Research Article

Discrete Optimal Control Method Based on the Optimal Strategy of Fishing

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Consideration was given to the discrete optimal control method for the optimal fishing strategy. Our method is new and efficient for discrete optimal control problem, which is different from the other optimal methods such as the traditional variational method, the Pontryagin principle of maximum, and the dynamic programming. The basic construction of the model is the traditional logistic function relating to the growth of fry. The discrete optimal control method for optimal fishing strategy was used to construct the optimal rate of each fishing strategy; the main focus of our work is on the rigorous mathematical analysis of the optimal control problem. The analysis allows one to obtain the optimal initial investment amount of the fry and the optimal size of the total catch. Furthermore, when the initial investment amount of the fry is below or above the optimal value and the intrinsic growth rate of fish R is too small, we derive that fishing operations should not be started in the last few years to make the overall fishing amount optimal. At last, several typical examples are given to illustrate the obtained results.

1. Introduction

Optimal control problem has been studied for many years, and in recent years, the optimal harvesting strategy of some species has been studied extensively by many authors [1–3]. An optimal harvesting policy is given using Pontryagin's maximum principle by many authors [1, 2]. They consider the following fishery model:

$$\dot{x} = G(x) - qEx, \quad (1)$$

where x is biomass of fish population, $G(x)$ is the biological net growth rate, q is catch ability coefficient, and E is fishing effort. This model was originally developed by Schaefer [4] as a management tool for the Eastern Tropical Pacific Tuna Fishery. In the original Schaefer model, $G(x)$ was specified in "logistic" form

$$G(x) = rx \left(1 - \frac{x}{K} \right), \quad (2)$$

where r and K are positive parameters called the "intrinsic growth rate" and the "carrying capacity," respectively.

The term $h = qEx$ represents the rate of mortality imposed by the fishery, that is, the rate of catch, corresponding to a given input of fishing "effort" E . Their objective is to determine

$$\max h = qEx. \quad (3)$$

The study of predator-prey models with harvesting has also attracted the attention of researchers; see, for example, Brauer and Soudack [5–7], Beddington and Cooke [8], Dai and Tang [9], Hogarth et al. [10], Myerscough et al. [11], and Goh [12].

Extensive and unregulated harvesting of fishes can even lead to the dropping of the overall number of fishing. Some associated documents have shown that the optimal initial investment amount of the fry $N/2$ is the optimal value to guarantee the optimal overall fishing amount [2], but they did not give the optimal rate of each fishing strategy and more important, however, is that when the initial investment amount of the fry is not $N/2$, how can we decide the optimal fishing strategy?

In this paper, we will give the discrete optimal control method of the optimal fishing strategy. This method is new and efficient for discrete optimal control problem, which

is different from the other optimal methods such as the traditional variational method, the Pontryagin principle of maximum, and the discrete dynamic programming. The basic construction of the model is the traditional logistic function relating to the growth of fry. The discrete optimal control method for optimal fishing strategy was used to construct the optimal rate of each fishing strategy; the main focus of our work is on the rigorous mathematical analysis of the optimal control problem. The analysis allows one to obtain the optimal initial investment amount of the fry and the overall optimal fishing amount. Furthermore, when the initial investment amount of the fry is below or above the optimal value and the intrinsic growth rate of fish R is too small, we derive that fishing operations should not be started in the last few years to make the overall fishing amount optimal. However, the other optimal methods such as the traditional variational method require the controlled variable u_n not to be restricted; hence the variational method is not available. The discrete dynamic programming method is not convenient in the calculation for the logistic model is complex and not linear. The maximum principle in the calculation of the discrete control optimization is not convenient too. Hence, we pointed out that our method is new and efficient for discrete optimal control problem. Finally, several typical examples are given to illustrate the obtained results.

The aim of this paper is to undertake the mathematical analysis of the optimal control problem introduced above, namely, to maximize (7) subject to (4)–(6). We also provide some numerical results that illustrate the theoretical results by several typical examples. We remark that there are few optimal control studies in the literature for discrete optimal control problem.

An outline of the paper is as follows. In Section 2, discrete optimal control method of the optimal fishing strategy is given. This method is used to construct the optimal rate of each fishing strategy and then obtain the optimal initial investment amount of the fry and the overall optimal fishing amount. In Section 3, we discuss that when the initial investment amount of the fry is not the optimal value and the intrinsic growth rate of fish R is too small, fishing operations should not be started in the last few years to make the overall fishing amount optimal, and then, several typical examples are given to illustrate the obtained results. In Section 4, we give a detailed table to show some figures to illustrate the obtained results. Finally, in Section 5 we draw a conclusion.

2. Discrete Optimal Control Method to Construct the Optimal Fishing Rate

Let x_n denote the original amount of fry, which is the density of fish in the n th years. Let

$$F(x_n) = R \left(1 - \frac{x_n}{N} \right) x_n \quad (4)$$

be the natural growth rate of fish, where $R > 0$ is the intrinsic growth rate of fish, $N > 0$ is the maximum carrying capacity

of environment, which is assumed to be fixed within three years, and $n = 1, 2, 3$ is the year under consideration. Let

$$x_{n+1} = (1 - u_n)F(x_n) \quad (5)$$

be the remaining amount of fish after n years, where $u_n \in [0, 1]$ is the degree of each fishing strategy in the n th years; choose the control set $U \subset R^n$, defined as

$$U = \{(u_1, u_2, u_3) \mid 0 \leq u_n \leq 1, n = 1, 2, 3\}. \quad (6)$$

We define the objective functional

$$J(u_n) = \sum_{n=1}^3 u_n F(x_n), \quad (7)$$

where $u_n F(x_n)$ represents the amount of fishing in the n th years. Our goal is to maximize $J(u_n)$, that is, to find a u_n such that

$$J(u_n^*) = \max_u J(u_n), \quad (8)$$

where u_n satisfies conditions (4)–(6).

