left. \cup \left(\frac{L_i^{(0,\dots,2r)}}{a_0 \cdots a_{2\xi} a_0 \cdots a_{2\eta} \delta^n}, \right. \right. \\
 &\quad \left. \left. \frac{G_i^{(0,\dots,2r-2)}}{a_0 \cdots a_{2\xi} a_0 \cdots a_{2\eta} \delta^n} \right) \right\},
 \end{aligned}$$

$$\xi, \eta, n \in \mathbb{N},$$

$(-\infty, 0]$

$$= \bigcup_{i=0}^{\infty} \bigcup_{r=0}^{k-1} \left\{ \bigcup_{(j,m) \in \Xi} \left(\left(\frac{R_{i,j}^{(1,\dots,2r+1;1,\dots,2m+1)}}{a_1 \cdots a_{2\xi+1} a_1 \cdots a_{2\eta+1} \rho^n}, \frac{R_{i,j}^{(1,\dots,2r+1;1,\dots,2m-1)}}{a_1 \cdots a_{2\xi+1} a_1 \cdots a_{2\eta+1} \rho^n} \right) \cup \left(\frac{M_i^{(1,\dots,2r+1)}}{a_1 \cdots a_{2\xi+1} a_1 \cdots a_{2\eta+1} \rho^n}, \frac{H_i^{(1,\dots,2r-1)}}{a_1 \cdots a_{2\xi+1} a_1 \cdots a_{2\eta+1} \rho^n} \right) \right\},$$

$$\xi, \eta, n \in \mathbb{N}.$$

(A.20)

We also note that

$$\frac{\overline{L}_p^{(0,\dots,2t)}}{a_0 \cdots a_{2n} a_0 \cdots a_{2t} \delta^{\tau+i}} = 0,$$

$$\frac{\overline{M}_q^{(1,\dots,2s+1)}}{a_1 \cdots a_{2n+1} a_1 \cdots a_{2s+1} \rho^{\tau+i}} = 0,$$

$$\frac{\overline{L}_0^{(-2)}}{a_0 \cdots a_{2n} a_0 \cdots a_{2t} \delta^{\tau+i}} = \frac{\lambda}{a_0 \cdots a_{2n} a_0 \cdots a_{2t} \delta^{\tau+i}}$$

$$> \frac{\lambda}{a_0 \cdots a_{2n} a_1 \cdots a_{2t-2} \delta^{\tau+i}} > \cdots$$

$$> \frac{\lambda}{\delta^{\tau+i}} > \cdots > D_0^{(-2)} = \lambda,$$

$$\frac{\overline{M}_0^{(-1)}}{a_1 \cdots a_{2n+1} a_1 \cdots a_{2s+1} \rho^{\tau+i}} = \frac{\lambda}{a_1 \cdots a_{2n+1} a_1 \cdots a_{2s+1} \rho^{\tau+i}}$$

$$> \frac{\lambda}{a_1 \cdots a_{2n+1} a_1 \cdots a_{2s-1} \rho^{\tau+i}} > \cdots$$

$$> \frac{\lambda}{\rho^{\tau+i}} > \cdots > E_0^{(-1)} = \lambda.$$

(A.21)

Therefore,

 $(\lambda, +\infty)$

$$= \bigcup_{\tau=0}^{\infty} \bigcup_{n=0}^{k-1} \bigcup_{(i,m) \in \Theta} \left(\frac{\overline{L}_i^{(0,\dots,2m)}}{a_0 \cdots a_{2n} a_0 \cdots a_{2r} \delta^{\tau+i}}, \frac{\overline{L}_i^{(0,2,\dots,2m)}}{a_0 \cdots a_{2n} a_0 \cdots a_{2r} \rho^{\tau+i}} \right)$$

$$\cap \left(D_{\tau}^{(0,\dots,2n-2)}, D_{\tau}^{(0,\dots,2n)} \right],$$

 $(\lambda, +\infty)$

$$= \bigcup_{\tau=0}^{\infty} \bigcup_{n=0}^{k-1} \left(\bigcup_{(i,m) \in \Xi} \left(\frac{\overline{M}_i^{(1,\dots,2m+1)}}{a_1 \cdots a_{2n+1} a_1 \cdots a_{2r+1} \rho^{\tau+i}}, \frac{\overline{M}_i^{(1,\dots,2m-1)}}{a_1 \cdots a_{2n+1} a_1 \cdots a_{2r+1} \rho^{\tau+i}} \right) \cap \left(E_{\tau}^{(1,\dots,2n-1)}, E_{\tau}^{(1,\dots,2n+1)} \right) \right).$$

(A.22)

Lemma A.1. A solution $\{\langle x_{2kn} \rangle\}_{n=0}^{\infty}$ of (7) with $(x_{-2}, x_{-1}) \in (0, \lambda] \times (\lambda, +\infty)$ (or $(x_{-2}, x_{-1}) \in (\lambda, +\infty) \times (0, \lambda]$) tends towards the limit 2-cycle $\langle 0, 1 \rangle$ (or $\langle 1, 0 \rangle$).

