Research Article

Approximate Kelvin-Voigt Fluid Driven by an External Force Depending on Velocity with Distributed Delay

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We consider the approximate 3D Kelvin-Voigt fluid driven by an external force depending on velocity with distributed delay. We investigate the long time behavior of solutions to Navier-Stokes-Voigt equation with a distributed delay external force depending on the velocity of fluid on a bounded domain. By a prior estimate and a contractive function, we give a sufficient condition for the existence of pullback attractor of NSV equation.

1. Introduction

In this paper, we consider 3D Navier-Stokes-Voigt (NSV) equation with a distributed delay external force depending on the velocity of the fluid:

$$\begin{aligned} u_t - \alpha^2 \Delta u_t - \nu \Delta u + (u \cdot \nabla) u + \nabla p \\ = \int_{-h}^0 G(s, u(t+s)) ds, \\ \operatorname{div} u(t, x) = 0, \quad (t, x) \in [\tau, +\infty) \times \Omega, \\ u(t, x) = 0, \quad (t, x) \in [\tau, +\infty) \times \partial\Omega, \\ u(t, x) = \phi(t - \tau, x), \quad t \in [\tau - h, \tau], \quad x \in \Omega, \end{aligned} \quad (1)$$

where $u = (u_1, u_2, u_3)$ is the velocity field of the fluid, p is the pressure, $\nu > 0$ is the kinematic viscosity, $\alpha > 0$ is the length scale parameter of the elasticity of the fluid, the external force G and initial velocity field ϕ are defined in the interval of time $[-h, 0]$, where h is a fixed positive number and Ω is a bounded smooth domain of $\mathbb{R}^3$.

The NSV equation was introduced by Oskolkov [1] to give an approximate description of the Kelvin-Voigt fluid and was proposed as a regularization of 3D Navier-Stokes equation for the purpose of direct numerical simulations in

[2]. Since the term $-\alpha^2 \Delta u_t$ changes the parabolic character of the equation, the NSV equation being well posed in 3D, many authors have studied the long time dynamics of this model. Kalantarov and Titi [3] investigated the existence of the global attractor, the estimation of the upper bounds for the number of determining modes, and the dimension of global attractor of the semigroup generated by the equations. By a useful decomposition method, Yue and Zhong [4] proved the asymptotic regularity of solution of NSV equation and obtained the existence of the uniform attractor; they also described the structure of the uniform attractor and its regularity. García-Luengo et al. [5] investigated the existence and relationship between minimal pullback attractor for the universe of fixed bounded sets and universe given by a tempered condition.

Partial differential equations with delays arise from various fields, like physics, control theory, and so on (see, e.g., [6–10]); the unknown functions depend on not only present stage but also some past stage. The existence and stability of solution and global attractor for Navier-Stokes equation with discrete delay were established in [11–13]. The existence of pullback attractors in $C_{H_0^1}$ and $C_{H_0^1 \cap H^2}$ was proved for the processes associated with nonclassical diffusion equations with variable bounded delay in [14, 15]. Delay effect has been considered on an unbounded domain in [16]. The existence of

pullback attractor for a Navier-Stokes equation with infinite discrete delay effect was studied in [17].

The aim of this paper is to investigate the NSV equation with a distributed delay, instead of the discussions with finite delays in the references. Our purpose is twofold. We first show the existence and uniqueness of solution to NSV equation (1) with a distributed delay; then we prove the existence of pullback attractor for the process generated by the NSV equation (1).

This paper is organized as follows. In Section 2, we give some preliminary results and prove existence of solution to NSV equation with a distributed delay. In Section 3, we derive the existence of pullback attractor by prior estimates and contractive functions.

2. Existence of Solutions

In order to prove the existence of solutions to problem (1), we define the function spaces

$$\mathcal{V} = \{u \in (C_0^\infty(\Omega))^3, \operatorname{div} u = 0\}. \quad (2)$$

H is the closure of $\mathcal{V}$ in $(L^2(\Omega))^3$ with the inner product $(\cdot, \cdot)$ and associate norm $|\cdot|$, V is the closure of $\mathcal{V}$ in $(H_0^1(\Omega))^3$ with scalar product $((\cdot, \cdot))$ and associate norm $\|\cdot\|$, where

$$((u, v)) = \sum_{i,j=1}^3 \int_{\Omega} \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i} dx, \quad \forall u, v \in (H_0^1(\Omega))^3; \quad (3)$$

it follows that $V \subset H \equiv H' \subset V'$, where the injections are dense and compact. We will use $\|\cdot\|_*$ for the norm in V' and $\langle \cdot, \cdot \rangle$ for the duality pairing between V and V' .

Define the linear continuous operator $A : V \rightarrow V'$ as

$$\langle Au, v \rangle = ((u, v)), \quad \forall u, v \in V. \quad (4)$$

We denote $D(A) = \{u \in V, Au \in H\}$; one has that $D(A) = (H^2(\Omega))^3 \cap V$ and $Au = -P\Delta u$, for all $u \in D(A)$ is the Stokes operator, where P is the orthoprojector from $(L^2(\Omega))^3$ onto H ; also denote $C_H = C^0([-h, 0]; H)$ and $C_V = C^0([-h, 0]; V)$.