Theorem 1. Assume that $n = 3, R \geq 2$, and conditions (4)–(6) hold; then there exists the optimal initial investment amount of the fry $x_1^* = N/2$, and the optimal rate of each fishing strategy is $u_1^* = u_2^* = 1 - 2/R, u_3^* = 1$ making the overall amount of fishing the optimal value $J^* = 3RN/4 - N$.

Proof. The way we consider this problem is similar to the dynamic programming; to make the overall amount of fishing the optimal value, we let $u_3^* = 1$, which means that we fish all in the last year:

$$J_3 = u_3 F(x_3) = F(x_3) = R \left(1 - \frac{x_3}{N} \right) x_3. \quad (9)$$

It is easy to see that J_3 is a quadratic function about x_3 , when $x_3^* = N/2$, J_3 to be the maximum $J_3^* = RN/4$, we have completed the third fishing progress.

From condition (5), we have

$$\frac{N}{2} = (1 - u_2)F(x_2) \quad (10)$$

and then we can derive that

$$J_2 = u_2 F(x_2) = u_2 \frac{N/2}{1 - u_2}. \quad (11)$$

Derive $J_2, J_2' = (N/2)/(1 - u_2)^2 > 0$. Obviously, J_2 is an increasing function on u_2 . Since

$$F(x_2) = R \left(1 - \frac{x_2}{N} \right) x_2 \leq \frac{RN}{4} \quad (12)$$

and when $x_2 = N/2$, the inequality could be equality. Adjoining this inequality with (10) gives

$$\frac{N}{2} = (1 - u_2)F(x_2) \leq (1 - u_2) \frac{RN}{4} \quad (13)$$

and this relation yields $u_2 \leq 1 - 2/R$. When $u_2^* = 1 - 2/R$, then

$$J_2^* = u_2 F(x_2) = \frac{R-2}{R} \frac{RN}{4} = \frac{(R-2)N}{4}, \quad (14)$$

where $x_2^* = N/2$; we have completed the second fishing progress.

Similarly, from condition (5), we have

$$\frac{N}{2} = (1 - u_1)F(x_1). \quad (15)$$

We can derive that

$$J_1 = u_1 F(x_1) = u_1 \frac{N/2}{1 - u_1}. \quad (16)$$

Derive $J_1, J_1' = (N/2)/(1 - u_1)^2 > 0$, where J_1 is an increasing function on u_1 . Since

$$F(x_1) = R \left(1 - \frac{x_1}{N}\right) x_1 \leq \frac{RN}{4}, \quad (17)$$

then

$$\frac{N}{2} = (1 - u_1)F(x_1) \leq (1 - u_1) \frac{RN}{4}. \quad (18)$$

We are able to obtain that $u_1 \leq 1 - 2/R$. When $u_1^* = 1 - 2/R$, then

$$J_1^* = u_1 F(x_1) = \frac{R-2}{R} \frac{RN}{4} = \frac{(R-2)N}{4}, \quad (19)$$

where $x_1^* = N/2$; we have completed the first fishing progress.

To sum up, we can determine the optimal rate of each fishing strategy as

$$u_1^* = u_2^* = 1 - \frac{2}{R}, \quad u_3^* = 1 \quad (20)$$

and the overall amount of fishing to be the optimal value as

$$\begin{aligned} J^* &= J_1^* + J_2^* + J_3^* \\ &= \frac{(R-2)N}{4} + \frac{(R-2)N}{4} + \frac{RN}{4} = \frac{3RN}{4} - N. \end{aligned} \quad (21)$$

The optimal initial investment amount of the fry is $x_1^* = N/2$. $\square$

Corollary 2. Assume that $n = k, R \geq 2$, and conditions (4)–(6) hold; then there exists the optimal initial investment amount of the fry $x_1^* = N/2$, and the optimal degree of each fishing strategy is

$$u_1^* = u_2^* = \dots = u_{k-1}^* = 1 - \frac{2}{R}, \quad u_k^* = 1 \quad (22)$$

making the overall amount of fishing the optimal value

$$J^* = J_1^* + J_2^* + \dots + J_{k-1}^* + J_k^* = (k-1) \frac{(R-2)N}{4} + \frac{RN}{4}. \quad (23)$$

Note 1. In Theorem 1 we take N to be fixed, since the capacity of environment cannot change greatly in three years. When time is more than three years or when it has been a long time, we can consider that N is possible to be changed, and recounting is necessary.