The proof of Lemma A.1 is the same as those discussed in the proofs of Theorems 6 and 7.

Lemma A.2. Suppose $\{\langle x_{2kn} \rangle\}_{n=0}^{\infty}$ is a solution of (7) with $(x_{-2}, x_{-1}) \in (D_i^{(0,2,\dots,2m-2)}, D_i^{(0,2,\dots,2m)}) \times (E_i^{(1,3,\dots,2l-1)}, E_i^{(1,3,\dots,2l+1)})$, where $i, j \in \{0, 1, \dots\}$ and $m, l \in \{0, 1, 2, \dots, k-1\}$. Then,

- (i) $(x_{2ki+2m}, x_{2ki+2m+1}) \in (0, \lambda] \times (\lambda, +\infty)$ for $0 \leq i = j$, $0 \leq m \leq l$;
- (ii) $(x_{2ki+2l}, x_{2ki+2l+1}) \in (\lambda, +\infty) \times (0, \lambda]$ for $0 \leq i = j$, $0 \leq l < m$;
- (iii) $(x_{2ki+2m}, x_{2ki+2m+1}) \in (0, \lambda] \times (\lambda, +\infty)$ for $0 \leq i < j$;
- (iv) $(x_{2kj+2l}, x_{2kj+2l+1}) \in (\lambda, +\infty) \times (0, \lambda]$ for $0 \leq j < i$.

Lemma A.3. Suppose $\{\langle x_{2kn} \rangle\}_{n=0}^{\infty}$ is a solution of (7) with $(x_{-2}, x_{-1}) \in (\overline{L}_i^{(0,2,\dots,2m)}, \overline{L}_i^{(0,2,\dots,2m-2)}) \times (\overline{M}_j^{(1,\dots,2l+1)}, \overline{M}_j^{(1,\dots,2l-1)})$, where $i, j \in \{0, 1, \dots\}$ and $m, l \in \{0, 1, 2, \dots, k-1\}$. Then,

- (i) $(x_{2ki+2m}, x_{2ki+2m+1}) \in (\lambda, +\infty) \times (0, \lambda]$ for $0 \leq i = j$, $0 \leq m \leq l$;
- (ii) $(x_{2ki+2l}, x_{2ki+2l+1}) \in (0, \lambda] \times (\lambda, +\infty)$ for $0 \leq i = j$, $0 \leq l < m$;
- (iii) $(x_{2ki+2m}, x_{2ki+2m+1}) \in (\lambda, +\infty) \times (0, \lambda]$ for $0 \leq i < j$;
- (iv) $(x_{2kj+2l}, x_{2kj+2l+1}) \in (0, \lambda] \times (\lambda, +\infty)$ for $0 \leq j < i$.

Lemma A.4. Suppose $0 < \lambda < 1$ and $\{\langle x_{2kn} \rangle\}_{n=0}^{\infty}$ is a solution of (7) with $(x_{-2}, x_{-1}) \in (0, \lambda] \times (-\infty, 0]$. If $(x_{-2}, x_{-1}) \in (0, \lambda] \times (M_i^{(1,\dots,2r+1)}, H_i^{(1,\dots,2r-1)}) \neq \emptyset$, then

$$(x_{2ki+2r}, x_{2ki+2r+1}) \in (0, \lambda] \times (\lambda, \infty). \quad (\text{A.23})$$

If

 (x_{-2}, x_{-1})

$$\in \left(\left(\frac{\overline{L}_l^{(0,\dots,2m)}}{a_0 a_2 \cdots a_{2r} \delta^i}, \frac{\overline{L}_l^{(0,\dots,2m-2)}}{a_0 a_2 \cdots a_{2r} \delta^i} \right) \cap (0, \lambda] \right) \quad (\text{A.24})$$

$$\times \left(R_{ij}^{(1,3,\dots,2r+1;1,\dots,2n+1)}, R_{ij}^{(1,3,\dots,2r+1;1,\dots,2n-1)} \right] \neq \emptyset,$$

then

$$(x_{2ki+2r}, x_{2ki+2r+1}) \in \left(\bar{L}_l^{(0,\dots,2m)}, \bar{L}_l^{(0,2,\dots,2m-2)} \right] \times \left(\bar{M}_j^{(1,\dots,2n+1)}, \bar{M}_j^{(1,\dots,2n-1)} \right]. \quad (\text{A.25})$$