Define the trilinear form b on $V \times V \times V$ by

$$b(u, v, w) = \sum_{i,j=1}^3 \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j dx, \quad \forall u, v, w \in V \quad (5)$$

and the operator $B : V \times V \rightarrow V'$ as

$$\langle B(u, v), w \rangle = b(u, v, w), \quad \forall u, v, w \in V, \quad (6)$$

and denote $B(u) = B(u, u)$.

The trilinear form b satisfies that

$$\begin{aligned} b(u, v, w) &= -b(u, w, v), \\ b(u, v, v) &= 0, \\ \forall u, v, w &\in V. \end{aligned} \quad (7)$$

We also recall that there exists a constant C depending only on Ω such that

$$\begin{aligned} |b(u, v, w)| &\leq C \|u\|^{1/2} |Au|^{1/2} \|v\| |w|, \\ u &\in D(A), \quad v \in V, \quad w \in H, \\ |b(u, v, w)| &\leq C \|u\| \|v\| |w|^{1/2} \|w\|^{1/2}, \\ u &\in V, \quad v \in V, \quad w \in V, \\ |b(u, v, w)| &\leq C |u|^{1/2} \|u\|^{1/2} \|v\| \|w\|, \\ u &\in V, \quad v \in V, \quad w \in V. \end{aligned} \quad (8)$$

For the term containing the time delay, $G : R \times H \rightarrow H$ satisfies that

$$\begin{aligned} (H_1) \quad &G(\cdot, u) : R \rightarrow H \text{ is a measurable function,} \\ (H_2) \quad &G(t, 0) = 0 \text{ for all } t \in R, \\ (H_3) \quad &\text{there exists a positive constant } L, \text{ such that } \forall R > 0; \text{ if } |u| < R \text{ and } |v| < R, \text{ then} \\ &|G(t, u) - G(t, v)|^2 \leq L |u - v|^2. \end{aligned} \quad (9)$$

Remark 1. Hypotheses (H_2) – (H_3) imply that $|G(t, u)|^2 \leq L|u|^2$, so we have $|G(t, u)|^2 \in L^\infty(\tau, T)$ for $|u| < R$.

Problem (1) can be rewritten as

$$\begin{aligned} &\frac{\partial}{\partial t} (u + \alpha^2 Au) + \nu Au + B(u, u) \\ &= \int_{-h}^0 G(s, u(t+s)) ds, \\ u(t, x) &= \phi(t - \tau, x), \quad t \in [\tau - h, \tau], \quad x \in \Omega; \end{aligned} \quad (10)$$

then we get the existence of solution to problem (10).

Theorem 2. Let $\phi \in C_V$, let $G : R \times H \rightarrow H$ satisfy the hypotheses (H_1) – (H_3) , and let $\tau \in R$. Then, $\forall T > \tau$; there exists a unique weak solution to (10) such that

$$\begin{aligned} u(t, x) &\in C([\tau - h, T]; V), \\ \frac{\partial u(t, x)}{\partial t} &\in L^2(\tau, T; V). \end{aligned} \quad (11)$$

Moreover, if $\phi \in C_{D(A)}$, then problem (10) admits a strong solution.

Proof. Consider the Galerkin approximations for problem (10):

$$\begin{aligned} &\frac{d}{dt} (u^m + \alpha^2 Au^m) + \nu Au^m + B(u^m, u^m) \\ &= \int_{-h}^0 G(s, u^m(t+s)) ds, \end{aligned} \quad (12)$$

where $u^m = \sum_{j=1}^m u_j^m e_j$, $Au^m = \sum_{j=1}^m \lambda_j u_j^m e_j$, and e_j and λ_j are the corresponding orthonormal eigenfunctions and eigenvalues of operator A , respectively; then,

$$\begin{aligned} \|u^m\|^2 &= \sum_{j=1}^m \lambda_j (u_j^m)^2, \\ |Au^m|^2 &= \sum_{j=1}^m \lambda_j^2 (u_j^m)^2, \\ |u^m|^2 &= \sum_{j=1}^m (u_j^m)^2. \end{aligned} \quad (13)$$

We now derive a prior estimate for the Galerkin approximate solution. Multiplying (12) by u_j^m , summing from $j = 1$ to m and using the fact

$$(B(u^m, u^m), u^m) = b(u^m, u^m, u^m) = 0, \quad (14)$$

we obtain that, for a.e. $t > \tau$,

$$\begin{aligned} \frac{d}{dt} (|u^m|^2 + \alpha^2 \|u^m\|^2) + 2\nu \|u^m\|^2 \\ = 2 \left(\int_{-h}^0 G(s, u^m(t+s)) ds, u^m \right) \\ \leq \left| \int_{-h}^0 G(s, u^m(t+s)) ds \right|^2 + |u^m|^2. \end{aligned} \quad (15)$$