Note 2. In front of the proof, we say that the way we consider this problem is similar to the discrete dynamic programming. Actually, this problem cannot be solved by other optimal methods such as the traditional variational method, the Pontryagin principle of maximum, and the dynamic programming. This is because we need a hypothesis in discrete optimal control problem: $\delta x(k)$ and $\delta u(k)$ are linearly independent, which can be assumed for discrete optimal control problem of free terminal, but for discrete optimal control problem of fixed terminal, this assumption is not established for the state equation $x(k+1) = f(x(k), u(k), k)$.

From Theorem 1 we can derive that the optimal initial investment amount of the fry is $x_1^* = N/2$, and the optimal rate of each fishing strategy is $u_1^* = u_2^* = 1 - 2/R, u_3^* = 1$, and then the overall amount of fishing is the optimal value $J^* = 3RN/4 - N$. However, in realistic application the initial investment amount of the fry may be above or below the optimal value $x_1^* = N/2$ and $R \geq 2$ may not hold; then how can we decide the optimal degree of each fishing strategy u_n to make the overall amount of fishing the optimal value? These questions are investigated below.

3. Optimal Fishing Strategy When the Initial Investment Amount of the Fry Is Not the Optimal Value $N/2$

Theorem 3. Suppose that $n = 3, x_1 = cN, c \in (0, 1), 0 < R_1 < 1/2a$, and $a = c(1 - c)$ where

$$\begin{aligned} R_1 &= \frac{1}{3a} \left[\frac{\sqrt{(27a-4)^2 - 4^2} - (27a-4)}{4} \right]^{1/3} \\ &+ \frac{1}{3a} \left[\frac{\sqrt{(27a-4)^2 - 4^2} + (27a-4)}{4} \right]^{1/3} \end{aligned} \quad (24)$$

and conditions (4)–(7) hold. Then one has the following optimal fishing strategy.

- (1) When $R \in (0, R_1)$, one does not fish in the two previous kinds of fishing progress.
- (2) When $R \in [R_1, 1/2a)$ ($R_1 < 1/2a$), one does not fish in the first fishing progress.
- (3) When $R \in [1/2a, +\infty)$, one fishes in each kind of progress.

In every kind of fishing progress, one can determine the optimal fishing degree u_n and obtain the overall optimal amount of fishing J .

Proof. Assume $x_1 = cN$, where $c \in (0, 1)$ is the density coefficient of the initial fry. From condition (5),

$$F(x_1) = R\left(1 - \frac{x_1}{N}\right)x_1 = R(1 - c)cN = RaN, \quad (25)$$

where $a = c(1 - c) \in (0, 1/4)$. From condition (5),

$$x_2 = (1 - u_1)F(x_1). \quad (26)$$

Suppose that $u_1 = 0$; then $x_2 = F(x_1) = RaN$; when $x_2 = RaN < N/2$ (i.e., $R < 1/2a$), u_1 should be 0, or $x_2 < F(x_1) = RaN < N/2$ does not reach the optimal investment value in the next fishing process. $\square$

Step 1. In the first fishing process, we divided the situation into two circumstances.

- (1) When $x_2 = RaN < N/2$ (i.e., $R < 1/2a$), we have $u_1^* = 0$, $J_1^* = u_1^*F(x_1) = 0$ ($R < 1/2a$).
- (2) When $x_2 = RaN \geq N/2$ (i.e., $R \geq 1/2a$), we have $u_1^* \neq 0$; to decide u_1^* we set $x_2 = x_2^* = N/2$, from (5), where

$$\frac{N}{2} = (1 - u_1)RaN. \quad (27)$$

We get

$$u_1^* = 1 - \frac{1}{2Ra} \quad (28)$$

and then

$$\begin{aligned} J_1^* &= u_1^*F(x_1) = \left(1 - \frac{1}{2Ra}\right)RaN \\ &= \frac{N}{2}(2Ra - 1) \left(R \geq \frac{1}{2a}\right). \end{aligned} \quad (29)$$

It is easy to see that when $x_2 = RaN \geq N/2$ (i.e., $R \geq 1/2a$), our fishing amount $J_1^* = (N/2)(2Ra - 1) = RaN - N/2$ is exactly equal to $x_2 - x_2^* = RaN - N/2$, which is just the exceeded amount.

Step 2. In the second fishing process, we divided the situation into two circumstances too.

- (1) When $R < 1/2a$, $x_2 = RaN < N/2$, from condition (4), we obtain

$$F(x_2) = R\left(1 - \frac{x_2}{N}\right)x_2 = R(1 - Ra)RaN = R^2aN(1 - Ra). \quad (30)$$

From condition (5),

$$x_3 = (1 - u_2)F(x_2). \quad (31)$$

Suppose that $u_2 = 0$, and then $x_3 = F(x_2) = R^2aN(1 - Ra)$; when $x_3 = R^2aN(1 - Ra) < N/2$ (i.e., $R < R_1 < 1/2a$), u_2 should be 0; otherwise $x_3 < F(x_2) < N/2$ does not attain the optimal investment value in the next fishing process.

- (a) When $x_3 = R^2aN(1 - Ra) < N/2$ ($R < R_1 < 1/2a$), we have

$$u_2^* = 0, \quad J_2^* = u_2^*F(x_2) = 0 \quad \left(R < R_1 < \frac{1}{2a}\right), \quad (32)$$

where

$$\begin{aligned} R_1 &= \frac{1}{3a} \left[\frac{\sqrt{(27a - 4)^2 - 4^2} - (27a - 4)}{4} \right]^{1/3} \\ &+ \frac{1}{3a} \left[\frac{\sqrt{(27a - 4)^2 - 4^2} + (27a - 4)}{4} \right]^{1/3} \end{aligned} \quad (33)$$

which is the solution of $x_3 = R^2aN(1 - Ra) = N/2$ (i.e., $a^2R^3 - aR^2 + 1/2 = 0$).

