Research Article

A New Approach to Global Stability of Discrete Lotka-Volterra Predator-Prey Models

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An Euler difference scheme for a three-dimensional predator-prey model is considered and we introduce a new approach to show the global stability of the scheme. For this purpose, we partition the three-dimensional space and calculate the sign of the rate change of population of species in each partitioned region. Our method is independent of dimension and then can be applicable to other dimensional discrete models. Numerical examples are presented to verify the results in this paper.

1. Introduction

Vito Volterra proposed a differential equation model to explain the observed increase in predator fish and corresponding decrease in prey fish in 1926. At the same time, the equations studied by Volterra were derived independently by Alfred Lotka (1925) to describe a chemical reaction. Many predator-prey models have been studied and a classic predator-prey model is given by

$$\begin{aligned}\frac{dx}{dt} &= x(r_1 - a_{11}x - a_{12}y), \\ \frac{dy}{dt} &= y(-r_2 + a_{21}x - a_{22}y),\end{aligned}\tag{1}$$

where $r_i > 0$, $a_{ij} > 0$, and x and y denote the population sizes of the prey and predator, respectively.

It is recognized that the rate of prey capture per predator cannot increase indefinitely as the number of prey increases. Instead, the rate of prey capture is saturated when the population of prey is relatively large. Then such nonlinear functional responses are employed to describe the phenomena of predation including the Holling types [1–5], Beddington-DeAngelis type [6–8], Crowley-Martin type [9–11], and Ivlev type of functional response [12–14].

In particular, similar phenomena are observed in the interactions in chemical reactions and molecular events when one species is abundant. Thus linear response function, Michaelis-Menten kinetics, and Hill function are related to Holling types I, II, and III, respectively. Holling type IV is also called the Monod-Haldane function [4, 5].

The functional responses have been applied to predator-prey models to express the Allee effect [15–19], which describes a positive relation between population density and the per capita growth rate.

Most of researches on the predator-prey models assume that the distribution of the predators and prey is homogeneous, which leads to ordinary differential equations. However both predators and prey have the natural tendency to diffuse, so that there have been models to take into account the inhomogeneous distribution of the predators and prey [20–22].

On the other hand, population is inevitably affected by environmental noise in nature. Therefore, many authors have taken stochastic perturbation into deterministic models [23–25].

There are a number of works investigating continuous time predator-prey models, but relatively few theoretical papers are published on their discretized models [26, 27]. As far as we know, there is no theoretical research on the

global stability of the discrete-time models of type (1) with more than two species except for [28]. The author in [28] introduced a method to present global stability for the case that all species coexist at a unique equilibrium. Then a new approach needs to be developed for the other cases.

For explaining our new approach, we consider a model with one prey and two predators:

$$\frac{dx^i}{dt} = x^i \left(\sigma_i r_i + \sum_{1 \leq j \leq i-1} a_{ij} x^j - \sum_{i \leq j \leq 3} a_{ij} x^j \right), \quad (2)$$

where $i = 1, 2, 3$, $\sigma_1 = 1$, $\sigma_2 = \sigma_3 = -1$, and

$$\frac{r_3}{a_{31}} < \frac{r_1}{a_{11}} < \frac{r_2}{a_{21}}, \quad (3)$$

$$\frac{r_1}{a_{12}} < \frac{r_3}{a_{32}}, \quad (4)$$

$$\frac{a_{21}}{a_{23}} < \frac{a_{31}}{a_{33}}. \quad (5)$$

Here x^1 denotes the population number of the prey; x^2 and x^3 denote the population numbers of the predators. Letting

$$f_i(x^1, x^2, x^3) = \sigma_i r_i + \sum_{1 \leq j \leq i-1} a_{ij} x^j - \sum_{i \leq j \leq 3} a_{ij} x^j, \quad (6)$$

the Euler difference scheme for (2) is as follows:

$$\frac{\Delta x_k^i}{\Delta t} = x_k^i f_i(x_k^1, x_k^2, x_k^3), \quad k = 0, 1, \dots, \quad (7)$$

where $x_0^i > 0$ ($i = 1, 2, 3$), $\Delta x_k^i = x_{k+1}^i - x_k^i$, and Δt is a step size.

The three conditions (3)–(5) mean that the two planes $f_1(x, y, z) = 0$ and $f_3(x, y, z) = 0$ intersect in the first octant, and the plane $f_2(x, y, z) = 0$ is not intersected with either $f_1(x, y, z) = 0$ or $f_3(x, y, z) = 0$ in the first octant. For example, let $(r_i, a_{i1}, a_{i2}, a_{i3})$ ($1 \leq i \leq 3$) be

$$\begin{aligned} &(2, 1, 2, 2), \\ &(3, 1, 1, 1), \\ &(2, 2, 1, 1), \end{aligned} \quad (8)$$

respectively. Then the three conditions (3)–(5) are satisfied and Figure 1 shows the three planes and regions with the globally stable point of scheme (7). The global stability of the point will be shown in Section 3.

