Research Article

The Time Discontinuous H^1 -Galerkin Mixed Finite Element Method for Linear Sobolev Equations

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We combine the H^1 -Galerkin mixed finite element method with the time discontinuous Galerkin method to approximate linear Sobolev equations. The advantages of these two methods are fully utilized. The approximate schemes are established to get the approximate solutions by a piecewise polynomial of degree at most $q - 1$ with the time variable. The existence and uniqueness of the solutions are proved, and the optimal H^1 -norm error estimates are derived. We get high accuracy for both the space and time variables.

1. Introduction

We consider the following linear Sobolev equations to find $u = u(x, t)$ such that

$$u_t - \nabla \cdot (a(x, t) \nabla u + b(x, t) \nabla u_t) = f(x, t), \quad (1a)$$

$$(x, t) \in \Omega \times (0, T],$$

$$u = 0, \quad (x, t) \in \partial\Omega \times [0, T], \quad (1b)$$

$$u(x, 0) = u_0(x), \quad x \in \Omega, \quad t = 0, \quad (1c)$$

where Ω is a convex domain in R^d ($d \leq 3$) with a smooth boundary $\partial\Omega$. Functions $u_0(x)$, $f(x, t)$, $a(x, t)$, and $b(x, t)$ are known. Assume that $a(x, t)$ and $b(x, t)$ have bounded derivatives with respect to the variables x and t , respectively, and satisfy

$$\begin{aligned} 0 < a_0 \leq a(x, t) \leq a_1, \\ 0 < b_0 \leq b(x, t) \leq b_1, \\ \left| a - \frac{1}{2} b_t \right| \leq \gamma_0, \\ \forall (x, t) \in \Omega \times [0, T], \end{aligned} \quad (2)$$

where a_0 , a_1 , b_0 , b_1 , and γ_0 are positive constants.

There has been a wide range of practical applications of Sobolev equations in engineering, such as the porous theories concerned with percolation into rocks with cracks [1], the transport problems of humidity in soil [2], and the heat conduction problems between different media [3, 4]. So many numerical simulation methods have been proposed to solve these equations; for example, see [5–11].

The time discontinuous Galerkin method was proposed by Delfour et al. in 1981 [12] to analyze the ordinary differential equations. The purpose of this method is to find the approximate solutions as a piecewise polynomial of degree at most $q - 1$ with time variable by discretizing the time variable with Galerkin method. So the similar processing method can be used to analyze the space and time variables in the definition and analysis. This method considered the space and time variables together in order to utilize the advantages of the finite element method on both the space and time variables. Hence, the accuracy of numerical solutions is improved. Jamet [13] and Lesaint and Raviart [14] used the time discontinuous Galerkin method to study the partial differential equations. Eriksson et al. promoted and enriched this method further; for example, see [15–19].

It is well known that standard mixed finite element spaces have to satisfy the LBB-consistency condition (also called Inf-Sup condition), which restricts the choice of finite element

spaces. For example, in [20], the Raviart-Thomas spaces of index $r \geq 1$ are used for the second-order elliptic problems.

In order to circumvent the LBB-consistency condition, H^1 -Galerkin mixed finite element method was introduced by Pani [21] to parabolic equations. By introducing a flux variable σ of the primary variable u , parabolic equations are changed into a first-order system. Then, H^1 -Galerkin finite element method is used to approximate the system. The finite element spaces $V_h \times \mathbf{W}_h$ for variable u and flux σ are allowed to be of different polynomial degrees and not subject to the LBB-consistency condition. Hence, the obtained estimations can distinguish the better approximation properties of V_h and $\mathbf{W}_h$. Moreover, the quasi-uniformity condition was not imposed on the finite element mesh for L^2 and H^1 -norm error estimates. More applications of H^1 -Galerkin mixed finite element method had been done to, for example, the parabolic integro-differential equations [22], the second-order hyperbolic equations [23], and Sobolev equations [8]. Recently, some researchers applied this method to other types of problems, such as RLW equation [24], heat transport equations [25], and pseudo-parabolic equation [26].

In this paper, we combine H^1 -Galerkin mixed finite element method and the time discontinuous Galerkin method to approximate linear Sobolev equations. We establish the time discontinuous H^1 -Galerkin mixed finite element schemes and expect to utilize the advantages of the two above-mentioned methods to obtain a high-accuracy numerical method.

The rest of this paper is organized as follows. In Section 2, we present the time discontinuous H^1 -Galerkin mixed finite element schemes. In Section 3, we prove the existence and uniqueness of the solutions. In Section 4, the optimal H^1 -norm error estimates are derived. In Section 5, we draw some conclusions of this paper.

2. Time Discontinuous H^1 -Galerkin Mixed Finite Element Schemes

In this section, we establish the approximate schemes for (1a), (1b), and (1c). Let us introduce an intermediate variable $\sigma = a(x, t)\nabla u + b(x, t)\nabla u_t$. Then, (1a), (1b), and (1c) can be rewritten as the following first-order system:

$$\begin{aligned} a(x, t)\nabla u + b(x, t)\nabla u_t &= \sigma, \\ (x, t) &\in \Omega \times (0, T], \end{aligned} \quad (3a)$$

$$u_t - \nabla \cdot \sigma = f, \quad (x, t) \in \Omega \times (0, T]. \quad (3b)$$

Let Sobolev space $\mathbf{L}^2(\Omega) = (L^2(\Omega))^d$, equipped with the norm and inner product defined by

$$(\sigma, \mathbf{w}) = \sum_{i=1}^d (\sigma_i, w_i), \quad \|\mathbf{w}\| = \left(\sum_{i=1}^d \|w_i\|^2 \right)^{1/2}, \quad (4)$$

respectively.

Similarly, let $\mathbf{H}^m(\Omega) = (H^m(\Omega))^d$, $\mathbf{H}(\text{div}; \Omega) = \{\mathbf{w} \in \mathbf{L}^2(\Omega) : \nabla \cdot \mathbf{w} \in L^2(\Omega)\}$. The norms on these two spaces are defined, respectively, by

$$\|\mathbf{w}\|_m = \left(\sum_{i=1}^d \|w_i\|_m^2 \right)^{1/2}, \quad (5)$$

$$\|\mathbf{w}\|_{\mathbf{H}(\text{div}; \Omega)} = \left(\|\mathbf{w}\|^2 + \|\nabla \cdot \mathbf{w}\|^2 \right)^{1/2}.$$

Then, the variational form of (3a) and (3b) is to find $\{u, \sigma\} : [0, T] \rightarrow H_0^1 \times \mathbf{H}(\text{div}; \Omega)$ such that

$$\begin{aligned} (a\nabla u, \nabla v) + (b\nabla u_t, \nabla v) &= (\sigma, \nabla v), \\ \forall v &\in H_0^1, \end{aligned} \quad (6a)$$

$$\begin{aligned} (\alpha\sigma, \mathbf{w}) + (\nabla \cdot \sigma, \nabla \cdot \mathbf{w}) &= (\beta\nabla u, \mathbf{w}) - (f, \nabla \cdot \mathbf{w}), \\ \forall \mathbf{w} &\in \mathbf{H}(\text{div}; \Omega), \end{aligned} \quad (6b)$$

where $\alpha(x, t) = b^{-1}(x, t)$, $\beta(x, t) = b^{-1}(x, t)a(x, t)$.