To sum up, when $R \in (0, R_1)$ ($R_1 < 1/2a$), we have

$$\begin{aligned} u_1^* &= u_2^* = 0, \quad J_1^* = J_2^* = 0 \\ x_1 &= cN, \quad x_2 = RaN < \frac{N}{2}, \\ x_3 &= R^2aN(1 - Ra) < \frac{N}{2}. \end{aligned} \quad (34)$$

- (b) When $x_3 = R^2aN(1 - Ra) \geq N/2$ ($R_1 \leq R \leq 1/2a$), we have $u_2^* \neq 0$; similar to Step 1 we set $x_3 = x_3^* = N/2$; from (5),

$$\frac{N}{2} = (1 - u_2)R^2aN(1 - Ra). \quad (35)$$

We get

$$u_2^* = 1 - \frac{1}{2R^2a(1 - Ra)} \quad (36)$$

and then

$$\begin{aligned} J_2^* &= u_2^*F(x_2) = \left(1 - \frac{1}{2R^2a(1 - Ra)}\right)R^2aN(1 - Ra) \\ &= \frac{N}{2} [2R^2a(1 - Ra) - 1] \quad \left(R_1 \leq R \leq \frac{1}{2a}\right). \end{aligned} \quad (37)$$

Obviously, when $x_3 = R^2aN(1 - Ra) \geq N/2$ ($R_1 \leq R \leq 1/2a$), we can verify that $J_2^* = x_3 - x_3^* = R^2aN(1 - Ra) - N/2$, which is just the exceeded amount.

To sum up, when $R \in [R_1, 1/2a)$ ($R_1 < 1/2a$), we have

$$\begin{aligned} u_1^* &= 0, \quad J_1^* = 0; \\ u_2^* &= 1 - \frac{1}{2R^2a(1 - Ra)}, \quad J_2^* = \frac{N}{2} [2R^2a(1 - Ra) - 1] \\ x_1 &= cN, \quad x_2 = RaN < \frac{N}{2}, \quad x_3^* = \frac{N}{2}. \end{aligned} \quad (38)$$

(2) When $R \geq 1/2a$, $x_2 = N/2$ along with condition (4), we obtain

$$F(x_2) = R\left(1 - \frac{x_2}{N}\right)x_2 = \frac{RN}{4}. \quad (39)$$

From condition (5),

$$x_3 = (1 - u_2)F(x_2). \quad (40)$$

Suppose that $u_2 = 0$, and then $x_3 = F(x_2) = RN/4$; when $x_3 = RN/4 < N/2$ (i.e., $R < 2$), u_2 should be 0; otherwise $x_3 < F(x_2) < N/2$ does not attain the optimal investment value in the next fishing process.

(a) When $x_3 = RN/4 < N/2$ ($R < 2$), we have

$$u_2^* = 0, \quad J_2^* = u_2^*F(x_2) = 0 \quad (R < 2). \quad (41)$$

Since $c \in (0, 1)$, then $a = c(1 - c) \in (0, 1/4)$; we get $1/2a \geq 2$, and therefore, $R < 2$ and $R \geq 1/2a$ have no intersection set.

(b) When $x_3 = RN/4 \geq N/2$ ($R \geq 2$), we have $u_2^* \neq 0$; we set $x_3 = x_3^* = N/2$; from (5),

$$\frac{N}{2} = (1 - u_2)\frac{RN}{4}. \quad (42)$$

We get

$$u_2^* = 1 - \frac{2}{R} \quad (43)$$

and then

$$J_2^* = u_2^*F(x_2) = \left(1 - \frac{2}{R}\right)\frac{RN}{4} = \frac{R-2}{4}N \quad (R \geq 2). \quad (44)$$

Since $1/2a \geq 2$, therefore, $R \geq 2$ and $R \geq 1/2a$ have the intersection set $R \geq 1/2a$.

Obviously, when $x_3 = RN/4 \geq N/2$ ($R \geq 2$), we can verify that $J_2^* = x_3 - x_3^* = RN/4 - N/2$, which is just the exceeded amount.

To sum up, when $R \in [1/2a, +\infty)$, we have

$$\begin{aligned} u_1^* &= 1 - \frac{1}{2Ra}, & J_1^* &= \frac{N}{2}(2Ra - 1); \\ u_2^* &= 1 - \frac{2}{R}, & J_2^* &= \frac{R-2}{4}N \\ x_1 &= cN, & x_2^* &= \frac{N}{2}, & x_3^* &= \frac{N}{2}. \end{aligned} \quad (45)$$

Step 3. In the third fishing process, we divided the situation into two circumstances.

