Research Article

Existence of Positive Solution for BVP of Nonlinear Fractional Differential Equation

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We consider the following boundary value problem of nonlinear fractional differential equation: $({}^C D_{0+}^\alpha u)(t) = f(t, u(t))$, $t \in [0, 1]$, $u(0) = 0$, $u'(0) + u''(0) = 0$, $u'(1) + u''(1) = 0$, where $\alpha \in (2, 3]$ is a real number, ${}^C D_{0+}^\alpha$ denotes the standard Caputo fractional derivative, and $f : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$ is continuous. By using the well-known Guo-Krasnoselskii fixed point theorem, we obtain the existence of at least one positive solution for the above problem.

1. Introduction

Fractional calculus has demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering, for example, fluid flow, rheology, electrical networks, chemical physics, control theory of dynamical systems, optics, and signal processing [1].

Since the discussion of many problems can be summed up in the study of boundary value problems (BVPs for short) for nonlinear fractional differential equations, recently, the existence or uniqueness of solutions or positive solutions of BVPs for nonlinear fractional differential equations has received much attention from many authors; see [2–13] and the references therein.

In particular, Zhang [6] studied the existence and multiplicity of positive solutions to the following BVP of nonlinear fractional differential equation:

$$\begin{aligned} ({}^C D_{0+}^\alpha u)(t) &= f(t, u(t)), \quad t \in (0, 1), \\ u(0) + u'(0) &= 0, \\ u(1) + u'(1) &= 0, \end{aligned} \quad (1)$$

where $\alpha \in (1, 2]$ is a real number and ${}^C D_{0+}^\alpha$ denotes the standard Caputo fractional derivative. The main tools used were the Guo-Krasnoselskii and Leggett-Williams fixed point theorems.

In 2009, by using the nonlinear alternative of Leray-Schauder type and Guo-Krasnoselskii fixed point theorem, Bai and Qiu [12] discussed the existence of a positive solution to the following singular BVP of nonlinear fractional differential equation:

$$\begin{aligned} ({}^C D_{0+}^\alpha u)(t) + f(t, u(t)) &= 0, \quad t \in (0, 1), \\ u(0) = u'(1) = u''(0) &= 0, \end{aligned} \quad (2)$$

where $\alpha \in (2, 3]$ is a real number, ${}^C D_{0+}^\alpha$ denotes the standard Caputo fractional derivative, and $f(t, u)$ is singular at $t = 0$.

Motivated greatly by the above-mentioned works, in this paper, we consider the following BVP of nonlinear fractional differential equation:

$$\begin{aligned} ({}^C D_{0+}^\alpha u)(t) &= f(t, u(t)), \quad t \in [0, 1], \\ u(0) &= 0, \\ u'(0) + u''(0) &= 0, \\ u'(1) + u''(1) &= 0, \end{aligned} \quad (3)$$

where $\alpha \in (2, 3]$ is a real number, ${}^C D_{0+}^\alpha$ denotes the standard Caputo fractional derivative, and $f : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$ is continuous. First, Green's function $G(t, s)$ for the associated linear BVP is constructed. It is necessary to point

out that $G(t, s)$ is singular at $s = 1$. Next, some useful properties of $G(t, s)$ are studied. Finally, existence results of at least one positive solution for BVP (3) are obtained.

In order to obtain our main results, we need the following Guo-Krasnoselskii fixed point theorem [14, 15].

Theorem 1. *Let E be a Banach space and let K be a cone in E . Assume that Ω_1 and Ω_2 are bounded open subsets of E such that $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$, and let $T : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \rightarrow K$ be a completely continuous operator such that either*

- (1) $\|Tu\| \leq \|u\|$ for $u \in K \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|$ for $u \in K \cap \partial\Omega_2$ or
- (2) $\|Tu\| \geq \|u\|$ for $u \in K \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|$ for $u \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

2. Preliminaries

For the convenience of the reader, we present here some necessary definitions and results from fractional calculus theory. These definitions and results can be found in the recent literature [1].

In this section, we always assume that $\mathbb{N} = \{1, 2, 3, \dots\}$, $\alpha > 0$ and $[\alpha]$ denotes the integer part of α .