Using Theorem 4.1 in [28], we have the positivity and boundedness of the solutions of scheme (7). Letting χ_i ($1 \leq i \leq 3$) satisfy

$$\begin{aligned} \chi_i &< \frac{1 + \sigma_i r_i \Delta t - \sum_{i+1 \leq j \leq n} a_{ij} \chi_j \Delta t}{2a_{ii} \Delta t}, \\ \frac{\sigma_i r_i + \sum_{1 \leq j \leq i-1} a_{ij} \chi_j}{a_{ii}} &< \chi_i, \\ \left(\sigma_i r_i + \sum_{1 \leq j \leq i-1} a_{ij} \chi_j \right) \Delta t &< 1, \end{aligned} \quad (9)$$

we have that for all k

$$(x_k^1, x_k^2, x_k^3) \in (0, \chi_1) \times (0, \chi_2) \times (0, \chi_3). \quad (10)$$

For small values of Δt , there exist infinitely many χ_i ($1 \leq i \leq 3$) satisfying (9).

Consider the five regions

$$\begin{aligned} \text{I} &= \{(x, y, z) \in \mathcal{S} \mid f_1(x, y, z) \geq 0, f_2(x, y, z) < 0, f_3(x, y, z) \leq 0\}, \\ \text{II} &= \{(x, y, z) \in \mathcal{S} \mid f_1(x, y, z) \geq 0, f_2(x, y, z) < 0, f_3(x, y, z) \geq 0\}, \\ \text{III} &= \{(x, y, z) \in \mathcal{S} \mid f_1(x, y, z) \leq 0, f_2(x, y, z) < 0, f_3(x, y, z) \leq 0\}, \\ \text{IV} &= \{(x, y, z) \in \mathcal{S} \mid f_1(x, y, z) \leq 0, f_2(x, y, z) \leq 0, f_3(x, y, z) \geq 0\}, \\ \text{V} &= \{(x, y, z) \in \mathcal{S} \mid f_1(x, y, z) < 0, f_2(x, y, z) \geq 0, f_3(x, y, z) > 0\}, \end{aligned} \quad (11)$$

where $\mathcal{S} = \{(x, y, z) \in \mathbb{R}^3 \mid x > 0, y > 0, z > 0\} \cap \prod_{1 \leq i \leq 3} (0, \chi_i)$ (see Figure 1).

For convenience we denote the set I by

$$\begin{aligned} &\text{I}(+, -, -) \\ &\text{or } (+, -, -), \end{aligned} \quad (12)$$

and then

$$\begin{aligned} &P \in \text{I}(+, -, -) \\ &\text{iff } f_1(P) \geq 0, f_2(P) < 0, f_3(P) \leq 0. \end{aligned} \quad (13)$$

We adapt similar notations for the other regions II to V.

Note that region I has the property

$$\sup \{x \mid (x, y, z) \in \text{I}(+, -, -)\} \leq \frac{r_1}{a_{11}}, \quad (14)$$

since $f_1(x, y, z) = r_1 - a_{11}x - a_{12}y - a_{13}z \geq 0$ for all $(x, y, z) \in \text{I}(+, -, -)$.

2. Method

In this section we present theorems that describe our approach. The theorems will be used to obtain the global stability of scheme (7) in the next section.

Assume that Δt satisfies

$$\begin{aligned} &\Delta t \left\{ \chi_1 \left(a_{11} + a_{13} \frac{a_{21}}{a_{23}} \right) + \chi_2 a_{22} + \chi_3 \left(a_{33} + a_{32} \frac{a_{13}}{a_{12}} \right) \right\} \\ &< 1, \end{aligned} \quad (15)$$

$$\frac{\Delta t}{1 - a_{33} \chi_3 \Delta t} a_{31} \chi_1^2 < \frac{a_{31} r_2 - a_{21} r_3}{a_{21}}. \quad (16)$$

FIGURE 1: Three planes and regions with the globally stable point. (a) The three planes $f_i(x, y, z) = 0$ ($1 \leq i \leq 3$) in $(0, 4) \times (0, 3) \times (0, 2)$. The magenta circle denotes the globally stable point $(6/5, 0, 2/5)$ of scheme (7). (b) Five regions projected on the xy -plane. (c) Five regions projected on the xz -plane.