Clearly (6a) is obtained by multiplying (3a) by ∇v and integrating the resulting equation with respect to x . Multiplying (3b) by $\nabla \cdot \mathbf{w}$ and integrating the first term by parts give

$$(\nabla u_t, \mathbf{w}) + (\nabla \cdot \sigma, \nabla \cdot \mathbf{w}) = -(f, \nabla \cdot \mathbf{w}), \quad (7)$$

where the condition $u_t|_{\partial\Omega} = 0$ is used. From (3a), we know $\nabla u_t = \alpha\sigma - \beta\nabla u$. Substituting this expression into the above equation yields (6b).

Assume that V_h and $\mathbf{W}_h$ are the finite dimensional subspaces of H_0^1 and $\mathbf{H}(\text{div}; \Omega)$, respectively, with the following approximation properties: for k, r positive integers

$$\inf_{v_h \in V_h} \{\|v - v_h\| + h\|v - v_h\|\} \leq Ch^{k+1}\|v\|_{k+1}, \quad (8)$$

$$\forall v \in H_0^1 \cap H^{k+1},$$

$$\inf_{\mathbf{w}_h \in \mathbf{W}_h} \{\|\mathbf{w} - \mathbf{w}_h\| + h\|\mathbf{w} - \mathbf{w}_h\|_{\mathbf{H}(\text{div}; \Omega)}\} \leq Ch^{r+1}\|\mathbf{w}\|_{r+1}, \quad \forall \mathbf{w} \in \mathbf{H}^{r+1}. \quad (9)$$

Let $0 = t^0 < t^1 < \dots < t^N = T$ be an unnecessarily uniform subdivision of $[0, T]$ and

$$\begin{aligned} I_n &= (t_n, t_{n+1}], \quad t_{n+1} - t_n = \tau_n, \quad \tau = \max_n \tau_n, \\ 0 &\leq n \leq N-1. \end{aligned} \quad (10)$$

With a given positive integer $q \geq 1$, we will look for the approximation solutions $\{U, \Sigma\}$ of (6a) and (6b), which reduce to a polynomial of degree at most $q-1$ with time variable t on each subinterval I_n with coefficients in V_h and

$\mathbf{W}_h$, respectively. That is to say, they belong to the following finite element spaces:

$$\begin{aligned} V_{h,\tau}^n &= \left\{ v_h; v_h = \sum_{j=0}^{q-1} v_{h,j} t^j, \omega_{h,j} \in V_h \right\}, \\ \mathbf{W}_{h,\tau}^n &= \left\{ \mathbf{w}_h; \mathbf{w}_h = \sum_{j=0}^{q-1} \mathbf{w}_{h,j} t^j, \mathbf{w}_{h,j} \in \mathbf{W}_h \right\}. \end{aligned} \quad (11)$$

Note that the functions of these two spaces are allowed to be discontinuous at the time nodal points but are taken to be continuous to the left there. For $U \in V_{h,\tau}^n$, we denote $U^n = U|_{t=t_n}$, $U_+^n = \lim_{t \rightarrow t_n^+} U$.
Let

$$\begin{aligned} V_{h,\tau} &= \{v_h; v_h|_{I_n} \in V_{h,\tau}^n\}, \\ \mathbf{W}_{h,\tau} &= \{\mathbf{w}_h; \mathbf{w}_h|_{I_n} \in \mathbf{W}_{h,\tau}^n\}. \end{aligned} \quad (12)$$

Notice that

$$\begin{aligned} \int_0^T (b \nabla u_t, \nabla v) dt &= \sum_{n=0}^{N-1} \int_{I_n} (b \nabla u_t, \nabla v) dt \\ &= \sum_{n=0}^{N-1} \left[(b^{n+1} \nabla u^{n+1}, \nabla v^{n+1}) - (b^n \nabla u^n, \nabla v_+^n) \right. \\ &\quad \left. - \int_{I_n} (\nabla u, (b \nabla v)_t) dt \right]. \end{aligned} \quad (13)$$

Here, we do not consider the continuity of v at $t = t^n$ in the above equation, which means the time discontinuous Galerkin finite element space can be adopted.

Hence, the time discontinuous H^1 -Galerkin mixed finite element scheme of (6a) and (6b) is to find $\{U, \Sigma\} \in V_{h,\tau} \times \mathbf{W}_{h,\tau}$ such that

$$\begin{aligned} \sum_{n=0}^{N-1} \left[(b^{n+1} \nabla U^{n+1}, \nabla v^{n+1}) - (b^n \nabla U^n, \nabla v_+^n) \right. \\ \left. - \int_{I_n} (\nabla U, (b \nabla v)_t) dt + \int_{I_n} (a \nabla U, \nabla v) dt \right] \end{aligned} \quad (14)$$

$$= \sum_{n=0}^{N-1} \int_{I_n} (\Sigma, \nabla v) dt, \quad \forall v \in V_{h,\tau},$$

$$\begin{aligned} \sum_{n=0}^{N-1} \int_{I_n} [(\alpha \Sigma, \mathbf{w}) + (\nabla \cdot \Sigma, \nabla \cdot \mathbf{w})] dt \\ = \sum_{n=0}^{N-1} \int_{I_n} [(\beta \nabla U, \mathbf{w}) - (f, \nabla \cdot \mathbf{w})] dt, \end{aligned} \quad (15)$$

$$\forall \mathbf{w} \in \mathbf{W}_{h,\tau}.$$

Here, we take $U^0 = u_{0,h}$ and $u_{0,h}$ is a certain initial approximation to $u_0(x)$ in V_h .