Lemma A.5. Suppose $0 < \lambda < 1$ and $\{\langle x_{2kn} \rangle\}_{n=0}^{\infty}$ is a solution of (7) with $(x_{-2}, x_{-1}) \in (-\infty, 0] \times (0, \lambda]$. If $(x_{-2}, x_{-1}) \in (L_i^{(0,\dots,2r)}, G_i^{(0,\dots,2r-2)}) \times (0, \lambda] \neq \emptyset$, then

$$(x_{2ki+2r}, x_{2ki+2r+1}) \in (\lambda, \infty) \times (0, \lambda]. \quad (\text{A.26})$$

If

$$(x_{-2}, x_{-1}) \in (Q_{ij}^{(0,\dots,2r;0,\dots,2m)}, Q_{ij}^{(0,\dots,2r;0,\dots,2m-2)}) \times \left(\left(\frac{\bar{M}_\tau^{(1,\dots,2n+1)}}{a_1 \cdots a_{2r+1} \rho^i}, \frac{\bar{M}_\tau^{(1,\dots,2n-1)}}{a_1 \cdots a_{2r+1} \rho^i} \right) \cap (0, \lambda] \right) \neq \emptyset, \quad (\text{A.27})$$

then

$$(x_{2ki+2r}, x_{2ki+2r+1}) \in \left(\bar{L}_j^{(0,\dots,2m)}, \bar{L}_j^{(0,\dots,2m-2)} \right] \times \left(\bar{M}_\tau^{(1,\dots,2n+1)}, \bar{M}_\tau^{(1,\dots,2n-1)} \right]. \quad (\text{A.28})$$

Lemma A.6. Suppose $0 < \lambda < 1$ and $\{\langle x_{2kn} \rangle\}_{n=0}^{\infty}$ is a solution of (7) with $(x_{-2}, x_{-1}) \in (-\infty, 0] \times (\lambda, \infty)$. If

$$(x_{-2}, x_{-1}) \in \left(\frac{L_j^{(0,\dots,2r)}}{a_0 \cdots a_{2\xi} a_0 \cdots a_{2\eta} \delta^{\tau+i}}, \frac{G_j^{(0,\dots,2r-2)}}{a_0 \cdots a_{2\xi} a_0 \cdots a_{2\eta} \delta^{\tau+i}} \right] \times \left(\left(\frac{\bar{M}_i^{(1,\dots,2m+1)}}{a_1 a_3 \cdots a_{2\xi+1} a_1 a_3 \cdots a_{2\eta+1} \rho^{\tau+i}}, \frac{\bar{M}_i^{(1,\dots,2m-1)}}{a_1 a_3 \cdots a_{2\xi+1} a_1 a_3 \cdots a_{2\eta+1} \rho^{\tau+i}} \right) \cap (E_\tau^{(1,\dots,2\xi-1)}, E_\tau^{(1,\dots,2\xi+1)}) \right) \neq \emptyset, \quad (\text{A.29})$$

then

$$(x_{2k(\tau+i)+2\xi+2\eta}, x_{2k(\tau+i)+2\xi+2\eta+1}) \in (L_j^{(0,\dots,2r)}, G_j^{(0,\dots,2r-2)}) \times \left(\bar{M}_i^{(1,\dots,2m+1)}, \bar{M}_i^{(1,\dots,2m-1)} \right]. \quad (\text{A.30})$$

If

$$(x_{-2}, x_{-1}) \in \left(\frac{Q_{ij}^{(0,\dots,2r;0,\dots,2m)}}{a_0 a_2 \cdots a_{2\xi} a_0 \cdots a_{2\eta} \delta^{\tau+i}}, \frac{Q_{ij}^{(0,\dots,2r;0,\dots,2m-2)}}{a_0 \cdots a_{2\xi} a_0 \cdots a_{2\eta} \delta^{\tau+i}} \right] \times \left(\left(\frac{\bar{M}_t^{(1,\dots,2s+1)}}{a_1 \cdots a_{2\xi+1} a_1 \cdots a_{2\eta+1} \rho^{i+\tau}}, \frac{\bar{M}_t^{(1,\dots,2s-1)}}{a_1 \cdots a_{2\xi+1} a_1 \cdots a_{2\eta+1} \rho^{i+\tau}} \right) \cap (E_\tau^{(1,\dots,2\xi-1)}, E_\tau^{(1,\dots,2\xi+1)}) \right) \neq \emptyset, \quad (\text{A.31})$$