Let $P = (P_1, P_2, P_3)$ and $T = (T_1, T_2, T_3)$ be the vector function defined on $\mathcal{S}$ by

$$T_i(P) = P_i \{1 + \Delta t f_i(P)\}, \quad i = 1, 2, 3. \quad (17)$$

Then scheme (7) can be written as

$$(x_{k+1}^1, x_{k+1}^2, x_{k+1}^3) = T(x_k^1, x_k^2, x_k^3). \quad (18)$$

Note that the map T has the three fixed points with all nonnegative components, $(0, 0, 0)$, $(r_1/a_{11}, 0, 0)$, and $\vartheta = (\vartheta_1, 0, \vartheta_3)$, satisfying

$$f_1(\vartheta) = f_3(\vartheta) = 0. \quad (19)$$

Then the fixed point ϑ with $\vartheta_1 = (r_1 a_{33} + r_3 a_{13}) / (a_{11} a_{33} + a_{31} a_{13})$ and $\vartheta_3 = (r_1 a_{31} - r_3 a_{11}) / (a_{13} a_{31} + a_{33} a_{11})$ is locally stable as follows.

Lemma 1. *Let ϑ be the fixed point of the map T satisfying (19). Assume that Δt satisfies (15). Then the fixed point ϑ is asymptotically stable.*

Proof. Applying the linearization method to the discrete system of (7) at ϑ , the matrix of the linearized system has the three eigenvalues

$$\begin{aligned} &1 + f_2(\vartheta) \Delta t, \\ &1 - \frac{1}{2} (\vartheta_1 a_{11} + \vartheta_3 a_{33}) \Delta t \\ &\pm \frac{1}{2} \left\{ (\vartheta_1 a_{11} - \vartheta_3 a_{33})^2 - 4 \vartheta_1 \vartheta_3 a_{13} a_{31} \right\}^{1/2} \Delta t. \end{aligned} \quad (20)$$

Substituting $\vartheta_1 = (r_1 a_{33} + r_3 a_{13}) / (a_{11} a_{33} + a_{31} a_{13})^{-1}$ to

$$f_2(\vartheta) = -r_2 + a_{21} \vartheta_1 - a_{23} \vartheta_3 < -r_2 + a_{21} \vartheta_1, \quad (21)$$

and using (3), we obtain

$$f_2(\vartheta) < 0. \quad (22)$$

In addition, (15) gives that the other two eigenvalues have magnitudes less than 1. Hence the spectral radius of the matrix is less than 1, which completes the proof. $\square$

The two fixed points $(0, 0, 0)$ and $(r_1/a_{11}, 0, 0)$ are unstable since the matrices of the linearized system at $(0, 0, 0)$ and $(r_1/a_{11}, 0, 0)$ have the eigenvalues $1 + f_1(0, 0, 0) \Delta t$ and $1 + f_3(r_1/a_{11}, 0, 0) \Delta t$ with $f_1(0, 0, 0) > 0$ and $f_3(r_1/a_{11}, 0, 0) > 0$, respectively. Then we only consider the fixed point $\vartheta = (\vartheta_1, 0, \vartheta_3)$ of the map T to show global stability in the next section.

Our methodology to obtain the global stability is based on the approach to determine regions among regions I to V in which $T(P)$ is contained by calculating the sign of $f_i(T(P))$ for P in each region. Then we use the sign symbol $P \in (+, *, *)$ as follows.

$P \in (+, *, *)$ if and only if $f_1(P) \geq 0$ and the signs of $f_i(P)$ ($i = 2, 3$) are unknown.

Other sign symbols are similarly defined.

Theorem 2. *Suppose that Δt satisfies (9) and (15). Let $s \in \{+, -\}$:*

$$\text{If } P \in (s, -s, -s), \text{ then the sign of } f_1(T(P)) \text{ is } s. \quad (23)$$

Proof. Let $P = (P_1, P_2, P_3)$. Using the definition of f_1 , we have

$$\begin{aligned} f_1(T(P)) &= r_1 - a_{11} T_1(P) - a_{12} T_2(P) - a_{13} T_3(P) \\ &= r_1 - \sum_{1 \leq i \leq 3} a_{1i} P_i \{1 + \Delta t f_i(P)\} \\ &= r_1 - \sum_{1 \leq i \leq 3} a_{1i} P_i - \sum_{1 \leq i \leq 3} \Delta t a_{1i} P_i f_i(P) \\ &= f_1(P) - \sum_{1 \leq i \leq 3} \Delta t a_{1i} P_i f_i(P) \\ &= (1 - \Delta t a_{11} P_1) f_1(P) \\ &\quad - \Delta t \{a_{12} P_2 f_2(P) + a_{13} P_3 f_3(P)\}. \end{aligned} \quad (24)$$