Since

$$\begin{aligned} & - \sum_{n=0}^{N-1} \int_{I_n} (\nabla U, (b \nabla v)_t) dt \\ &= \sum_{n=0}^{N-1} \left[\int_{I_n} (b \nabla U_t, \nabla v) dt - (b^{n+1} \nabla U^{n+1}, \nabla v^{n+1}) \right. \\ &\quad \left. + (b^n \nabla U_+^n, \nabla v_+^n) \right], \end{aligned} \quad (16)$$

(14) can be rewritten as

$$\begin{aligned} & \sum_{n=0}^{N-1} \int_{I_n} \{[(b \nabla U_t, \nabla v) + (a \nabla U, \nabla v)] dt \\ &\quad + (b^n \nabla U_+^n, \nabla v_+^n) - (b^n \nabla U^n, \nabla v_+^n)\} \\ &= \sum_{n=0}^{N-1} \int_{I_n} (\Sigma, \nabla v) dt, \quad \forall v \in V_{h,\tau}. \end{aligned} \quad (17)$$

Due to the discontinuities of $V_{h,\tau}$ and $\mathbf{W}_{h,\tau}$ with respect to t , the local forms of (17) and (15) can be given as follows:

$$\begin{aligned} & \int_{I_n} [(b \nabla U_t, \nabla v) + (a \nabla U, \nabla v)] dt \\ &\quad + (b^n \nabla U_+^n, \nabla v_+^n) - (b^n \nabla U^n, \nabla v_+^n) \\ &= \int_{I_n} (\Sigma, \nabla v) dt, \quad \forall v \in V_{h,\tau}^n, \end{aligned} \quad (18)$$

$$\begin{aligned} & \int_{I_n} [(\alpha \Sigma, \mathbf{w}) + (\nabla \cdot \Sigma, \nabla \cdot \mathbf{w})] dt \\ &= \int_{I_n} [(\beta \nabla U, \mathbf{w}) - (f, \nabla \cdot \mathbf{w})] dt, \quad \forall \mathbf{w} \in \mathbf{W}_{h,\tau}^n. \end{aligned} \quad (19)$$

The advantage of the local forms (18)-(19) is that they can be solved locally in each time interval I_n ($n = 0, \dots, N-1$) gradually.

3. Existence and Uniqueness of the Solutions

In this section, we prove the existence and uniqueness of the solutions of (18)-(19). First, we adopt the following lemma [27].

Lemma 1 (Gronwall's inequality). Assume that $y(t)$ is a continuous function in $[0, T)$ satisfying

$$y(t) \leq y_0 + \int_0^t \lambda(\tau) y(\tau) d\tau, \quad (20)$$

where $\lambda(\tau) \geq 0$ and $\lambda(\tau) \in L^1(0, T)$. Then, there holds

$$y(t) \leq y_0 \exp \left(\int_0^t \lambda(\tau) d\tau \right). \quad (21)$$

Similarly, we have the following.

Lemma 2. Assume g_0 is a nonnegative real number, $c > 0$, $h(t)$ is a continuous function in $[t_n, t_{n+1})$ that satisfies $h(t_n) = 0$, and

$$h(t) \leq g_0 + c \int_{t_n}^t h(\tau) d\tau, \quad t_n \leq t \leq t_{n+1}. \quad (22)$$

Then, there exists

$$h(t) \leq g_0 e^{c(t-t_n)} \leq g_0 e^{c\tau_n}, \quad t_n \leq t \leq t_{n+1}, \quad (23)$$

where $\tau_n = t_{n+1} - t_n$.

Proof. Let $Y(t) = c \int_{t_n}^t h(\tau) d\tau$, $\forall t_n \leq t \leq t_{n+1}$. Using (22), we have

$$Y'(t) = ch(t) \leq cg_0 + cY(t). \quad (24)$$

Multiplying both sides of (24) by $e^{-c(t-t_n)}$ leads to

$$\left[Y(t) e^{-c(t-t_n)} \right]' \leq cg_0 e^{-c(t-t_n)} = -g_0 \left(e^{-c(t-t_n)} \right)'. \quad (25)$$

Integrating (25) in $[t_n, t]$, $t \leq t_{n+1}$, and using $Y(t_n) = 0$, we can obtain

$$Y(t) \leq g_0 e^{c(t-t_n)} - g_0. \quad (26)$$

Substituting (26) into (22) completes the existence of (23). $\square$

By Lemma 2, we can have the following theorem.

Theorem 3. The solutions $\{U, \Sigma\}$ of (18)-(19) are existent and unique.

Proof. Since (18)-(19) are linear equations about U and Σ , the existence and uniqueness of the solutions are equivalent to the homogenous linear equations only that have zero solutions on each time interval I_n [28]. These proofs are different from well-known Brouwer's fixed point theorem [29], which is used to consider the nonlinear boundary problems.

Letting $f = 0$, $U^n = 0$ and taking $v = U$ in (18), we hold

$$\begin{aligned} & \int_{I_n} \left[\frac{1}{2} \frac{d}{dt} \|b^{1/2} \nabla U\|^2 dt + \left(\left(a - \frac{1}{2} b_t \right) \nabla U, \nabla U \right) \right] dt \\ & + (b^n \nabla U_+^n, \nabla U_+^n) = \int_{I_n} (\Sigma, \nabla U) dt. \end{aligned} \quad (27)$$

Using Cauchy inequality, Poincaré inequality, and (2), we have

$$\begin{aligned} & \frac{1}{2} \left\| (b^{n+1})^{1/2} \nabla U^{n+1} \right\|^2 + \frac{1}{2} \left\| (b^n)^{1/2} \nabla U_+^n \right\|^2 \\ & \leq \frac{3\gamma_0}{2} \int_{I_n} \|\nabla U\|^2 dt + \frac{1}{2\gamma_0} \int_{I_n} \|\Sigma\|^2 dt. \end{aligned} \quad (28)$$

This yields

$$b_0 \|\nabla U^{n+1}\|^2 \leq 3\gamma_0 \int_{I_n} \|\nabla U\|^2 dt + \frac{1}{\gamma_0} \int_{I_n} \|\Sigma\|^2 dt. \quad (29)$$

Using Lemma 2, we derive

$$\|\nabla U^{n+1}\|^2 \leq C \int_{I_n} \|\Sigma\|^2 dt. \quad (30)$$

Taking $\mathbf{w} = \Sigma$ in (19), we find

$$\int_{I_n} [(\alpha \Sigma, \Sigma) + (\nabla \cdot \Sigma, \nabla \cdot \Sigma)] dt = \int_{I_n} (\beta \nabla U, \Sigma) dt. \quad (31)$$

Using Cauchy inequality, ε -inequality, and (2), we have

$$\begin{aligned} & \int_{I_n} \left[\|\alpha^{1/2} \Sigma\|^2 + \|\nabla \cdot \Sigma\|^2 \right] dt \\ & \leq \varepsilon \int_{I_n} \|\alpha^{1/2} \Sigma\|^2 dt + \frac{1}{4\varepsilon} \int_{I_n} \|a^{1/2} \nabla U\|^2 dt. \end{aligned} \quad (32)$$

Therefore, by taking $\varepsilon = 1/2$ in (32), we obtain

$$\int_{I_n} \left[\|\Sigma\|^2 + 2b_1 \|\nabla \cdot \Sigma\|^2 \right] dt \leq a_1 b_1 \int_{I_n} \|\nabla U\|^2 dt. \quad (33)$$

Taking (33) into (30) leads to

$$\|\nabla U^{n+1}\|^2 \leq C \int_{I_n} \|\nabla U\|^2 dt. \quad (34)$$

Then, using Lemma 2, we know $\|\nabla U^{n+1}\| = 0$. That is to say, $\nabla U^{n+1} = 0$. So we can obtain

$$\int_{I_n} \nabla U_t dt = \nabla U^{n+1} - \nabla U^n = 0, \quad \int_{I_n} \|\nabla U_t\|^2 dt = 0. \quad (35)$$

Taking (35) into the following inequality

$$\|\nabla U\|^2 \leq 2 \left(\|\nabla U^{n+1}\|^2 + \tau_n \int_{I_n} \|\nabla U_t\|^2 dt \right), \quad t_n \leq t \leq t_{n+1}, \quad (36)$$

we obtain $\|\nabla U\| = 0$; that is, $\nabla U = 0$. Then, using Poincaré inequality, we have $\|U\| = 0$; that is, $U = 0$.