(1) When $R \in (0, R_1)$ ($R_1 < 1/2a$), we have $x_3 = R^2aN(1 - Ra) < N/2$ along with condition (4); we obtain

$$\begin{aligned} F(x_3) &= R\left(1 - \frac{x_3}{N}\right)x_3 \\ &= R[1 - R^2a(1 - Ra)]R^2a(1 - Ra)N \\ &= R^3a(1 - Ra)N[1 - R^2a(1 - Ra)]. \end{aligned} \quad (46)$$

In the last fishing process, we let $u_3^* = 1$, and then

$$J_3^* = u_3^*F(x_3) = R^3a(1 - Ra)N[1 - R^2a(1 - Ra)]. \quad (47)$$

(2) When $R \in [R_1, 1/2a)$ or $R \in [1/2a, +\infty)$ ($R_1 < 1/2a$), $x_3^* = N/2$ along with condition (4); we obtain

$$F(x_3) = R\left(1 - \frac{x_3}{N}\right)x_3 = \frac{RN}{4}. \quad (48)$$

In the last fishing process, we let $u_3^* = 1$, and then

$$J_3^* = u_3^*F(x_3) = \frac{RN}{4}. \quad (49)$$

In summary, we obtain the following conclusion.

(1) When $R \in (0, R_1)$ ($R_1 < 1/2a$), we have

$$\begin{aligned} u_1^* &= u_2^* = 0, & u_3^* &= 1; \\ J_1^* &= J_2^* = 0, \\ J_3^* &= u_3^*F(x_3) = R^3a(1 - Ra)N[1 - R^2a(1 - Ra)] \\ J^* &= J_1^* + J_2^* + J_3^* = 0 + 0 + u_3^*F(x_3) \\ &= R^3a(1 - Ra)N[1 - R^2a(1 - Ra)] \\ &\left(x_1 = cN, x_2 = RaN < \frac{N}{2}, x_3 = R^2aN(1 - Ra) < \frac{N}{2}\right). \end{aligned} \quad (50)$$

It means that when $R \in (0, R_1)$ ($R_1 < 1/2a$), we do not fish in the two previous kinds of fishing progress and fish all in the last fishing progress. This has no sense from a practical point of view. Therefore, we consider that not going in for fish farming is the better strategy.

(2) When $R \in [R_1, 1/2a)$ ($R_1 < 1/2a$), we have

$$\begin{aligned} u_1^* &= 0, & u_2^* &= 1 - \frac{1}{2R^2a(1 - Ra)}, & u_3^* &= 1; \\ J_1^* &= 0, & J_2^* &= \frac{N}{2}[2R^2a(1 - Ra) - 1], \\ J_3^* &= u_3^*F(x_3) = \frac{RN}{4} \\ J^* &= J_1^* + J_2^* + J_3^* = 0 + \frac{N}{2}[2R^2a(1 - Ra) - 1] + \frac{RN}{4} \\ &\left(x_1 = cN, x_2 = RaN < \frac{N}{2}, x_3^* = \frac{N}{2}\right). \end{aligned} \quad (51)$$

It means that when $R \in [R_1, 1/2a)$ ($R_1 < 1/2a$), we do not fish in the first fishing progress, among which it is not difficult to verify $J_2^* = x_3 - x_3^*$ for $x_3 \geq x_3^*$.

(3) When $R \in [1/2a, +\infty)$, we have

$$\begin{aligned} u_1^* &= 1 - \frac{1}{2Ra}, & u_2^* &= 1 - \frac{2}{R}, & u_3^* &= 1; \\ J_1^* &= \frac{N}{2}(2Ra - 1), & J_2^* &= \frac{R-2}{4}N, & J_3^* &= \frac{RN}{4} \\ J^* &= J_1^* + J_2^* + J_3^* = \frac{N}{2}(2Ra - 1) + \frac{R-2}{4}N + \frac{RN}{4}. \end{aligned} \quad (52)$$

$$\left(x_1 = cN, x_2^* = \frac{N}{2}, x_3^* = \frac{N}{2} \right).$$

It means that when $R \in [1/2a, +\infty)$, we fish in each kind of progress. In every kind of fishing progress, we can determine the optimal fishing degree u_n and obtain the overall optimal amount of fishing J , among which it is not difficult to verify $J_1^* = x_2 - x_2^*$, $J_2^* = x_3 - x_3^*$ for $x_2 \geq x_2^*$ and $x_3 \geq x_3^*$, respectively.

Example 4. Consider Theorem 3 $c = 1/2$ and $x_1 = cN$; determine the optimal strategy of fishing using the discrete optimal control method.

Since $c = 1/2$, then $a = c(1-c) = 1/4$, $1/2a = 2$; we obtain $R_1 = 2$ by calculating the function $f(R) = R^3 - 4R^2 + 8 = 0$, which come from the equality $x_3 = (R^2N/4)(1 - R/4) = N/2$. A more detailed procedure is described below.

Since $x_1 = cN = N/2$,

$$F(x_1) = R\left(1 - \frac{x_1}{N}\right)x_1 = R(1-c)cN = RaN = \frac{RN}{4} \quad (53)$$

and then

$$x_2 = (1 - u_1)F(x_1). \quad (54)$$

Suppose that $u_1 = 0$; then $x_2 = F(x_1) = RaN = RN/4$; when $x_2 = RaN = RN/4 < N/2$ (i.e., $R < 2$), u_1 should be 0.

Step 1. In the first fishing process, we divided the situation into two circumstances.