Note that if $f_2(x, y, z) = 0$ for nonnegative x, y , and z , then

$$x = \frac{1}{a_{21}} (r_2 + a_{22} y + a_{23} z), \quad (25)$$

with which the nonnegativity of x, y , and z gives

$$\begin{aligned} f_1(x, y, z) &= r_1 - a_{11} x - a_{12} y - a_{13} z \\ &= r_1 - a_{11} \frac{1}{a_{21}} (r_2 + a_{22} y + a_{23} z) - a_{12} y \\ &\quad - a_{13} z \leq r_1 - a_{11} \frac{1}{a_{21}} r_2 \\ &= a_{11} \left(\frac{r_1}{a_{11}} - \frac{r_2}{a_{21}} \right) < 0, \end{aligned} \quad (26)$$

and similarly

$$f_3(x, y, z) > 0, \quad (27)$$

due to (3). Hence there exist no nonnegative numbers x, y , and z such that $f_1(x, y, z) = f_2(x, y, z) = f_3(x, y, z) = 0$. Since $P \in (s, -s, -s)$ and it is impossible that $f_1(P) = f_2(P) = f_3(P) = 0$, (24) gives that the sign of $f_1(T(P))$ is s . $\square$

Using Theorem 2, we have the following:

$$\begin{aligned} \text{If } P \in \text{I}(+, -, -), \\ \text{then } T(P) \notin \text{III} \cup \text{IV} \cup \text{V}(-, *, *). \end{aligned} \quad (28)$$

$$\text{If } P \in \text{V}(-, +, +), \text{ then } T(P) \notin \text{I}(+, -, -). \quad (29)$$

Remark 3. Suppose that Δt satisfies (9) and (15). Let $s \in \{+, -\}$. Then the definitions of f_2 and f_3 yield

$$\begin{aligned} f_2(T(P)) &= (1 - \Delta t a_{22} P_2) f_2(P) \\ &\quad + \Delta t \{a_{21} P_1 f_1(P) - a_{23} P_3 f_3(P)\}, \end{aligned} \quad (30)$$

$$\begin{aligned} f_3(T(P)) &= (1 - \Delta t a_{33} P_3) f_3(P) \\ &\quad + \Delta t \{a_{31} P_1 f_1(P) + a_{32} P_2 f_2(P)\}, \end{aligned} \quad (31)$$

and hence we can also obtain the property as in Theorem 2.

- (a) If $P \in (s, s, -s)$, then the sign of $f_2(T(P))$ is s .
 (b) If $P \in (s, s, s)$, then the sign of $f_3(T(P))$ is s .

Using (a) and (b), we can obtain the following:

$$\text{If } P \in \text{IV}(-, -, +), \quad \text{then } T(P) \notin V(-, +, +), \quad (32)$$

$$\begin{aligned} \text{If } P \in \text{III}(-, -, -), \\ \text{then } T(P) \notin \text{II} \cup \text{IV} \cup V(*, *, +). \end{aligned} \quad (33)$$

It follows from (28) and (29) that every point P in a region cannot move by the map T to regions with three different signs. In the case of regions with two different signs, it is also impossible by the following theorem.

Theorem 4. Suppose that Δt satisfies (9), (15), and (16).

- (a) If $P \in \text{II}(+, -, +)$, then $T(P) \notin V(-, +, +)$.
 (b) If $P \in V(-, +, +)$, then $T(P) \notin \text{II}(+, -, +)$.
 (c) If $P \in \text{II}(+, -, +)$, then $T(P) \notin \text{III}(-, -, -)$.
 (d) If $P \in \text{III}(-, -, -)$, then $T(P) \notin \text{II}(+, -, +)$.
 (e) If $P \in \text{III}(-, -, -)$, then $T(P) \notin V(-, +, +)$.
 (f) If $P \in V(-, +, +)$, then $T(P) \notin \text{III}(-, -, -)$.

Proof. (a) Suppose that $T(P) \in V(-, +, +)$. Then (24) and (30) with $P \in \text{II}(+, -, +)$ give

$$\begin{aligned} (1 - \Delta t a_{11} P_1) f_1(P) + \Delta t a_{12} P_2 |f_2(P)| \\ < \Delta t a_{13} P_3 f_3(P), \end{aligned} \quad (34)$$

$$a_{23} P_3 f_3(P) \leq a_{21} P_1 f_1(P), \quad (35)$$

respectively. Combining (34) and (35), we obtain

$$(1 - \Delta t a_{11} P_1) f_1(P) < \Delta t a_{13} \frac{a_{21} P_1}{a_{23}} f_1(P). \quad (36)$$

Since $f_1(P) \geq 0$, inequality (36) is a contradiction to (15). Therefore the proof of (a) is completed.