Integrating (15) from τ to t , we obtain that

$$\begin{aligned} |u^m(t)|^2 + \alpha^2 \|u^m(t)\|^2 + 2\nu \int_{\tau}^t \|u^m(s)\|^2 ds \\ \leq |u^m(\tau)|^2 + \alpha^2 \|u^m(\tau)\|^2 \\ + \int_{\tau}^t \left| \int_{-h}^0 G(s, u^m(r+s)) ds \right|^2 dr \\ + \int_{\tau}^t |u^m(r)|^2 dr. \end{aligned} \quad (16)$$

Remark 1 implies that

$$\begin{aligned} \int_{\tau}^t \left| \int_{-h}^0 G(s, u^m(r+s)) ds \right|^2 dr \\ \leq \int_{-h}^0 \int_{\tau}^t |G(s, u^m(r+s))|^2 dr ds \\ \leq \int_{-h}^0 \int_{\tau}^t L |u^m(r+s)|^2 dr ds \\ \leq L \int_{-h}^0 \int_{\tau+s}^{t+s} |u^m(r)|^2 dr ds \\ \leq L \int_{-h}^0 \left(\int_{\tau-h}^{\tau} |u^m(r)|^2 dr + \int_{\tau}^t |u^m(r)|^2 dr \right) ds \\ \leq Lh^2 \|\phi\|_{C_H}^2 + Lh \int_{\tau}^t |u^m(r)|^2 dr. \end{aligned} \quad (17)$$

Then, $\forall t \in (\tau, T)$ and

$$\begin{aligned} |u^m(t)|^2 + \alpha^2 \|u^m(t)\|^2 + 2\nu \int_{\tau}^t \|u^m(s)\|^2 ds \\ \leq (Lh+2) \int_{\tau}^t |u^m(s)|^2 ds + Lh^2 \|\phi\|_{C_H}^2 + |\phi|_{C_H}^2 \\ + \alpha^2 \|\phi\|_{C_V}^2. \end{aligned} \quad (18)$$

So, we have

$$\begin{aligned} |u^m(t)|^2 \leq C \int_{\tau}^t |u^m(s)|^2 ds + C \|\phi\|_{C_H}^2 + |\phi|_{C_H}^2 \\ + \alpha^2 \|\phi\|_{C_V}^2. \end{aligned} \quad (19)$$

The Gronwall inequality implies that

$$|u^m(t)|^2 \leq C. \quad (20)$$

Putting (20) into the right-hand side of (18), we have

$$\alpha^2 \|u^m(t)\|^2 + 2\nu \int_{\tau}^t \|u^m(s)\|^2 ds \leq C, \quad \forall t \in (\tau, T). \quad (21)$$

This implies that

$$\{u^m\} \text{ is bounded in } L^\infty(\tau, T; V) \cap L^2(\tau, T; V). \quad (22)$$

Now, multiplying (12) by $\partial_t u^m$ and integrating over Ω , we have

$$\begin{aligned} |\partial_t u^m|^2 + \alpha^2 \|\partial_t u^m\|^2 + \frac{\nu}{2} \frac{d}{dt} \|u^m\|^2 \\ \leq |b(u^m, u^m, \partial_t u^m)| \\ + \left| \left(\int_{-h}^0 G(s, u^m(t+s)) ds, \partial_t u^m \right) \right| \\ \leq c \|u^m\|^{3/2} |u^m|^{1/2} \|\partial_t u^m\| \\ + \frac{1}{2} \left| \int_{-h}^0 G(s, u^m(t+s)) ds \right|^2 + \frac{1}{2} |\partial_t u^m|^2; \end{aligned} \quad (23)$$

since

$$\|u^m\|^{3/2} |u^m|^{1/2} \|\partial_t u^m\| \leq c \|u^m\|^3 |u^m| + \frac{\alpha^2}{2} \|\partial_t u^m\|^2; \quad (24)$$

then

$$\begin{aligned} |\partial_t u^m|^2 + \alpha^2 \|\partial_t u^m\|^2 + \nu \frac{d}{dt} \|u^m\|^2 \\ \leq c \|u^m\|^3 |u^m| + \left| \int_{-h}^0 G(s, u^m(t+s)) ds \right|^2, \end{aligned} \quad (25)$$

integrating the above inequality from τ to t , by (17), (20), and (22) we have

$$\begin{aligned}
& \int_{\tau}^t \left(|\partial_t u^m|^2 + \alpha^2 \|\partial_t u^m\|^2 \right) ds + \nu \|u^m(t)\|^2 \\
& \leq \nu \|u^m(\tau)\|^2 + c \int_{\tau}^t \|u^m\|^3 |u^m| ds \\
& \quad + \int_{\tau}^t \left| \int_{-h}^0 G(s, u^m(r+s)) ds \right|^2 dr \\
& \leq \nu \|\phi(\tau)\|^2 + c \int_{\tau}^t \|u^m\|^3 |u^m| ds + c \|\phi\|_{C_H}^2 \\
& \quad + c \int_{\tau}^t |u^m(r)|^2 dr \\
& \leq \nu \|\phi\|_{C_V}^2 + c \int_{\tau}^t \|u^m\|^3 |u^m| ds + c \|\phi\|_{C_H}^2 \\
& \quad + c \int_{\tau}^t |u^m(r)|^2 dr.
\end{aligned} \tag{26}$$