(1) When $R < 2$, we have $u_1^* = 0$, $J_1^* = 0$.

(2) When $R \geq 2$, we have $u_1^* \neq 0$; we set $x_2 = x_2^* = N/2$; from (5),

$$\frac{N}{2} = (1 - u_1)\frac{RN}{4}. \quad (55)$$

We get

$$u_1^* = 1 - \frac{2}{R} \quad (56)$$

and then

$$J_1^* = u_1^*F(x_1) = \left(1 - \frac{2}{R}\right)\frac{RN}{4} = \frac{R-2}{4}N. \quad (57)$$

Step 2. In the second fishing process, we divided the situation into two circumstances too.

(1) When $R < 2$, $x_2 = RN/4 < N/2$; from condition (4), we obtain

$$F(x_2) = R\left(1 - \frac{x_2}{N}\right)x_2 = \frac{R^2N}{4}\left(1 - \frac{R}{4}\right). \quad (58)$$

From condition (5),

$$x_3 = (1 - u_2)F(x_2). \quad (59)$$

Suppose that $u_2 = 0$, and then $x_3 = F(x_2) = (R^2N/4)(1 - R/4)$; when $x_3 = (R^2N/4)(1 - R/4) < N/2$ (i.e., $R < 2$), u_2 should be 0.

Note. To solve $x_3 = (R^2N/4)(1 - R/4) < N/2$, we only find the solution of the following problem:

$$f(R) = R^3 - 4R^2 + 8 > 0. \quad (60)$$

In the derivatives of R ,

$$f'(R) = 3R^2 - 8R = R(3R - 8) \quad (61)$$

and when $R < 0$ or $R > 8/3$, then $f(R)$ is an increasing function; it is easy to see that $f(2) = 0$, when $0 < R < 2$, and the above inequality is right. It means that $R_1 = 1/2a = 2$.

To sum up, when $R \in (0, 2)$, we have

$$\begin{aligned} u_1^* &= u_2^* = 0, & J_1^* &= J_2^* = 0 \\ x_1 &= \frac{1}{2}N, & x_2 &= \frac{RN}{4} < \frac{N}{2}, \\ x_3 &= \frac{R^2N}{4}\left(1 - \frac{R}{4}\right) < \frac{N}{2}. \end{aligned} \quad (62)$$

(2) When $R \geq 2$, we have $u_2^* \neq 0$, and $x_2^* = N/2$ along with condition (4); we obtain

$$F(x_2) = R\left(1 - \frac{x_2}{N}\right)x_2 = \frac{RN}{4} \quad (63)$$

and we set $x_3 = x_3^* = N/2$; from condition (5),

$$\frac{N}{2} = (1 - u_2)\frac{RN}{4}. \quad (64)$$

We get

$$u_2^* = 1 - \frac{2}{R} \quad (65)$$

and then

$$J_2^* = u_2^*F(x_2) = \frac{(R-2)N}{4}. \quad (66)$$

We have

$$\begin{aligned} u_1^* &= 1 - \frac{2}{R}, & J_1^* &= \frac{(R-2)N}{4}; \\ u_2^* &= 1 - \frac{2}{R}, & J_2^* &= \frac{(R-2)N}{4} \\ x_1^* &= \frac{N}{2}, & x_2^* &= \frac{N}{2}, & x_3^* &= \frac{N}{2}. \end{aligned} \quad (67)$$

Step 3. In the third fishing process, we divided the situation into two circumstances.

- (1) When $R \in (0, 2)$, we have $x_3 = (R^2N/4)(1 - R/4) < N/2$ along with condition (4); we obtain

$$\begin{aligned} F(x_3) &= R\left(1 - \frac{x_3}{N}\right)x_3 \\ &= \frac{R^3(1 - R/4)N}{4} \left[1 - \frac{R^2(1 - R/4)}{4}\right]. \end{aligned} \quad (68)$$

In the last fishing process, we let $u_3^* = 1$, and then

$$J_3^* = u_3 F(x_3) = \frac{R^3(1 - R/4)N}{4} \left[1 - \frac{R^2(1 - R/4)}{4}\right]. \quad (69)$$

- (2) When $R \in [2, +\infty)$, $x_3 = N/2$ along with condition (4); we obtain

$$F(x_3) = \frac{RN}{4}. \quad (70)$$

In the last fishing process, we let $u_3^* = 1$, and then

$$J_3^* = u_3 F(x_3) = \frac{RN}{4}. \quad (71)$$

In summary, we obtain the following conclusion.

- (1) When $R \in (0, 2)$, we have

$$\begin{aligned} u_1^* &= u_2^* = 0, \quad u_3^* = 1; \\ J_1^* &= J_2^* = 0, \quad J_3^* = \frac{R^3(1 - R/4)N}{4} \left[1 - \frac{R^2(1 - R/4)}{4}\right]; \\ J^* &= J_1^* + J_2^* + J_3^* = 0 + 0 + u_3 F(x_3) \\ &= \frac{R^3(1 - R/4)N}{4} \left[1 - \frac{R^2(1 - R/4)}{4}\right]; \\ x_1 &= \frac{1}{2}N, \quad x_2 = \frac{RN}{4} < \frac{N}{2}, \\ x_3 &= \frac{R^2N}{4} \left(1 - \frac{R}{4}\right) < \frac{N}{2}. \end{aligned} \quad (72)$$