(b) Suppose that $T(P) \in \text{II}(+, -, +)$. Then the inclusion of $P \in V(-, +, +)$ gives a contradiction:

$$\begin{aligned} 0 \leq f_1(T(P)) \\ = (1 - \Delta t a_{11} P_1) f_1(P) \\ - \Delta t \{a_{12} P_2 f_2(P) + a_{13} P_3 f_3(P)\} < 0. \end{aligned} \quad (37)$$

(c) Suppose that $T(P) \in \text{III}(-, -, -)$. Then (24) and (31) with $P \in \text{II}(+, -, +)$ give

$$\begin{aligned} (1 - \Delta t a_{11} P_1) f_1(P) + \Delta t a_{12} P_2 |f_2(P)| \\ \leq \Delta t a_{13} P_3 f_3(P), \\ (1 - \Delta t a_{33} P_3) f_3(P) \leq \Delta t a_{32} P_2 |f_2(P)|, \end{aligned} \quad (38)$$

which yield

$$(1 - \Delta t a_{33} P_3) f_3(P) \leq \Delta t a_{32} \frac{a_{13} P_3}{a_{12}} f_3(P). \quad (39)$$

Hence, if $f_3(P) > 0$ or $f_3(P) = 0$, then

$$\begin{aligned} (1 - \Delta t a_{33} P_3) < \Delta t a_{32} \frac{a_{13} P_3}{a_{12}} \\ \text{or } f_i(P) = 0 \quad (1 \leq i \leq 3), \end{aligned} \quad (40)$$

respectively; these are contradictions due to (15) and the fact that there is no solution of the system of equations $f_i(x, y, z) = 0$ ($i = 1, 2, 3$).

(d) and (e) are proved by (33).

(f) Suppose that $T(P) \in \text{III}(-, -, -)$. Then (31) with $P \in V(-, +, +)$ gives

$$0 < f_3(P) \leq \frac{\Delta t}{1 - \Delta t a_{33} P_3} a_{31} P_1 |f_1(P)|. \quad (41)$$

It follows from $f_2(P) = -r_2 + a_{21} P_1 - a_{22} P_2 - a_{23} P_3 \geq 0$, (5), and (3) that

$$\begin{aligned} f_3(P) &\geq -r_3 + a_{31} \frac{1}{a_{21}} (r_2 + a_{22} P_2 + a_{23} P_3) + a_{32} P_2 \\ &\quad - a_{33} P_3 \\ &> \frac{1}{a_{21}} \{-a_{21} r_3 + a_{31} r_2 + (a_{31} a_{23} - a_{21} a_{33}) P_3\} \\ &> \frac{1}{a_{21}} (-a_{21} r_3 + a_{31} r_2). \end{aligned} \quad (42)$$

Since $-a_{21} r_3 + a_{31} r_2 > 0$ by (3), inequality (41) with (42) is a contradiction to (16), which completes the proof. $\square$

Theorem 5. Suppose that Δt satisfies (9). If $P \in V$, then $T^{m_1}(P) \notin V$ for some m_1 .

Proof. Suppose that $T^m(P) \in V(-, +, +)$ for all m . Then there exist constants x_* , y_* , and z_* such that

$$\begin{aligned} x_* &= \lim_{k \rightarrow \infty} x_k^1 \geq 0, \\ y_* &= \lim_{k \rightarrow \infty} x_k^2 > 0, \\ z_* &= \lim_{k \rightarrow \infty} x_k^3 > 0, \end{aligned} \quad (43)$$

and so

$$f_2(x_*, y_*, z_*) = f_3(x_*, y_*, z_*) = 0. \quad (44)$$

This is a contradiction since the system of (44) gives that

$$\begin{aligned} 0 &< (a_{22} a_{31} + a_{32} a_{21}) y_* \\ &= -(a_{31} r_2 - a_{21} r_3) - (a_{23} a_{31} - a_{33} a_{21}) z_* < 0, \end{aligned} \quad (45)$$

due to both (3) and (4) with $z_* > 0$. $\square$

The results we obtained are summarized in Table 1.

Finally, using Table 1, we can obtain the following theorem.

TABLE 1: The regions containing $T(P)$. The symbols I to V denote the regions defined in (11). The region I in the second column denotes $P \in I$, the symbol $\times$ at $(P, T(P)) = (I, III)$ means $T(P) \notin III$, and then the two circles $\circ$ together at both (I, I) and (I, II) denote that $T(P) \in I \cup II$ by (28). The symbol $\times$ denotes that $P \in V$ and $T^{m_1}(P) \notin V$ for some m_1 by Theorem 5.

	$T(P)$					Equation and theorem
	I	II	III	IV	V	
P	I	$\circ$	$\circ$	$\times$	$\times$	Equation (28)
	II	$\circ$	$\circ$	$\times$	$\circ$	Theorem 4(c) and (a)
	III	$\circ$	$\times$	$\circ$	$\times$	Equation (33)
	IV	$\circ$	$\circ$	$\circ$	$\times$	Equation (32)
	V	$\times$	$\times$	$\times$	$\circ$	Equation (29); Theorem 4(b) and (f); and Theorem 5

Theorem 6. Suppose that Δt satisfies (9), (15), and (16). If $P \in I \cup II \cup III \cup IV \cup V$, then for all sufficiently large m

$$T^m(P) \in I \cup II \cup III \cup IV (*, -, *). \quad (46)$$

Remark 7. Table 1 and Theorem 6 are obtained for the case that only two planes $f_i(x, y, z) = 0$ ($i = 1, 3$) intersect in the first octant. We can also apply the approach to the other cases and then have a table and a theorem similar to Table 1 and Theorem 6.