Since $\{u^m\}$ is bounded in $L^\infty(\tau, T; V) \cap L^2(\tau, T; V)$, we obtain that

$$\{\partial_t u^m\} \text{ is bounded in } L^2(\tau, T; V). \tag{27}$$

By the Faedo-Galerkin scheme, for example, see [14, 18], according to the estimates (22) and (27), we can get existence of the weak solution; here we omit the details.

We next consider the uniqueness of solution. Let u, v be two solutions to problem (10) corresponding the initial data ϕ and ψ , respectively.

Denote $w = u - v$; then, we have

$$\begin{aligned}
& \frac{\partial}{\partial t} (w + \alpha^2 Aw) + \nu Aw + B(u, u) - B(v, v) \\
& = \int_{-h}^0 G(s, u(t+s)) ds - \int_{-h}^0 G(s, v(t+s)) ds.
\end{aligned} \tag{28}$$

Multiplying (28) by w and integrating over Ω , we obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} (|w|^2 + \alpha^2 \|w\|^2) + \nu \|w\|^2 + (B(u, u), w) \\
& \quad - (B(v, v), w) \\
& = \left(\int_{-h}^0 G(s, u(t+s)) ds - \int_{-h}^0 G(s, v(t+s)) ds, w \right).
\end{aligned} \tag{29}$$

Notice that

$$\begin{aligned}
& (B(u, u), w) - (B(v, v), w) \\
& = b(u, u, u-v) - b(v, u, u-v) + b(v, u, u-v) \\
& \quad - b(v, v, u-v) \\
& = b(u-v, u, u-v) - b(v, u-v, u-v) \\
& = b(w, u, w).
\end{aligned} \tag{30}$$

Substituting (30) into (29) and integrating from τ to t , we get

$$\begin{aligned}
& |w(t)|^2 + \alpha^2 \|w(t)\|^2 + 2\nu \int_{\tau}^t \|w(s)\|^2 ds - |w(\tau)|^2 \\
& \quad - \alpha^2 \|w(\tau)\|^2 \leq \int_{\tau}^t |b(w, u, w)| ds + \int_{\tau}^t |w(s)|^2 ds \\
& \quad + \int_{\tau}^t \left| \int_{-h}^0 G(s, u(r+s)) ds \right. \\
& \quad \left. - \int_{-h}^0 G(s, v(r+s)) ds \right|^2 dr.
\end{aligned} \tag{31}$$

(H_3) implies that

$$\begin{aligned}
& \int_{\tau}^t \left| \int_{-h}^0 G(s, u(r+s)) ds - \int_{-h}^0 G(s, v(r+s)) ds \right|^2 dr \\
& \leq \int_{-h}^0 \int_{\tau}^t |G(s, u(r+s)) - G(s, v(r+s))|^2 dr ds \\
& \leq \int_{-h}^0 \int_{\tau}^t |G(s, u(r+s)) - G(s, v(r+s))|^2 dr ds \\
& \leq L \int_{-h}^0 \int_{\tau}^t |u(r+s) - v(r+s)|^2 dr ds \\
& \leq L \int_{-h}^0 \int_{\tau+s}^{t+s} |u(r) - v(r)|^2 dr ds \\
& \leq L \int_{-h}^0 \left(\int_{\tau-h}^{\tau} |u(r) - v(r)|^2 dr \right. \\
& \quad \left. + \int_{\tau}^t |u(r) - v(r)|^2 dr \right) ds \leq Lh^2 \|\phi - \psi\|_{C_H}^2 \\
& \quad + Lh \int_{\tau}^t |w(s)|^2 ds.
\end{aligned} \tag{32}$$

As the property of operator b and Poincaré, we have

$$\begin{aligned}
& \int_{\tau}^t |b(w, u, w)| ds \leq C \int_{\tau}^t \|w\|^{3/2} |w|^{1/2} \|u\| ds \\
& \leq C \int_{\tau}^t \|w\|^2 \|u\| ds.
\end{aligned} \tag{33}$$

Substituting (32) and (33) into (31), we get

$$\begin{aligned}
& |w(t)|^2 + \alpha^2 \|w(t)\|^2 + 2\nu \int_{\tau}^t \|w(s)\|^2 ds \\
& \leq \int_{\tau}^t \|w\|^2 \|u\| ds + Lh^2 \|\phi - \psi\|_{C_H}^2 \\
& \quad + (Lh + 1) \int_{\tau}^t |w(s)|^2 ds + |w(\tau)|^2 \\
& \quad + \alpha^2 \|w(\tau)\|^2 \\
& \leq \int_{\tau}^t \|w\|^2 \|u\| ds + Lh^2 \|\phi - \psi\|_{C_H}^2 \\
& \quad + (Lh + 1) \int_{\tau}^t |w(s)|^2 ds + \|\phi - \psi\|_{C_H}^2 \\
& \quad + \alpha^2 \|\phi - \psi\|_{C_V}^2 \\
& \leq C \int_{\tau}^t \|w(s)\|^2 ds + C \|\phi - \psi\|_{C_V}^2.
\end{aligned} \tag{34}$$