Finally, taking $\|\nabla U\| = 0$ in (33), we have $\|\Sigma\|_{H(\text{div}; \Omega)} = 0$; that is to say, $\Sigma = 0$. So the existence and uniqueness of the solutions are proved. $\square$

4. H^1 -Norm Error Estimates

Now, we turn to consider the error estimates of (18)-(19). In order to derive H^1 -norm error estimates, we adopt the nonstandard elliptic projection $P_h : u \in [0, T] \rightarrow V_h$ such that [9]

$$\begin{aligned} & (a \nabla (P_h u - u), \nabla v) + (b \nabla (P_h u - u)_t, \nabla v) \\ & + \lambda ((P_h u - u)_t, v) = 0, \quad \forall v \in V_h, \end{aligned} \quad (37)$$

where constant $\lambda > 0$.

Then, we have some properties of $P_h u \in V_h$ as the following lemmas.

Lemma 4 (see [9]). There exists a unique solution $P_h u$ of (37), for any $\lambda > 0$.

Lemma 5 (see [9]). Let $\pi = P_h u - u$. Assume that $\|u_{0,h} - u_0\| + h\|u_{0,h} - u_0\|_1 \leq Ch^{k+1}\|u_0\|_{k+1}$, $u, u_t \in H^{k+1}(\Omega)$, and $u, \nabla u, u_t$, and ∇u_t are bounded. Then, there exists a positive constant $C = C(u)$ such that

$$\begin{aligned} \|\pi\| + h \|\nabla \pi\| & \leq Ch^{k+1} \|u\|_{H^{k+1}}, \\ \|\pi_t\| + h \|\nabla \pi_t\| & \leq Ch^{k+1} \|u_t\|_{H^{k+1}}. \end{aligned} \quad (38)$$

Lemma 6 (see [9]). Assume that u , ∇u , u_t , and ∇u_t are bounded. Then, there exists a positive constant $C = C(u)$ such that

$$\|P_h u\|_{L^\infty} + \|\nabla P_h u\|_{L^\infty} + \|(P_h u)_t\|_{L^\infty} + \|\nabla (P_h u)_t\|_{L^\infty} \leq C(u). \quad (39)$$

Lemma 7 (see [9]). If the order of approximate accuracy of space V_h is $k+1$, there exists a positive constant C such that

$$\begin{aligned} \|\pi\|_{L^2(0,T;L^2(\Omega))} &\leq Ch^{k+1} \|u\|_{L^2(0,T;H^{k+1}(\Omega))}, \\ \|\pi_t\|_{L^2(0,T;L^2(\Omega))} &\leq Ch^{k+1} \|u_t\|_{L^2(0,T;H^{k+1}(\Omega))}. \end{aligned} \quad (40)$$

Let $I_h \sigma$ be the standard finite element interpolation of σ and $\chi = I_h \sigma - \sigma$. By the Lagrange interpolation theory [28, 30], we have the following.

Lemma 8. If the order of approximate accuracy of space $\mathbf{W}_h$ is $r+1$, there exists a positive constant C such that

$$\begin{aligned} \|\chi\| + h \|\chi\|_{\mathbf{H}(\text{div};\Omega)} &\leq Ch^{r+1} \|\sigma\|_{\mathbf{H}^{r+1}}, \\ \|\chi\|_{L^2(0,T;L^2(\Omega))} + h \|\chi\|_{L^2(0,T;\mathbf{H}(\text{div};\Omega))} &\leq Ch^{r+1} \|\sigma\|_{L^2(0,T;\mathbf{H}^{r+1}(\Omega))}. \end{aligned} \quad (41)$$

Assume $\tilde{U} \in V_{h,\tau}^n$, $\tilde{\Sigma} \in \mathbf{W}_{h,\tau}^n$ are the following interpolations of $P_h u$ and $I_h \sigma$ with respect to t , respectively:

$$\begin{aligned} \tilde{U}\left(t_{n+1} - \frac{j}{q-1}\tau_n\right) &= P_h u\left(t_{n+1} - \frac{j}{q-1}\tau_n\right), \\ 0 \leq j &\leq q-1, \end{aligned} \quad (42)$$

$$\begin{aligned} \tilde{\Sigma}\left(t_{n+1} - \frac{j}{q-1}\tau_n\right) &= I_h \sigma\left(t_{n+1} - \frac{j}{q-1}\tau_n\right), \\ 0 \leq j &\leq q-1. \end{aligned} \quad (43)$$

By the Lagrange interpolation theory [28, 30], the following holds.

Lemma 9. Let $\rho = \tilde{U} - P_h u$ and $\varphi = \tilde{\Sigma} - P_h \sigma$, where $\tilde{U}$ satisfies (42) and $\tilde{\Sigma}$ satisfies (43). Then, there exist

$$\begin{aligned} \sup_{I_n} \|D_t^i \rho\|_{H^s} &\leq C \tau_n^{j-i} \sup_{I_n} \|D_t^j u\|_{H^s}, \\ \int_{I_n} \|D_t^i \rho\|_{H^s}^2 dt &\leq C \tau_n^{2(j-i)} \int_{I_n} \|D_t^j u\|_{H^s}^2 dt, \\ \sup_{I_n} \|D_t^i \varphi\|_{\mathbf{H}^s} &\leq C \tau_n^{j-i} \sup_{I_n} \|D_t^j \sigma\|_{\mathbf{H}^s}, \\ \int_{I_n} \|D_t^i \varphi\|_{\mathbf{H}^s}^2 dt &\leq C \tau_n^{2(j-i)} \int_{I_n} \|D_t^j \sigma\|_{\mathbf{H}^s}^2 dt, \end{aligned} \quad (44)$$

where $0 \leq i \leq q-1$, $i+1 \leq j \leq q$, $-1 \leq s \leq 1$, and D_t^i denotes the i th order derivative with respect to t and C is a positive constant depending only on q .