3. Global Stability

In this section we show that the fixed point $\vartheta = (\vartheta_1, 0, \vartheta_3)$ of the map T satisfying (19) is globally stable. Let Δt satisfy that

$$\{a_{i1}(\chi_1 + \vartheta_1) + a_{i2}\chi_2 + a_{i3}(\chi_3 + \vartheta_3)\} \Delta t < \frac{1}{2}, \quad (47)$$

$i = 1, 3,$

$$\Delta t < \frac{\min\{a_{31}a_{11}, a_{13}a_{33}\}}{2a_{31}(a_{12} + A\vartheta_1M) + 2a_{13}(a_{32} + A\vartheta_3M)}, \quad (48)$$

where $A = 3 \sum_{1 \leq j \leq 3} (a_{1j} + a_{3j})$ and $M = 6 \max_{1 \leq i, j \leq 3} a_{ij}$.

Theorem 8. Let ϑ be the point defined in (19). Suppose that Δt satisfies (9), (15), (16), (47), and (48). Assume that the initial point (x_0^1, x_0^2, x_0^3) of the Euler scheme (7) satisfies

$$(x_0^1, x_0^2, x_0^3) \in \prod_{1 \leq i \leq 3} (0, \chi_i). \quad (49)$$

Then

$$\lim_{k \rightarrow \infty} (x_k^1, x_k^2, x_k^3) = \vartheta. \quad (50)$$

Proof. Theorem 6 gives that for all k

$$(x_k^1, x_k^2, x_k^3) \in I \cup II \cup III \cup IV (*, -, *), \quad (51)$$

and then (10) gives that for a nonnegative constant y_* :

$$y_* = \lim_{k \rightarrow \infty} x_k^2. \quad (52)$$

Now we claim that $y_* = 0$: suppose, on the contrary, that

$$y_* > 0. \quad (53)$$

Applying both (52) and (53) to (7), we have

$$\lim_{k \rightarrow \infty} f_2(x_k^1, x_k^2, x_k^3) = 0, \quad (54)$$

which implies that there exists a constant ϵ_0 such that for all sufficiently large k

$$0 < \epsilon_0 < a_{21} \max \left\{ \frac{r_2}{a_{21}} - \frac{r_1}{a_{11}}, \frac{r_2}{a_{21}} - \frac{r_3}{a_{31}} \right\}, \quad (55)$$

$$f_2(x_k^1, x_k^2, x_k^3) > -\epsilon_0. \quad (56)$$

Then (56) and (55) give $x_k^1 > (1/a_{21})(r_2 - \epsilon_0) > r_1/a_{11}$, and so (14) gives

$$(x_k^1, x_k^2, x_k^3) \notin I(+, -, -). \quad (57)$$

Using (56), we have

$$x_k^1 > \frac{1}{a_{21}} (r_2 - \epsilon_0 + a_{22}x_k^2 + a_{23}x_k^3), \quad (58)$$

and so it follows from (42) and (55) that for all sufficiently large k

$$f_3(x_k^1, x_k^2, x_k^3) > \frac{1}{a_{21}} \{a_{31}(r_2 - \epsilon_0) - a_{21}r_3\} > 0, \quad (59)$$

which gives

$$(x_k^1, x_k^2, x_k^3) \notin III(-, -, -). \quad (60)$$

Hence (51) with (57) and (60) gives that for all sufficiently large k

$$(x_k^1, x_k^2, x_k^3) \in II \cup IV(*, -, +). \quad (61)$$

Both (61) and (53) yield that there exist the two limits:

$$x_* = \lim_{k \rightarrow \infty} x_k^1, \quad (62)$$

$$z_* = \lim_{k \rightarrow \infty} x_k^3 > 0,$$

and then

$$f_2(x_*, y_*, z_*) = f_3(x_*, y_*, z_*) = 0. \quad (63)$$

This is a contradiction due to (45) and finally we obtain the claim

$$\lim_{k \rightarrow \infty} x_k^2 = 0. \quad (64)$$

Consider the function V_k defined by

$$V_k = a_{31} (x_k^1 - \vartheta_1 \ln x_k^1) + a_{13} (x_k^3 - \vartheta_3 \ln x_k^3). \quad (65)$$