It means that when $R \in (0, 2)$, we do not fish in the two previous kinds of fishing progress. It is easy to see that

$$J^* = J_1^* + J_2^* + J_3^* = 0 + 0 + u_3 F(x_3) \leq \frac{RN}{4} < \frac{N}{2} = x_1^*. \quad (73)$$

It means that the maximum harvesting amount J^* is less than the initial optimal investment amount $x_1^* = N/2$ within three years; this has no sense from a practical point of view, let alone $x_1 \neq N/2$. Therefore, we consider that when the intrinsic growth rate of fish $R < 2$, not going in for fish farming is the better strategy. On the contrary, when $R \geq 2$, we consider that setting $x_1^* = N/2$ is the best policy decision and then determine u_n and J_n .

- (2) When $R \in [2, +\infty)$, we have

$$\begin{aligned} u_1^* &= 1 - \frac{2}{R}, \quad u_2^* = 1 - \frac{2}{R}, \quad u_3^* = 1; \\ J_1^* &= \frac{(R-2)N}{4}, \quad J_2^* = \frac{(R-2)N}{4}, \quad J_3^* = \frac{RN}{4}; \\ J^* &= J_1^* + J_2^* + J_3^* \\ &= \frac{(R-2)N}{4} + \frac{(R-2)N}{4} + \frac{RN}{4} = \frac{3RN}{4} - N. \\ x_1^* &= \frac{N}{2}, \quad x_2^* = \frac{N}{2}, \quad x_3^* = \frac{N}{2}. \end{aligned} \quad (74)$$

It means that when $R \in [2, +\infty)$, we fish in each kind of progress. In every kind of fishing progress, we can determine the optimal fishing degree u_n and obtain the overall optimal amount of fishing J_n .

Note. When $R \in [2, +\infty)$, $c = 1/2$, the result of Example 4 is just the very thing of Theorem 1. It means that Theorem 1 is a special case of Theorem 3.

Example 5. Consider Theorem 3 $x_1 = cN$; we set $c = 1/3$ and determine the optimal strategy of fishing using the discrete optimal control method.

For convenience, we only give the final result. Since $c = 1/3$, then $a = c(1 - c) = 2/9$, $1/2a = 9/4 = 2.25$; we obtain $R_1 \approx 2.02$ by calculating the function $f(R) = 2R^3 - 9R^2 + 20.25 = 0$, which come from the equality $x_3 = (2R^2(9 - 2R)/81)N = N/2$.

- (1) When $R \in (0, 2.02)$, we have

$$\begin{aligned} u_1^* &= u_2^* = 0, \quad u_3^* = 1; \\ J_1^* &= J_2^* = 0, \quad J_3^* = \frac{2R^3(9 - 2R)N[81 - 2R^2(9 - 2R)]}{81^2}; \\ J^* &= J_1^* + J_2^* + J_3^* = \frac{2R^3(9 - 2R)N[81 - 2R^2(9 - 2R)]}{81^2}; \\ \left(x_1 = \frac{N}{3}, x_2 = \frac{2RN}{9} < \frac{N}{2}, x_3 = \frac{2R^2N(9 - 2R)}{81} < \frac{N}{2}\right). \end{aligned} \quad (75)$$

It means that when $R \in (0, 2.02)$, we do not fish in the two previous kinds of fishing progress. It is easy to see that

$$J^* = J_1^* + J_2^* + J_3^* = 0 + 0 + u_3 F(x_3) \leq \frac{RN}{4}, \quad (76)$$

where J^* may be greater than $N/2 = x_1$ only when $R \in [2, 2.02)$. It means that the maximum harvesting amount $J_1^* + J_2^*$ is less than the initial optimal investment amount $x_1 = N/2$ within two years, and the maximum harvesting amount J^* is less likely to be more than the initial optimal investment amount $x_1 = N/2$ within three years; this has no sense from a practical point of view. Therefore, we consider

TABLE 1

x_1	$1/2a$	R_1	$R(0 < R_1 \leq 1/2a)$	J_1	J_2	J_3	J
$N/2$	2	2	$R = 1 \in (0, 2)$	0	0	$0.15N$	$0.15N = 150$
			$R = 2 \in [2, +\infty)$	0	0	$N/2$	$0.5N = 500$
			$R = 2.20 \in [2, +\infty)$	$0.05N$	$0.05N$	$0.55N$	$0.65N = 650$
			$R = 4 \in [2, +\infty)$	$N/2$	$N/2$	N	$2N = 2000$
$N/3$ or $2N/3$	2.25	2.02	$R = 1 \in (0, 2.02)$	0	0	$0.14N$	$0.14N = 140$
			$R = 2 \in (0, 2.02)$	0	0	$0.49N$	$0.49N = 499$
			$R = 2.20 \in [2.02, 2.25)$	0	$0.05N$	$0.55N$	$0.6N = 600$
			$R = 4 \in [2.25, +\infty)$	$7N/18$	$N/2$	N	$1.89N = 1890$

that when the intrinsic growth rate of fish $R < 2.02$, not going in for fish farming is the better strategy.