Let $\theta = U - \tilde{U}$ and $\psi = \Sigma - \tilde{\Sigma}$. Then, we have the decompositions

$$\begin{aligned} U - u &= U - \tilde{U} + \tilde{U} - P_h u + P_h u - u = \theta + \rho + \pi, \\ \Sigma - \sigma &= \Sigma - \tilde{\Sigma} + \tilde{\Sigma} - I_h \sigma + I_h \sigma - \sigma = \psi + \varphi + \chi. \end{aligned} \quad (45)$$

Combining (6a) and (6b) with (18)-(19), we achieve the error equations

$$\begin{aligned} &\int_{I_n} [(b \nabla \theta_t, \nabla v) + (a \nabla \theta, \nabla v)] dt \\ &\quad + (b^n \nabla \theta_+^n, \nabla v_+^n) - (b^n \nabla \theta^n, \nabla v_+^n) \\ &= \int_{I_n} [-(b \nabla \rho_t, \nabla v) - (a \nabla \rho, \nabla v) + \gamma(\pi_t, v)] dt \\ &\quad + \int_{I_n} (\psi + \varphi + \chi, \nabla v) dt, \quad \forall v \in V_{h,\tau}^n, \end{aligned} \quad (46)$$

$$\begin{aligned} &\int_{I_n} [(\alpha \psi, \mathbf{w}) + (\nabla \cdot \psi, \nabla \cdot \mathbf{w})] dt \\ &= \int_{I_n} [-(\alpha(\varphi + \chi), \mathbf{w}) - (\nabla \cdot (\varphi + \chi), \nabla \cdot \mathbf{w})] dt \\ &\quad + \int_{I_n} [(\beta \nabla(\theta + \rho + \pi), \mathbf{w})] dt, \quad \forall \mathbf{w} \in \mathbf{W}_{h,\tau}^n. \end{aligned} \quad (47)$$

In order to get the optimal order estimation of t , we still need some lemmas. First of all, it is Lemma 1.5.6 from [31]. The proof of this lemma is so long that we only adopt it here for the sake of brevity.

Lemma 10. Let $\theta \in V_{h,\tau}^n$, $\rho, \rho_t \in L^2(I_n, H^1(\Omega))$, $\rho^n = \rho^{n+1} = 0$; $\pi, \pi_t \in L^2(I_n, L^2(\Omega))$. They satisfy $\theta + \rho + \pi = U - u$, $\psi + \varphi + \chi = \Sigma - \sigma$, and

$$\begin{aligned} &\int_{I_n} [(b \nabla \theta_t, \nabla v) + (a \nabla \theta, \nabla v)] dt \\ &\quad + (b^n \nabla \theta_+^n, \nabla v_+^n) - (b^n \nabla \theta^n, \nabla v_+^n) \\ &= \int_{I_n} [-(b \nabla \rho_t, \nabla v) - (a \nabla \rho, \nabla v) + \lambda(\pi_t, v)] dt \\ &\quad + \int_{I_n} (\psi + \varphi + \chi, \nabla v) dt, \quad \forall v \in V_{h,\tau}^n. \end{aligned} \quad (48)$$

Then, we have the following estimates:

$$\begin{aligned} &\int_{I_n} b_0 \|\nabla \theta_t\|^2 dt \\ &\leq C \int_{I_n} [\|\nabla \rho_t\|^2 + \|\nabla \rho\|^2 + \|\pi_t\|^2 \\ &\quad + \|\psi\|^2 + \|\varphi\|^2 + \|\chi\|^2] dt, \end{aligned}$$

$$\begin{aligned}
& \left\| (b^{n+1})^{1/2} \nabla \theta^{n+1} \right\|^2 + a_0 \int_{I_n} \|\nabla \theta\|^2 dt \\
& \leq \left\| (b^n)^{1/2} \nabla \theta^n \right\|^2 + C(\varepsilon) \\
& \quad \cdot \int_{I_n} [\|\nabla \rho_t\|^2 + \|\nabla \rho\|^2 + \|\pi_t\|^2 \\
& \quad + \|\psi\|^2 + \|\varphi\|^2 + \|\chi\|^2] dt, \\
& \left\| \nabla \theta^{n+1} \right\|^2 \\
& \leq \left\| \nabla \theta^0 \right\|^2 + \int_{I_n} [\|\nabla \rho_t\|^2 + \|\nabla \rho\|^2 \\
& \quad + \|\pi_t\|^2 + \|\psi\|^2 + \|\varphi\|^2 + \|\chi\|^2] dt.
\end{aligned} \tag{49}$$

Lemma 11. Under the conditions in Lemma 10, there also exist

$$\begin{aligned}
& \left\| (b^{n+1})^{1/2} \nabla \theta^{n+1} \right\|^2 + a_0 \int_{I_n} \|\nabla \theta\|^2 dt \\
& \leq \left\| (b^n)^{1/2} \nabla \theta^n \right\|^2 \\
& \quad + C \int_{I_n} \left\{ \tau_n [\|\nabla \rho_t\|^2 + \|\nabla \rho\|^2 + \|\pi_t\|^2 + \|\nabla \pi\|^2 \right. \\
& \quad + \|\varphi\|_{\mathbf{H}(\text{div}; \Omega)}^2 + \|\chi\|_{\mathbf{H}(\text{div}; \Omega)}^2] \\
& \quad + \tau_n^{-1} [\|\nabla \rho\|^2 + \|\pi_t\|^2 + \|\nabla \pi\|^2 \\
& \quad + \|\varphi\|_{\mathbf{H}(\text{div}; \Omega)}^2 + \|\chi\|_{\mathbf{H}(\text{div}; \Omega)}^2] \Big\} dt,
\end{aligned} \tag{50}$$

$$\begin{aligned}
& \tau_n \int_{I_n} \|\nabla \theta_t\|^2 dt + \tau_n \left\| \nabla \theta^{n+1} \right\|^2 \\
& \leq C \left\{ \left\| \nabla \theta^n \right\|^2 \right. \\
& \quad + \int_{I_n} \left\{ \tau_n [\|\nabla \rho_t\|^2 + \|\nabla \rho\|^2 \right. \\
& \quad + \|\pi_t\|^2 + \|\nabla \pi\|^2 + \|\varphi\|_{\mathbf{H}(\text{div}; \Omega)}^2 \\
& \quad + \|\chi\|_{\mathbf{H}(\text{div}; \Omega)}^2] \\
& \quad + \tau_n^{-1} [\|\nabla \rho\|^2 + \|\pi_t\|^2 \\
& \quad + \|\nabla \pi\|^2 + \|\varphi\|_{\mathbf{H}(\text{div}; \Omega)}^2 + \|\chi\|_{\mathbf{H}(\text{div}; \Omega)}^2] \Big\} dt \Big\},
\end{aligned} \tag{51}$$