The last inequality comes from Poincaré inequality and the boundedness of $\|u\|$. Therefore, the Gronwall inequality implies the uniqueness of the solution. The proof is complete. $\square$

3. Existence of Pullback Attractor

In this section, we will prove the existence of pullback attractor to problem (10). First we give existence of pullback absorbing set for the process $\{U(t, \tau)\}$ generated by the global solution to problem (10).

Lemma 3. Assume (H_1) – (H_3) hold and $L \leq \nu^2 \lambda_1^2 \varepsilon / 4e^{\varepsilon h}$; then, the process $\{U(t, \tau)\}$ is pullback dissipative, where $0 < \varepsilon < \min\{\nu/\alpha^2, \nu\lambda_1/4\}$.

Proof. Multiplying (10) by u and integrating over Ω , we obtain

$$\begin{aligned}
& \frac{\partial}{\partial t} (|u|^2 + \alpha^2 \|u\|^2) + 2\nu \|u\|^2 \\
& \leq \eta |u|^2 + \frac{1}{\eta} \left| \int_{-h}^0 G(s, u(t+s)) ds \right|^2,
\end{aligned} \tag{35}$$

where η is a constant determined later.

By Poincaré inequality, we have

$$\begin{aligned}
& \frac{\partial}{\partial t} (|u|^2 + \alpha^2 \|u\|^2) + \nu \|u\|^2 + (\nu\lambda_1 - \eta) |u|^2 \\
& \leq \frac{1}{\eta} \left| \int_{-h}^0 G(s, u(t+s)) ds \right|^2.
\end{aligned} \tag{36}$$

Since

$$\begin{aligned}
& e^{-\varepsilon t} \frac{\partial}{\partial t} [e^{\varepsilon t} (|u|^2 + \alpha^2 \|u\|^2)] \\
& = \varepsilon (|u|^2 + \alpha^2 \|u\|^2) + \frac{\partial}{\partial t} (|u|^2 + \alpha^2 \|u\|^2),
\end{aligned} \tag{37}$$

then

$$\begin{aligned}
& e^{-\varepsilon t} \frac{\partial}{\partial t} [e^{\varepsilon t} (|u|^2 + \alpha^2 \|u\|^2)] \\
& \leq (\varepsilon + \eta - \nu\lambda_1) |u|^2 + (\varepsilon\alpha^2 - \nu) \|u\|^2 \\
& \quad + \frac{1}{\eta} \left| \int_{-h}^0 G(s, u(t+s)) ds \right|^2.
\end{aligned} \tag{38}$$

Integrating (38) from τ to t , we get

$$\begin{aligned}
& e^{\varepsilon t} (|u(t)|^2 + \alpha^2 \|u(t)\|^2) \\
& \quad - e^{\varepsilon \tau} (|u(\tau)|^2 + \alpha^2 \|u(\tau)\|^2) \\
& \leq \int_{\tau}^t e^{\varepsilon s} [(\varepsilon + \eta - \nu\lambda_1) |u|^2 + (\varepsilon\alpha^2 - \nu) \|u\|^2] ds \\
& \quad + \frac{1}{\eta} \int_{\tau}^t e^{\varepsilon r} \left| \int_{-h}^0 G(s, u(r+s)) ds \right|^2 dr.
\end{aligned} \tag{39}$$

Assumptions (H_1) – (H_3) imply that

$$\begin{aligned}
& \int_{\tau}^t e^{\varepsilon r} \left| \int_{-h}^0 G(s, u(r+s)) ds \right|^2 dr \\
& \leq \int_{\tau}^t e^{\varepsilon r} \int_{-h}^0 |G(s, u(r+s))|^2 ds dr \\
& \leq L \int_{-h}^0 \int_{\tau}^t e^{\varepsilon r} |u(r+s)|^2 dr ds \\
& \leq L \int_{-h}^0 \int_{\tau+s}^{t+s} e^{\varepsilon(r-s)} |u(r)|^2 dr ds \\
& \leq L \int_{-h}^0 \int_{\tau-h}^t e^{\varepsilon(r-s)} |u(r)|^2 dr ds \\
& \leq L \int_{-h}^0 e^{-\varepsilon s} \int_{\tau-h}^t e^{\varepsilon r} |u(r)|^2 dr ds \\
& \leq L \int_{-h}^0 e^{-\varepsilon s} \left(\int_{\tau-h}^{\tau} e^{\varepsilon r} |u(r)|^2 dr \right. \\
& \quad \left. + \int_{\tau}^t e^{\varepsilon r} |u(r)|^2 dr \right) ds \leq \frac{Le^{\varepsilon h}}{\varepsilon} \left(\frac{1}{\varepsilon} \|\phi\|_{C_H}^2 e^{\varepsilon \tau} \right. \\
& \quad \left. + \int_{\tau}^t e^{\varepsilon r} |u(r)|^2 dr \right) \leq \frac{Le^{\varepsilon(\tau+h)}}{\varepsilon^2} \|\phi\|_{C_H}^2 + \frac{Le^{\varepsilon h}}{\varepsilon} \\
& \quad \cdot \int_{\tau}^t e^{\varepsilon r} |u(r)|^2 dr.
\end{aligned} \tag{40}$$