(2) When $R \in [2.02, 2.25)$, we have

$$\begin{aligned}
 u_1^* &= 0, & u_2^* &= 1 - \frac{81}{4R^2(9-2R)}, & u_3^* &= 1; \\
 J_1^* &= 0, & J_2^* &= \frac{4R^2(9-2R)-81}{81 \times 2}N, & J_3^* &= \frac{RN}{4} \\
 J^* &= J_1^* + J_2^* + J_3^* = 0 + \frac{4R^2(9-2R)-81}{81 \times 2}N + \frac{RN}{4}. \\
 \left(x_1 &= \frac{N}{3}, x_2 = \frac{2RN}{9} < \frac{N}{2}, x_3 = \frac{N}{2}\right).
 \end{aligned} \tag{77}$$

It means that when $R \in [2.02, 2.25)$, we do not fish in the first fishing progress. It is easy to see that the maximum harvesting amount J_1^* is less than the initial optimal investment amount $x_1 = N/2$ within one year; this has no sense from a practical point of view.

(3) When $R \in [2.25, +\infty)$, we have

$$\begin{aligned}
 u_1^* &= 1 - \frac{9}{4R}, & u_2^* &= 1 - \frac{2}{R}, & u_3^* &= 1; \\
 J_1^* &= \frac{4R-9}{18}N, & J_2^* &= \frac{R-2}{4}N, & J_3^* &= \frac{RN}{4} \\
 J^* &= J_1^* + J_2^* + J_3^* = \frac{4R-9}{18}N + \frac{R-2}{4}N + \frac{RN}{4}. \\
 \left(x_1 &= \frac{N}{3}, x_2^* = \frac{N}{2}, x_3^* = \frac{N}{2}\right).
 \end{aligned} \tag{78}$$

It means that when $R \in [2.25, +\infty)$, we fish in each kind of progress. In every kind of fishing progress, we can determine the optimal fishing degree u_n and obtain the overall optimal amount of fishing J_n .

Note. When Example 5 $x_1 = cN$, $c = 1 - 1/3 = 2/3$, we can obtain the same optimal strategy of fishing as Example 5. This is because no matter whether $c = 1/3$ or $2/3$, $a = c(1-c) = (1/3)(2/3) = 2/9$ and $F(x_1) = R(1 - x_1/N)x_1 = 2N/9$ is unchanging.

Corollary 6. No matter whether Theorem 3 $x_1 = c_1N$ or $x_1 = c_2N$, one can obtain the same optimal strategy of fishing on condition that $c_1 = 1 - c_2$.

Corollary 7. Some people mistakenly think that “more is better” when they give the initial investment amount, but in fact this is not the case. Only when $x_1 = N/2$ can one obtain the optimal overall fishing amount.

4. Discussion

From Sections 2 and 3 and Examples 4 and 5, we set $n = 3$ and obtain the following conclusions.

When the initial investment amount of fry $x_1 = N/2$, $R \geq 2$, we can determine u_n ($n = 1, 2, 3$) to make the overall fishing amount optimal. It is easy to see that when $R < 2$, we cannot guarantee $RN/4 \geq N/2$. It means that the maximum harvesting amount $u_n F(x_n) = RN/4$ is less than the initial optimal investment amount $N/2$, which has no sense from a practical point of view, let alone $x_1 \neq N/2$. Therefore, we consider that when the intrinsic growth rate of fish $R < 2$, not going in for fish farming is the better strategy. On the contrary, when $R \geq 2$, setting $x_1^* = N/2$ is the best policy decision. Assume that $x_1 \neq N/2$; if R is large enough, we only have $J_1 < J_1^*$, and the other J_n is possible to reach J_n^* ($n = 2, 3$); we do not cause great losses except for the first year.

In Section 3, we set $n = 3$ and obtain a series of conclusions. If $n > 3$, we can also gain some similar conclusions. When the initial investment amount of the fry is below or above the optimal value and the intrinsic growth rate of fish R is too small, we derive that fishing operations should not be started in the last few years to make the overall fishing amount optimal. To illustrate our conclusions clearly, we list the optimal total fishing amount J as follows.

Assume that $n = 3$, $x_1 = cN$, and $a = c(1-c)$, when $c = 1/2, 1/3$ or $2/3$ and $N = 1000$, detailed in Table 1.

From Table 1 we see that the overall optimal amount J of fishing is different when the initial investment amount of fry x_1 is different; only when initial investment amount of fry is $x_1 = N/2$, is the overall fishing amount optimal. On the other hand, in every kind of fishing progress, we can determine the optimal fishing degree u_n and obtain the overall optimal amount of fishing J . It is easy to see that when $R < R_1$, we do not fish in the two previous kinds of fishing progress; when $R \in [R_1, 1/2a)$ ($R_1 < 1/2a$), we do not fish in the first

fishing progress; when $R \in [1/2a, +\infty)$, we fish in each kind of progress.

5. Conclusion

From the rigorous mathematical analysis of the optimal control problem, we obtain the optimal initial investment amount of the fry and the optimal size of the total catch. When the initial investment amount of fry attains the optimal value $x_1 = N/2$, we can determine u_n ($n = 1, 2, 3$) to make the overall fishing amount optimal. Furthermore, when the initial investment amount of the fry is below or above the optimal value $x_1 = N/2$ and the intrinsic growth rate of fish R is too small, we derive that fishing operations should not be started in the last few years to make the overall fishing amount optimal.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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