Letting $\hat{a}_{ij} = a_{ij}\Delta t$ and

$$\begin{aligned} \vartheta_{1k} &= \vartheta_1 - x_k^1, \\ \vartheta_{3k} &= \vartheta_3 - x_k^3, \end{aligned} \quad (66)$$

scheme (7) and the fixed point (19) yield

$$\begin{aligned} \Delta x_k^1 &= x_k^1 (\hat{a}_{11} \vartheta_{1k} - \hat{a}_{12} x_k^2 + \hat{a}_{13} \vartheta_{3k}), \\ \Delta x_k^3 &= x_k^3 (-\hat{a}_{31} \vartheta_{1k} + \hat{a}_{32} x_k^2 + \hat{a}_{33} \vartheta_{3k}). \end{aligned} \quad (67)$$

Then the mean value theorem gives that for some α, β with $0 < \alpha, \beta < 1$

$$\begin{aligned} V_{k+1} - V_k &= a_{31} \Delta x_k^1 \left(1 - \vartheta_1 \frac{\Delta \ln x_k^1}{\Delta x_k^1} \right) \\ &\quad + a_{13} \Delta x_k^3 \left(1 - \vartheta_3 \frac{\Delta \ln x_k^3}{\Delta x_k^3} \right) \\ &= a_{31} (\hat{a}_{11} \vartheta_{1k} - \hat{a}_{12} x_k^2 + \hat{a}_{13} \vartheta_{3k}) \\ &\quad \cdot \left(x_k^1 - \vartheta_1 \frac{x_k^1}{\alpha \Delta x_k^1 + x_k^1} \right) \\ &\quad + a_{13} (-\hat{a}_{31} \vartheta_{1k} + \hat{a}_{32} x_k^2 + \hat{a}_{33} \vartheta_{3k}) \\ &\quad \cdot \left(x_k^3 - \vartheta_3 \frac{x_k^3}{\beta \Delta x_k^3 + x_k^3} \right). \end{aligned} \quad (68)$$

Note that

$$\begin{aligned} \frac{x_k^1}{\alpha \Delta x_k^1 + x_k^1} &= 1 - \frac{\alpha (a_{11} \vartheta_{1k} - a_{12} x_k^2 + a_{13} \vartheta_{3k}) \Delta t}{\alpha (a_{11} \vartheta_{1k} - a_{12} x_k^2 + a_{13} \vartheta_{3k}) \Delta t + 1} \\ &\equiv 1 - (C_1 \vartheta_{1k} + C_2 x_k^2 + C_3 \vartheta_{3k}) \Delta t, \\ \frac{x_k^3}{\beta \Delta x_k^3 + x_k^3} &= 1 \\ &\quad - \frac{\beta (-a_{31} \vartheta_{1k} + a_{32} x_k^2 + a_{33} \vartheta_{3k}) \Delta t}{\beta (-a_{31} \vartheta_{1k} + a_{32} x_k^2 + a_{33} \vartheta_{3k}) \Delta t + 1} \\ &\equiv 1 - (C_4 \vartheta_{1k} + C_5 x_k^2 + C_6 \vartheta_{3k}) \Delta t, \end{aligned} \quad (69)$$

where (47) gives

$$\max_{1 \leq i \leq 6} |C_i| < 2 \max_{1 \leq i, j \leq 3} a_{ij}. \quad (70)$$

Now suppose, on the contrary, that (x_k^1, x_k^2, x_k^3) does not converge to $(\vartheta_1, 0, \vartheta_3)$.

Since ϑ is asymptotically stable by Lemma 1, the supposition with $\lim_{k \rightarrow \infty} x_k^2 = 0$ implies that

$$|\vartheta_{1k}| + |\vartheta_{3k}| \text{ has a positive lower bound.} \quad (71)$$

Again, using $\lim_{k \rightarrow \infty} x_k^2 = 0$ with (71), we can have that for all sufficiently large k

$$|x_k^2| < \Delta t (|\vartheta_{1k}| + |\vartheta_{3k}|). \quad (72)$$

Then (68) becomes

$$\begin{aligned} V_{k+1} - V_k &\leq -\Delta t (a_{31} a_{11} - C_7 \Delta t) \vartheta_{1k}^2 \\ &\quad - \Delta t (a_{13} a_{33} - C_8 \Delta t) \vartheta_{3k}^2, \end{aligned} \quad (73)$$

where

$$\begin{aligned} \max \{C_7, C_8\} &< 2a_{31} (a_{12} + A \vartheta_1 |C_{123}|) \\ &\quad + 2a_{13} (a_{32} + A \vartheta_3 |C_{456}|), \end{aligned} \quad (74)$$

with $A = 3 \sum_{1 \leq j \leq 3} (a_{1j} + a_{3j})$ and $|C_{ijk}| = |C_i| + |C_j| + |C_k|$. Hence (73) with (48) becomes

$$V_{k+1} - V_k \leq -C_9 (\vartheta_{1k}^2 + \vartheta_{3k}^2) \Delta t, \quad (75)$$

for a positive constant C_9 , so that there exists a positive constant $\mathcal{C}$ such that for all sufficiently large k