Substituting (40) into (39), we have

$$\begin{aligned}
& e^{\varepsilon t} (|u(t)|^2 + \alpha^2 \|u(t)\|^2) - e^{\varepsilon \tau} (|u(\tau)|^2 + \alpha^2 \|u(\tau)\|^2) \\
& \leq \int_{\tau}^t e^{\varepsilon s} \left[\left(\varepsilon + \eta - \nu\lambda_1 + \frac{Le^{\varepsilon h}}{2\eta\varepsilon} \right) |u|^2 \right. \\
& \quad \left. + (\varepsilon\alpha^2 - \nu) \|u\|^2 \right] ds + \frac{Le^{\varepsilon(\tau+h)}}{2\eta\varepsilon^2} \|\phi\|_{C_H}^2.
\end{aligned} \tag{41}$$

Let $\eta = (1/2)\nu\lambda_1$, choosing $0 < \varepsilon < \min\{\nu/\alpha^2, \nu\lambda_1/4\}$. $L \leq \nu^2\lambda_1^2\varepsilon/4e^{\varepsilon h}$ implies that

$$\max \left\{ \varepsilon + \eta - \nu\lambda_1 + \frac{\eta^{-1}Le^{\varepsilon h}}{2\varepsilon}, \varepsilon\alpha^2 - \nu \right\} < 0; \quad (42)$$

then,

$$\begin{aligned} & e^{\varepsilon t} (|u(t)|^2 + \alpha^2 \|u(t)\|^2) \\ & \leq \frac{Le^{\varepsilon(\tau+h)}}{\varepsilon^2\nu\lambda_1} \|\phi\|_{C_H}^2 + e^{\varepsilon\tau} (|u(\tau)|^2 + \alpha^2 \|u(\tau)\|^2), \end{aligned} \quad (43)$$

which implies

$$\begin{aligned} & |u(t)|^2 + \alpha^2 \|u(t)\|^2 \\ & \leq \frac{Le^{\varepsilon(\tau+h-t)}}{\varepsilon^2\nu\lambda_1} \|\phi\|_{C_H}^2 \\ & + e^{\varepsilon(\tau-t)} (|\phi(\tau)|^2 + \alpha^2 \|\phi(\tau)\|^2). \end{aligned} \quad (44)$$

Now, if we take $t \geq \tau + h$, then, for $\theta \in [-h, 0]$, we have

$$\begin{aligned} & |u(t+\theta)|^2 + \alpha^2 \|u(t+\theta)\|^2 \\ & \leq \frac{Le^{\varepsilon(\tau+h-t-\theta)}}{\varepsilon^2\nu\lambda_1} \|\phi\|_{C_H}^2 \\ & + e^{\varepsilon(\tau-t-\theta)} (\|\phi\|_{C_H}^2 + \alpha^2 \|\phi\|_{C_V}^2). \end{aligned} \quad (45)$$

We denote by $\mathcal{R}$ the set of all functions $r : (-\infty, +\infty) \rightarrow (0, +\infty)$ such that

$$\lim_{\tau \rightarrow -\infty} e^{\alpha\tau} r^2(\tau) = 0. \quad (46)$$

Then, the closed ball in C_V defined by

$$B = \{\varphi \in C_V : \|\varphi\|_{C_V}^2 \leq 1\} \quad (47)$$

is pullback absorbing set for $\{U(t, \tau)\}$. The proof is complete. $\square$

We next prove the asymptotic compactness of solution to problem (10) by contractive functions; see [19, 20].

Let X be a Banach space and let B be a bounded subset of X . We call a function $\Phi(\cdot, \cdot)$ which, defined on $X \times X$, is a contractive function on $B \times B$ if for any sequence $\{x_n\}_{n=1}^\infty \subset B$ there is a subsequence $\{x_{n_k}\}_{k=1}^\infty \subset \{x_n\}_{n=1}^\infty$ such that

$$\lim_{l \rightarrow \infty} \lim_{k \rightarrow \infty} \Phi(x_{n_k}, x_{n_l}) = 0. \quad (48)$$

Denote all such contractive functions on $B \times B$ by $C(B)$.