$$\begin{aligned}
& \int_{I_n} [\|\psi\|^2 + \|\nabla \cdot \psi\|^2] dt \leq \left\| \nabla \theta^n \right\|^2 \\
& \quad + \int_{I_n} \left\{ \|\varphi\|_{\mathbf{H}(\text{div}; \Omega)}^2 + \|\chi\|_{\mathbf{H}(\text{div}; \Omega)}^2 \right. \\
& \quad + \|\nabla \rho\|^2 + \|\nabla \pi\|^2 \\
& \quad + \tau_n [\|\nabla \rho_t\|^2 + \|\nabla \rho\|^2 + \|\pi_t\|^2 + \|\varphi\|^2 + \|\chi\|^2] \\
& \quad + \tau_n^{-1} [\|\nabla \rho\|^2 + \|\pi_t\|^2 + \|\varphi\|^2 + \|\chi\|^2] \Big\} dt.
\end{aligned} \tag{52}$$

Proof. By Lemma 10, we know that there exist

$$\begin{aligned}
& \left\| (b^{n+1})^{1/2} \nabla \theta^{n+1} \right\|^2 + a_0 \int_{I_n} \|\nabla \theta\|^2 dt \\
& \leq \left\| (b^n)^{1/2} \nabla \theta^n \right\|^2 \\
& \quad + C \int_{I_n} \left\{ \tau_n [\|\nabla \rho_t\|^2 + \|\nabla \rho\|^2 + \|\pi_t\|^2 \right. \\
& \quad + \|\psi\|^2 + \|\varphi\|^2 + \|\chi\|^2] \\
& \quad + \tau_n^{-1} [\|\nabla \rho\|^2 + \|\pi_t\|^2 \\
& \quad + \|\psi\|^2 + \|\varphi\|^2 + \|\chi\|^2] \Big\} dt, \\
& \tau_n \int_{I_n} \|\nabla \theta_t\|^2 dt + \tau_n \left\| \nabla \theta^{n+1} \right\|^2 \\
& \leq C \left\{ \left\| \nabla \theta^n \right\|^2 \right. \\
& \quad + \int_{I_n} \left\{ \tau_n [\|\nabla \rho_t\|^2 + \|\nabla \rho\|^2 + \|\pi_t\|^2 \right. \\
& \quad + \|\psi\|^2 + \|\varphi\|^2 + \|\chi\|^2] \\
& \quad + \tau_n^{-1} [\|\nabla \rho\|^2 + \|\pi_t\|^2 \\
& \quad + \|\psi\|^2 + \|\varphi\|^2 + \|\chi\|^2] \Big\} dt \Big\}.
\end{aligned} \tag{53}$$

Taking $\mathbf{w} = \psi$ in (47) and letting ε be small enough, we can get

$$\begin{aligned}
& \int_{I_n} [\|\psi\|^2 + \|\nabla \cdot \psi\|^2] dt \\
& \leq \int_{I_n} [\|\varphi\|^2 + \|\chi\|^2 + \|\nabla \cdot \varphi\|^2 + \|\nabla \cdot \chi\|^2 \\
& \quad + \|\nabla \rho\|^2 + \|\nabla \pi\|^2] dt + \varepsilon \int_{I_n} \|\nabla \theta\|^2 dt.
\end{aligned} \tag{54}$$

Taking (53) into (55) and letting ε be small enough, we obtain (52).

Then, taking (55) back into (53) and (54) and also letting ε be small enough, we finally achieve (50) and (51), respectively. The proof of Lemma 11 is completed. $\square$

Lemma 12. Using Lemmas 10 and 11, we can get the following estimates for the error equations (46)-(47):

$$\begin{aligned}
& \|\theta^{n+1}\|_1^2 + a_0 \int_0^{t_{n+1}} \|\theta\|_1^2 dt \\
& \leq \|\theta^0\|_1^2 + C \left\{ \tau^{2q} \left\{ \|\nabla (D_t^q u)\|_{L^\infty(0,t_{n+1};L^2(\Omega))}^2 \right. \right. \\
& \quad + \|u_t\|_{L^2(0,t_{n+1};H^{k+1}(\Omega))}^2 \\
& \quad + \|D_t^q \sigma\|_{L^\infty(0,t_{n+1};\mathbf{H}^1(\Omega))}^2 \\
& \quad \left. + \|\sigma\|_{L^2(0,t_{n+1};\mathbf{H}^{r+1}(\Omega))}^2 \right\} \\
& \quad + h^{2(k+1)} \|u_t\|_{L^\infty(0,t_{n+1};H^{k+1}(\Omega))}^2 \\
& \quad + h^{2r} \|\sigma\|_{L^\infty(0,t_{n+1};\mathbf{H}^{r+1}(\Omega))}^2 \Big\},
\end{aligned} \tag{56}$$

$$\begin{aligned}
& \|\theta\|_{L^\infty(0,T;H^1(\Omega))}^2 \\
& \leq \|\theta^0\|_1^2 + C \left\{ \tau^{2q} \left\{ \|\nabla (D_t^q u)\|_{L^\infty(0,T;L^2(\Omega))}^2 \right. \right. \\
& \quad + \|u_t\|_{L^2(0,T;H^{k+1}(\Omega))}^2 \\
& \quad + \|D_t^q \sigma\|_{L^\infty(0,T;\mathbf{H}^1(\Omega))}^2 \\
& \quad \left. + \|\sigma\|_{L^2(0,T;\mathbf{H}^{r+1}(\Omega))}^2 \right\} \\
& \quad + h^{2(k+1)} \|u_t\|_{L^\infty(0,T;H^{k+1}(\Omega))}^2 \\
& \quad + h^{2r} \|\sigma\|_{L^\infty(0,T;\mathbf{H}^{r+1}(\Omega))}^2 \Big\},
\end{aligned} \tag{57}$$

$$\begin{aligned}
& \|\psi\|_{L^2(0,T;\mathbf{H}(\text{div};\Omega))}^2 \\
& \leq \|\theta^0\|_1^2 + C \left\{ \tau^{2q} \left\{ \|\nabla (D_t^q u)\|_{L^\infty(0,T;L^2(\Omega))}^2 \right. \right. \\
& \quad + \|\nabla (D_t^q u)\|_{L^2(0,T;L^2(\Omega))}^2 \\
& \quad + \|u_t\|_{L^2(0,T;H^{k+1}(\Omega))}^2 \\
& \quad + \|D_t^q \sigma\|_{L^\infty(0,T;\mathbf{H}^1(\Omega))}^2 \\
& \quad + \|D_t^q \sigma\|_{L^2(0,T;\mathbf{H}^1(\Omega))}^2 \\
& \quad \left. + \|\sigma\|_{L^2(0,T;\mathbf{H}^{r+1}(\Omega))}^2 \right\} \\
& \quad + h^{2k} \|u_t\|_{L^2(0,T;H^{k+1}(\Omega))}^2 \\
& \quad + h^{2r} \left(\|\sigma\|_{L^2(0,T;\mathbf{H}^{r+1}(\Omega))}^2 \right. \\
& \quad \left. + \|\sigma\|_{L^\infty(0,T;\mathbf{H}^{r+1}(\Omega))}^2 \right) \Big\}.
\end{aligned} \tag{58}$$