then

$$(x_{2k(\tau+i)+2\xi+2\eta}, x_{2k(\tau+i)+2\xi+2\eta+1}) \in (Q_{ij}^{(0,\dots,2r;0,\dots,2m)}, Q_{ij}^{(0,\dots,2r;0,\dots,2m-2)}) \times \left(\bar{M}_t^{(1,\dots,2s+1)}, \bar{M}_t^{(1,\dots,2s-1)} \right]. \quad (\text{A.32})$$

Lemma A.7. Suppose $0 < \lambda < 1$ and $\{\langle x_{2kn} \rangle\}_{n=0}^{\infty}$ is a solution of (7) with $(x_{-2}, x_{-1}) \in (\lambda, \infty) \times (-\infty, 0]$. If

$$(x_{-2}, x_{-1}) \in \left(\left(\frac{\bar{L}_i^{(0,\dots,2m)}}{a_0 \cdots a_{2\xi} a_0 \cdots a_{2\eta} \delta^{\tau+i}}, \frac{\bar{L}_i^{(0,2,\dots,2m-2)}}{a_0 \cdots a_{2\xi} a_0 \cdots a_{2\eta} \rho^{\tau+i}} \right) \cap (D_\tau^{(0,\dots,2\xi-2)}, D_\tau^{(0,\dots,2\xi)}) \right) \times \left(\frac{M_\tau^{(1,\dots,2r+1)}}{a_1 \cdots a_{2\xi+1} a_1 \cdots a_{2\eta+1} \rho^{\tau+i}}, \frac{H_\tau^{(1,\dots,2r-1)}}{a_1 \cdots a_{2\xi+1} a_1 \cdots a_{2\eta+1} \rho^{\tau+i}} \right) \neq \emptyset, \quad (\text{A.33})$$

then

$$(x_{2k(\tau+i)+2\xi+2\eta}, x_{2k(\tau+i)+2\xi+2\eta+1}) \in (\bar{L}_i^{(0,\dots,2m)}, \bar{L}_i^{(0,2,\dots,2m-2)}) \times (M_\tau^{(1,\dots,2r+1)}, H_\tau^{(1,\dots,2r-1)}]. \quad (\text{A.34})$$

If

$$\begin{aligned}
 & (x_{-2}, x_{-1}) \\
 & \in \left(\left(\frac{\bar{L}_i^{(0, \dots, 2m)}}{a_0 \cdots a_{2\xi} a_0 \cdots a_{2\eta} \delta^{\tau+i}}, \right. \right. \\
 & \quad \left. \left. \frac{\bar{L}_i^{(0, 2, \dots, 2m)}}{a_0 \cdots a_{2\xi} a_0 \cdots a_{2\eta} \rho^{\tau+i}} \right] \right. \\
 & \quad \left. \cap \left(D_\tau^{(0, \dots, 2\xi-2)}, D_\tau^{(0, \dots, 2\xi)} \right] \right) \\
 & \times \left(\frac{R_{i,j}^{(1, \dots, 2\eta+1; 1, \dots, 2m+1)}}{a_1 \cdots a_{2\xi+1} \rho^{i+\tau}}, \frac{R_{i,j}^{(1, \dots, 2\eta+1; 1, \dots, 2m-1)}}{a_1 \cdots a_{2\xi+1} \rho^{i+\tau}} \right] \neq \emptyset,
 \end{aligned} \tag{A.35}$$

then

$$\begin{aligned}
 & (x_{2k(\tau+i)+2\xi+2\eta}, x_{2k(\tau+i)+2\xi+2\eta+1}) \\
 & \in \left(\bar{L}_i^{(0, \dots, 2m)}, \bar{L}_i^{(0, 2, \dots, 2m)} \right] \\
 & \times \left(R_{i,j}^{(1, \dots, 2m+1)}, R_{i,j}^{(1, \dots, 2m-1)} \right].
 \end{aligned} \tag{A.36}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This study is supported by the Natural Science Foundation of China (11161049).

References

- [1] H. Zhu and L. Huang, "Asymptotic behavior of solutions for a class of delay difference equation," *Annals of Differential Equations*, vol. 21, no. 1, pp. 99–105, 2005.
- [2] Y. Chen, "All solutions of a class of difference equations are truncated periodic," *Applied Mathematics Letters*, vol. 15, no. 8, pp. 975–979, 2002.