Theorem 4 (see [19]). *Let $\{S(t)\}_{t \geq \tau}$ be a semigroup on a Banach space $(X, \|\cdot\|)$ and have a bounded absorbing set B , $\forall \varepsilon > 0$; there exist $T = T(B, \varepsilon)$ and $\Phi \in C(B)$, such that*

$$\|S(T)x - S(T)y\| \leq \varepsilon + \Phi_T(x, y), \quad \forall x, y \in B, \quad (49)$$

where Φ_T depends on T . Then, $\{S(t)\}_{t \geq \tau}$ is asymptotically compact in X .

Lemma 5. *Assume that (H_1) – (H_3) hold; the process $\{U(t, \tau)\}_{t \geq \tau}$ generated by the global solution to problem (10) is asymptotically compact.*

Proof. Let $u^i(t)$ be the solution to problem (10) with initial data $\phi^i \in B$ ($i = 1, 2$), respectively. Denote $v(t) = u^1(t) - u^2(t)$; then, $v(t)$ satisfies the equivalent abstract equation

$$\begin{aligned} & \frac{\partial}{\partial t} (v + \alpha^2 Av) + \nu Av + B(u^1, u^1) - B(u^2, u^2) \\ & = \int_{-h}^0 [G(s, u^1(t+s)) - G(s, u^2(t+s))] ds \end{aligned} \quad (50)$$

with the initial condition $v(t) = \phi^1(t - \tau) - \phi^2(t - \tau)$, $t \in [\tau - h, \tau]$.

Set an energy function

$$E_v(t) = \frac{1}{2} \int_{\Omega} |v|^2 dx + \frac{\alpha^2}{2} \int_{\Omega} |\nabla v|^2 dx. \quad (51)$$

Multiplying (50) by v and integrating over $[s, T] \times \Omega$ with $T > t + \tau$, $s \geq \tau$, we have

$$\begin{aligned} & E_v(T) - E_v(s) + \nu \int_s^T \int_{\Omega} |\nabla v|^2 dx dr \\ & + \int_s^T \int_{\Omega} [B(u^1) - B(u^2)] v(r) dx dr \\ & = \int_s^T \int_{\Omega} \int_{-h}^0 [G(\sigma, u^1(r+\sigma)) - G(\sigma, u^2(r+\sigma))] v(r) d\sigma dx dr; \end{aligned} \quad (52)$$

then we have

$$\begin{aligned} & \nu \int_s^T \int_{\Omega} |\nabla v|^2 dx dr \leq E_v(s) - \int_s^T \int_{\Omega} [B(u^1) - B(u^2)] v(r) dx dr \\ & + \int_s^T \int_{\Omega} \int_{-h}^0 [G(\sigma, u^1(r+\sigma)) - G(\sigma, u^2(r+\sigma))] v(r) d\sigma dx dr. \end{aligned} \quad (53)$$

Using Poincaré inequality and (51) and (53), we have

$$\begin{aligned} \int_{\tau}^T E_{\nu}(s) ds &= \frac{1}{2} \int_{\tau}^T \int_{\Omega} |v|^2 dx dr + \frac{\alpha^2}{2} \int_{\tau}^T \int_{\Omega} |\nabla v|^2 dx dr \leq C \int_{\tau}^T \int_{\Omega} |\nabla v|^2 dx dr \\ &\leq CE_{\nu}(\tau) - C \int_{\tau}^T \int_{\Omega} [B(u^1) - B(u^2)] v(r) dx dr \\ &\quad + C \int_{\tau}^T \int_{\Omega} \int_{-h}^0 [G(\sigma, u^1(r+\sigma)) - G(\sigma, u^2(r+\sigma))] v(r) d\sigma dx dr. \end{aligned} \quad (54)$$

Integrating (52) from τ to T with respect to s , we obtain

$$\begin{aligned} TE_{\nu}(T) + \nu \int_{\tau}^T \int_s^T \int_{\Omega} |\nabla v|^2 dx dr ds &= \int_{\tau}^T E_{\nu}(s) ds - \int_{\tau}^T \int_s^T \int_{\Omega} [B(u^1) - B(u^2)] v(r) dx dr ds \\ &\quad + \int_{\tau}^T \int_s^T \int_{\Omega} \int_{-h}^0 [G(\sigma, u^1(r+\sigma)) - G(\sigma, u^2(r+\sigma))] v(r) d\sigma dx dr ds. \end{aligned} \quad (55)$$

Substituting (54) into (55), we get

$$\begin{aligned} TE_{\nu}(T) + \nu \int_{\tau}^T \int_s^T \int_{\Omega} |\nabla v|^2 dx dr ds &\leq CE_{\nu}(\tau) - C \int_{\tau}^T \int_{\Omega} [B(u^1) - B(u^2)] v(r) dx dr \\ &\quad + C \int_{\tau}^T \int_{\Omega} \int_{-h}^0 [G(\sigma, u^1(r+\sigma)) - G(\sigma, u^2(r+\sigma))] v(r) d\sigma dx dr \\ &\quad - \int_{\tau}^T \int_s^T \int_{\Omega} [B(u^1) - B(u^2)] v(r) dx dr ds \\ &\quad + \int_{\tau}^T \int_s^T \int_{\Omega} \int_{-h}^0 [G(\sigma, u^1(r+\sigma)) - G(\sigma, u^2(r+\sigma))] v(r) d\sigma dx dr ds. \end{aligned} \quad (56)$$