Proof. (1) Using (50), we have

$$\begin{aligned}
& \left\| (b^{n+1})^{1/2} \nabla \theta^{n+1} \right\|^2 + a_0 \int_0^{t_{n+1}} \|\nabla \theta\|^2 dt \\
& \leq \left\| (b^0)^{1/2} \nabla \theta^0 \right\|^2 \\
& \quad + C \sum_{j=0}^n \int_{I_j} \left\{ \tau_j \left[\|\nabla \rho_t\|^2 + \|\nabla \rho\|^2 + \|\pi_t\|^2 \right. \right. \\
& \quad \left. \left. + \|\nabla \pi\|^2 + \|\varphi\|_{\mathbf{H}(\text{div};\Omega)}^2 + \|\chi\|_{\mathbf{H}(\text{div};\Omega)}^2 \right] \right. \\
& \quad \left. + \tau_j^{-1} \left[\|\nabla \rho\|^2 + \|\pi_t\|^2 + \|\nabla \pi\|^2 \right. \right. \\
& \quad \left. \left. + \|\varphi\|_{\mathbf{H}(\text{div};\Omega)}^2 + \|\chi\|_{\mathbf{H}(\text{div};\Omega)}^2 \right] \right\} dt.
\end{aligned} \tag{59}$$

Then, using Lemmas 5, 8, and 9, we can derive

$$\begin{aligned}
& \int_{I_j} \left[\tau_j \|\nabla \rho_t\|^2 + \tau_j^{-1} \|\nabla \rho\|^2 \right] dt \leq C \tau_j^{2q} \sup_{I_j} \|\nabla (D_t^q u)\|^2, \\
& \int_{I_j} \tau_j \|\nabla \rho\|^2 dt \leq C \tau_j^{2q+1} \|\nabla (D_t^q u)\|_{L^2(I_j;L^2(\Omega))}^2, \\
& \int_{I_j} \tau_j \|\pi_t\|^2 dt \leq C \int_{I_j} \tau_j h^{2(k+1)} \|u_t\|_{H^{k+1}(\Omega)}^2 dt \\
& \leq C \left(\tau_j^{2q} + h^{2(k+1)(2q/(2q-1))} \right) \|u_t\|_{L^2(I_j;H^{k+1}(\Omega))}^2, \\
& \int_{I_n} \tau_j^{-1} \|\pi_t\|^2 dt \leq C h^{2(k+1)} \sup_{I_j} \|u_t\|_{H^{k+1}(\Omega)}^2, \\
& \int_{I_j} \left[\tau_j \|\varphi\|_{\mathbf{H}(\text{div};\Omega)}^2 + \tau_j^{-1} \|\varphi\|_{\mathbf{H}(\text{div};\Omega)}^2 \right] dt \\
& \leq C \tau_j^{2q+1} \|D_t^q \sigma\|_{L^2(I_j;\mathbf{H}^1(\Omega))}^2 + C \tau_j^{2q} \sup_{I_j} \|D_t^q \sigma\|_{\mathbf{H}^1(\Omega)}^2, \\
& \int_{I_j} \left[\tau_j \|\chi\|_{\mathbf{H}(\text{div};\Omega)}^2 + \tau_j^{-1} \|\chi\|_{\mathbf{H}(\text{div};\Omega)}^2 \right] dt \\
& \leq C \tau_j h^{2r} \|\sigma\|_{L^2(I_j;\mathbf{H}^{r+1}(\Omega))}^2 + C h^{2r} \sup_{I_j} \|\sigma\|_{\mathbf{H}^{r+1}(\Omega)}^2 \\
& \leq C \left(\tau_j^{2q} + h^{2r(2q/(2q-1))} \right) \|\sigma\|_{L^2(I_j;\mathbf{H}^{r+1}(\Omega))}^2 \\
& \quad + C h^{2r} \sup_{I_j} \|\sigma\|_{\mathbf{H}^{r+1}(\Omega)}^2.
\end{aligned} \tag{60}$$

Taking the above estimates into (59) leads to

$$\begin{aligned}
& \left\| (b^{n+1})^{1/2} \nabla \theta^{n+1} \right\|^2 + a_0 \int_0^{t_{n+1}} \|\nabla \theta\|^2 dt \\
& \leq \left\| (b^0)^{1/2} \nabla \theta^0 \right\|^2
\end{aligned}$$

$$\begin{aligned}
& + C \left\{ \tau^{2q} \|\nabla (D_t^q u)\|_{L^\infty(0,t_{n+1};L^2(\Omega))}^2 \right. \\
& \quad + \tau^{2q+1} \|\nabla (D_t^q u)\|_{L^2(0,t_{n+1};L^2(\Omega))}^2 \\
& \quad + \left(\tau^{2q} + h^{2(k+1)(2q/(2q-1))} \right) \|u_t\|_{L^2(0,t_{n+1};H^{k+1}(\Omega))}^2 \\
& \quad + h^{2(k+1)} \|u_t\|_{L^\infty(0,t_{n+1};H^{k+1}(\Omega))}^2 \\
& \quad + \tau^{2q+1} \|D_t^q \sigma\|_{L^2(0,t_{n+1};\mathbf{H}^1(\Omega))}^2 \\
& \quad + \tau^{2q} \|D_t^q \sigma\|_{L^\infty(0,t_{n+1};\mathbf{H}^1(\Omega))}^2 \\
& \quad + \left(\tau^{2q} + h^{2r(2q/(2q-1))} \right) \|\sigma\|_{L^2(0,t_{n+1};\mathbf{H}^{r+1}(\Omega))}^2 \\
& \quad \left. + h^{2r} \|\sigma\|_{L^\infty(0,t_{n+1};\mathbf{H}^{r+1}(\Omega))}^2 \right\}. \tag{61}
\end{aligned}$$

Then, using the Poincaré inequality $\|\theta\| \leq C\|\nabla\theta\|$, $\forall \theta \in H_0^1(\Omega)$, we obtain the estimate (56).