$$V_{k+1} - V_k \leq -\mathcal{C} \Delta t, \quad (76)$$

and then

$$V_k \leq V_0 - k \mathcal{C} \Delta t. \quad (77)$$

Hence we have

$$\lim_{k \rightarrow \infty} V_k = \lim_{k \rightarrow \infty} (V_0 - k \mathcal{C} \Delta t) = -\infty, \quad (78)$$

which is a contradiction due to (10). $\square$

Remark 9. The global stability in Theorem 8 is obtained for the discrete predator-prey model with one prey and two predators by using both Theorem 6 and the Lyapunov type function V_k . At first, we show that one predator is extinct by Theorem 6 and then we can apply the function V_k which was used in the lower dimensional case: the two-dimensional discrete model with one prey and one predator. Therefore our approach can utilize the methods used in lower dimensional models.

On the other hand, in the case that

$$\begin{aligned} \frac{r_1}{a_{11}} &< \frac{r_3}{a_{31}} < \frac{r_2}{a_{21}}, \\ \frac{r_1}{a_{12}} &< \frac{r_3}{a_{32}}, \\ \frac{a_{21}}{a_{23}} &< \frac{a_{31}}{a_{33}}, \end{aligned} \quad (79)$$

region II is empty and similarly we can obtain the global stability of the fixed point $(r_1/a_{11}, 0, 0)$ of the map T by using Theorems 2–5 without applying a Lyapunov type function like V_k .

FIGURE 2: Trajectories for five different initial points denoted by the black circles. The initial points are $(0.5, 0.375, 0.3562)$, $(1.15, 0.2125, 0.1281)$, $(0.125, 1.5, 1.5)$, $(1.5, 0.125, 0.2813)$, and $(4, 0.5, 0.25)$, which are contained in regions I, II, III, IV, and V defined in (11), respectively. The brown dotted lines mean the trajectories which converge to $\vartheta = (6/5, 0, 2/5)$, denoted by the magenta circle. The three planes $f_i = 0$ are depicted in Figure 1.

FIGURE 3: The region containing (x_k^1, x_k^2, x_k^3) at time $t = k\Delta t$ for the five trajectories in Figure 2. (a) The initial point (x_0^1, x_0^2, x_0^3) is $(0.5, 0.375, 0.3562)$, denoted by the black circle and located at $(0, I)$. The green segments denote the regions containing (x_k^1, x_k^2, x_k^3) . The region having P is connected to the region having $T(P)$ by the brown line. The other figures (b), (c), (d), and (e) are similarly obtained by using the different initial points in Figure 2.

4. Numerical Examples

In this section, we consider the Euler difference scheme for (2):

$$\begin{aligned}\Delta x_k^1 &= x_k^1 (2 - x_k^1 - 2x_k^2 - 2x_k^3) \Delta t, \\ \Delta x_k^2 &= x_k^2 (-3 + x_k^1 - x_k^2 - x_k^3) \Delta t, \\ \Delta x_k^3 &= x_k^3 (-2 + 2x_k^1 + x_k^2 - x_k^3) \Delta t,\end{aligned}\quad (80)$$

where $\Delta t = 0.001$ and $k = 0, 1, \dots$. Applying Theorem 4.1 in [28] to (80), we can obtain the positivity and boundedness of the solutions. For example,

$$(x_k^1, x_k^2, x_k^3) \in (0, 4) \times (0, 2) \times (0, 8) \quad \forall k. \quad (81)$$

The fixed point ϑ in (19) becomes $\vartheta = (6/5, 0, 2/5)$ which is denoted by the magenta circle in Figures 1 and 2, and the value $\Delta t = 0.001$ satisfies all conditions (9), (15), (16), (47), and (48). Consequently, $\vartheta = (6/5, 0, 2/5)$ is globally stable, which is demonstrated in Figure 2 for five different initial points (x_0^1, x_0^2, x_0^3) contained in regions I to V, respectively.

In order to verify the results in Table 1, we mark the regions containing $T(P)$ for P located in the five trajectories in Figure 2 and present the result in Figure 3, which demonstrates that the regions containing $T(P)$ follow the rule in Table 1. For example, we can see in Figure 3 that $T(P)$ cannot be contained in V for every $P \in \text{I} \cup \text{II} \cup \text{III} \cup \text{IV}$, and if $P \in \text{II}$, then $T(P)$ can be contained only in $\text{II} \cup \text{IV}$.

5. Conclusion

In this paper, we have developed a new approach to obtain the global stability of the fixed point of a discrete predator-prey system with one prey and two predators. The main idea of our approach is to describe how to trace the trajectories. In this process, we calculate the sign of the rate change of population of species, so that we call our method *the sign method*. Although we have applied our sign method for the three-dimensional discrete model, the sign method can be utilized for two-dimensional and other higher dimensional discrete models.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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