Set

$$C_0 = CE_{\nu}(\tau) = \frac{C}{2} \int_{\Omega} |\phi^1(0) - \phi^2(0)|^2 dx + \frac{C\alpha^2}{2} \int_{\Omega} |\nabla \phi^1(0) - \nabla \phi^2(0)|^2 dx, \quad (57)$$

$$\begin{aligned} \Phi(u^1, u^2) &= -C \int_{\tau}^T \int_{\Omega} [B(u^1) - B(u^2)] v(r) dx dr \\ &\quad + C \int_{\tau}^T \int_{\Omega} \int_{-h}^0 [G(\sigma, u^1(r+\sigma)) - G(\sigma, u^2(r+\sigma))] v(r) d\sigma dx dr \\ &\quad - \int_{\tau}^T \int_s^T \int_{\Omega} [B(u^1) - B(u^2)] v(r) dx dr ds \\ &\quad + \int_{\tau}^T \int_s^T \int_{\Omega} \int_{-h}^0 [G(\sigma, u^1(r+\sigma)) - G(\sigma, u^2(r+\sigma))] v(r) d\sigma dx dr ds; \end{aligned} \quad (58)$$

then, by (56), we have

$$E_v(T) \leq \frac{C_0}{T} + \frac{1}{T} \Phi(u^1, u^2). \quad (59)$$

One has $\forall \epsilon > 0$; we choose $T > 0$ large enough such that $C_0/T < \epsilon$. By Theorem 4, it suffices to prove that $\Phi(u^1, u^2)$ is

a contractive function, since B is a bounded positive invariant set.

If $u^m \rightarrow u$ ($m \rightarrow \infty$), we have the limits

$$\begin{aligned} & \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \int_{\tau}^T \int_{\Omega} (B(u^n) - B(u^m)) (u^n - u^m) dx dr \\ &= \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \int_{\tau}^T \int_{\Omega} [((u^n - u^m) \cdot \nabla) u^n - (u^m \cdot \nabla) (u^n - u^m)] (u^n - u^m) dx dr = 0, \\ & \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \int_{\tau}^T \int_s^T \int_{\Omega} (B(u^n) - B(u^m)) (u^n - u^m) dx dr ds \\ &= \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \int_{\tau}^T \int_s^T \int_{\Omega} [((u^n - u^m) \cdot \nabla) u^n - (u^m \cdot \nabla) (u^n - u^m)] (u^n - u^m) dx dr ds = 0. \end{aligned} \quad (60)$$

By (H_1) – (H_3) , we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \int_{\tau}^T \int_{\Omega} \int_{-h}^0 (G(\sigma, u^n(r+\sigma)) - G(\sigma, u^m(r+\sigma))) v(r) d\sigma dx dr \\ & \leq \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \int_{\tau}^T \int_{\Omega} \int_{-h}^0 L |u^n(r+\sigma) - u^m(r+\sigma)| |u^n(r) - u^m(r)| d\sigma dx dr = 0, \\ & \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \int_{\tau}^T \int_s^T \int_{\Omega} \int_{-h}^0 (G(\sigma, u^n(r+\sigma)) - G(\sigma, u^m(r+\sigma))) v(r) d\sigma dx dr ds \\ & \leq \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \int_{\tau}^T \int_s^T \int_{\Omega} \int_{-h}^0 L |(u^n - u^m)(r+\sigma)| |(u^n - u^m)(r)| d\sigma dx dr ds = 0. \end{aligned} \quad (61)$$

Combining (60)–(61) with (58), we have that $\Phi(u^1, u^2)$ is a contractive function.

The proof is complete. $\square$

Theorem 6. Assume that (H_1) – (H_3) hold and $L \leq \nu^2 \lambda_1^2 \epsilon / 4e^{\epsilon h}$; then, the process $\{U(t, \tau)\}$ generated by the global solution for problem (10) has a pullback attractor.

Proof. Lemma 3 implies $\{U(t, \tau)\}$ has a pullback absorbing set B and Lemma 5 implies $\{U(t, \tau)\}$ is asymptotically compact; we obtain the conclusion immediately. $\square$

According to Theorem 6, $0 < \epsilon < \min\{\nu/\alpha^2, \nu\lambda_1/4\}$ and $L \leq \nu^2 \lambda_1^2 \epsilon / 4e^{\epsilon h}$. Under the assumptions (H_1) – (H_3) , the NSV equation (1) with a distributed delay has a pullback attractor.

Conflict of Interests

The authors declare that they have no competing interests.

Authors' Contribution

Yantao Guo carried out the long time behavior of solutions. Shuilin Cheng carried out the pullback attractor. Yanbin Tang carried out the distributed delay. All authors read and approved the final paper.

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