(2) For $\forall t \in I_n$, we know

$$\|\nabla\theta\|^2 \leq 2 \left\{ \|\nabla\theta^{n+1}\|^2 + \tau_n \int_{I_n} \|\nabla\theta_t\|^2 dt \right\}. \tag{62}$$

Further, use (51) and (59) to get

$$\begin{aligned}
& \|\nabla\theta\|^2 \\
& \leq \left\| (b^0)^{1/2} \nabla\theta^0 \right\|^2 \\
& \quad + C \sum_{j=0}^n \int_{I_j} \left\{ \tau_j \left[\|\nabla\rho_t\|^2 + \|\nabla\rho\|^2 \right. \right. \\
& \quad \quad \left. \left. + \|\pi_t\|^2 + \|\varphi\|^2 + \|\chi\|^2 \right] \right. \\
& \quad \quad \left. + \tau_j^{-1} \left[\|\nabla\rho\|^2 + \|\pi_t\|^2 \right. \right. \\
& \quad \quad \left. \left. + \|\varphi\|^2 + \|\chi\|^2 \right] \right\} dt. \tag{63}
\end{aligned}$$

Similar to the analysis of getting (56), we obtain the estimate (57).

(3) Similarly, using (52), (56) and Lemmas 8 and 9, we have

$$\begin{aligned}
& \int_{I_n} \left[\|\psi\|^2 + \|\nabla \cdot \psi\|^2 \right] dt \\
& \leq \|\theta^0\|_1^2 + C \left\{ \tau^{2q} \left\{ \|\nabla (D_t^q u)\|_{L^\infty(0,t_{n+1};L^2(\Omega))}^2 \right. \right. \\
& \quad \left. \left. + \|\nabla (D_t^q u)\|_{L^2(0,t_{n+1};L^2(\Omega))}^2 \right. \right. \\
& \quad \left. \left. + \|u_t\|_{L^2(0,t_{n+1};H^{k+1}(\Omega))}^2 \right. \right.
\end{aligned}$$

$$\begin{aligned}
& \quad \left. + \|D_t^q \sigma\|_{L^\infty(0,t_{n+1};\mathbf{H}^1(\Omega))}^2 \right. \\
& \quad \left. + \|D_t^q \sigma\|_{L^2(0,t_{n+1};\mathbf{H}^1(\Omega))}^2 \right. \\
& \quad \left. + \|\sigma\|_{L^2(0,t_{n+1};\mathbf{H}^{r+1}(\Omega))}^2 \right\} \\
& \quad + h^{2k} \|u_t\|_{L^2(0,t_{n+1};H^{k+1}(\Omega))}^2 \\
& \quad + h^{2r} \left(\|\sigma\|_{L^2(0,t_{n+1};\mathbf{H}^{r+1}(\Omega))}^2 \right. \\
& \quad \left. + \|\sigma\|_{L^\infty(0,t_{n+1};\mathbf{H}^{r+1}(\Omega))}^2 \right) \}. \tag{64}
\end{aligned}$$

So, we obtain the estimate (58). Then, the proof of Lemma 12 is completed. $\square$

Finally, by the triangle inequality and Lemmas 5, 7, 8, 9, and 12, we can establish the following theorem.

Theorem 13. *Let u be the solution of the original problem (1a), (1b), and (1c); U and Σ are the finite element solutions of the approximate schemes (14)-(15). Assume that u is smooth enough to satisfy the required regularities in the analyses, and $\|u_{0,h} - u_0\| + h\|u_{0,h} - u_0\|_1 \leq Ch^{k+1}\|u_0\|_{k+1}$. Then, we have the following error estimates:*

$$\begin{aligned}
& \|u - U\|_{L^\infty(0,T;H^1(\Omega))}^2 + a_0 \|u - U\|_{L^2(0,T;H^1(\Omega))}^2 \\
& \leq C \left\{ \tau^{2q} \left\{ \|D_t^q u\|_{L^\infty(0,T;H^1(\Omega))}^2 \right. \right. \\
& \quad \left. \left. + \|u_t\|_{L^2(0,T;H^{k+1}(\Omega))}^2 + \|D_t^q \sigma\|_{L^\infty(0,T;L^2(\Omega))}^2 \right. \right. \\
& \quad \left. \left. + \|\sigma\|_{L^2(0,T;\mathbf{H}^{r+1}(\Omega))}^2 \right\} \right. \\
& \quad \left. + h^{2k} \left\{ \|u_t\|_{L^\infty(0,T;H^{k+1}(\Omega))}^2 + \|u\|_{L^\infty(0,T;H^{k+1}(\Omega))}^2 \right. \right. \\
& \quad \left. \left. + \|u\|_{L^2(0,T;H^{k+1}(\Omega))}^2 + \|u_0\|_{H^{k+1}(\Omega)}^2 \right\} \right. \\
& \quad \left. + h^{2r} \|\sigma\|_{L^2(0,T;\mathbf{H}^{r+1}(\Omega))}^2 \right\}, \\
& \|\sigma - \Sigma\|_{L^2(0,T;\mathbf{H}(\text{div};\Omega))}^2 \\
& \leq C \left\{ \tau^{2q} \left\{ \|\nabla (D_t^q u)\|_{L^\infty(0,T;L^2(\Omega))}^2 + \|\nabla (D_t^q u)\|_{L^2(0,T;L^2(\Omega))}^2 \right. \right. \\
& \quad \left. \left. + \|u_t\|_{L^2(0,T;H^{k+1}(\Omega))}^2 + \|D_t^q \sigma\|_{L^\infty(0,T;L^2(\Omega))}^2 \right. \right. \\
& \quad \left. \left. + \|D_t^q \sigma\|_{L^2(0,T;\mathbf{H}^1(\Omega))}^2 + \|\sigma\|_{L^2(0,T;\mathbf{H}^{r+1}(\Omega))}^2 \right\} \right. \\
& \quad \left. + h^{2k} \left(\|u_t\|_{L^2(0,T;H^{k+1}(\Omega))}^2 + \|u_0\|_{H^{k+1}(\Omega)}^2 \right) \right. \\
& \quad \left. + h^{2r} \left(\|\sigma\|_{L^\infty(0,T;\mathbf{H}^{r+1}(\Omega))}^2 + \|\sigma\|_{L^2(0,T;\mathbf{H}^{r+1}(\Omega))}^2 \right) \right\}. \tag{65}
\end{aligned}$$

From this theorem, we see that with respect to space variable x the optimal H^1 -norm error estimates of both u and σ are derived. Also, q -order accuracy for time variable

t is obtained. That is to say, we get high accuracy for both the space and time variables.

5. Conclusions

In this paper, we have solved the linear Sobolev equations (1a), (1b), and (1c) by the time discontinuous H^1 -Galerkin mixed finite element method. We have presented the approximate schemes, proved the existence and uniqueness of the solutions, and derived the optimal H^1 -norm error estimates for both the primary variable u and intermediate variable σ . In particular, high-order accuracy for time variable also is obtained. In our forthcoming work, some numerical experiments will be considered to verify our analysis for the schemes.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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