Definition 2. The Riemann-Liouville fractional integral $I_{0+}^\alpha u$ of order α is defined by

$$(I_{0+}^\alpha u)(t) := \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(s) ds}{(t-s)^{1-\alpha}}, \quad t > 0. \quad (4)$$

Definition 3. The Riemann-Liouville fractional derivative $D_{0+}^\alpha u$ of order α is defined by

$$\begin{aligned} (D_{0+}^\alpha u)(t) &:= \frac{d^n}{dt^n} (I_{0+}^{n-\alpha} u)(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{u(s) ds}{(t-s)^{\alpha-n+1}}, \end{aligned} \quad (5)$$

$$n = [\alpha] + 1, \quad t > 0.$$

Definition 4. The Caputo fractional derivative ${}^C D_{0+}^\alpha u$ of order α is defined via the above Riemann-Liouville fractional derivative by

$$({}^C D_{0+}^\alpha u)(t) := \left(D_{0+}^\alpha \left[u(s) - \sum_{k=0}^{n-1} \frac{u^{(k)}(0)}{k!} s^k \right] \right)(t), \quad (6)$$

where

$$n = \begin{cases} [\alpha] + 1, & \alpha \notin \mathbb{N}, \\ \alpha, & \alpha \in \mathbb{N}. \end{cases} \quad (7)$$

Lemma 5. *Let n be given by (7) and $u \in AC^n[0, 1]$ or $u \in C^n[0, 1]$. Then*

$$(I_{0+}^\alpha {}^C D_{0+}^\alpha u)(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1}, \quad (8)$$

where $c_i \in \mathbb{R}$, $i = 0, 1, \dots, n-1$.

3. Main Results

In the remainder of this paper, we always assume that $\alpha \in (2, 3]$ is a real number and $f : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$ is continuous.

Lemma 6. *Let $y \in C[0, 1]$ be a given function. Then the BVP*

$$\begin{aligned} ({}^C D_{0+}^\alpha u)(t) &= y(t), \quad t \in [0, 1], \\ u(0) &= 0, \\ u'(0) + u''(0) &= 0, \\ u'(1) + u''(1) &= 0 \end{aligned} \quad (9)$$

has a unique solution:

$$u(t) = \int_0^1 G(t, s) y(s) ds, \quad t \in [0, 1], \quad (10)$$

where

$$G(t, s) = \begin{cases} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(t-t^2/2)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{(t-t^2/2)(1-s)^{\alpha-3}}{\Gamma(\alpha-2)}, & 0 \leq s < t \leq 1, \\ \frac{(t-t^2/2)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{(t-t^2/2)(1-s)^{\alpha-3}}{\Gamma(\alpha-2)}, & 0 \leq t \leq s < 1. \end{cases} \quad (11)$$

Proof. In view of the equation in (9) and Lemma 5, we know that

$$\begin{aligned} u(t) &= (I_{0+}^\alpha y)(t) - c_0 - c_1 t - c_2 t^2 \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds - c_0 - c_1 t - c_2 t^2, \end{aligned} \quad (12)$$

and so,

$$\begin{aligned} u'(t) &= \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} y(s) ds - c_1 - 2c_2 t, \\ u''(t) &= \frac{1}{\Gamma(\alpha-2)} \int_0^t (t-s)^{\alpha-3} y(s) ds - 2c_2, \end{aligned} \quad (13)$$

which together with the boundary conditions in (9) implies that

$$\begin{aligned} c_0 &= 0, \\ c_1 &= -\left[\left(I_{0+}^{\alpha-1} \gamma\right)(1) + \left(I_{0+}^{\alpha-2} \gamma\right)(1)\right], \\ c_2 &= \frac{1}{2} \left[\left(I_{0+}^{\alpha-1} \gamma\right)(1) + \left(I_{0+}^{\alpha-2} \gamma\right)(1)\right]. \end{aligned} \quad (14)$$

Therefore, BVP (9) has a unique solution:

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \gamma(s) ds + \frac{t}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} \gamma(s) ds \\ &\quad + \frac{t}{\Gamma(\alpha-2)} \int_0^1 (1-s)^{\alpha-3} \gamma(s) ds \\ &\quad - \frac{t^2}{2\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} \gamma(s) ds - \frac{t^2}{2\Gamma(\alpha-2)} \int_0^1 (1-s)^{\alpha-3} \gamma(s) ds \\ &\quad = \int_0^1 \left[\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(t-t^2/2)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{(t-t^2/2)(1-s)^{\alpha-3}}{\Gamma(\alpha-2)} \right] \\ &\quad \cdot \gamma(s) ds + \int_t^1 \left[\frac{(t-t^2/2)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{(t-t^2/2)(1-s)^{\alpha-3}}{\Gamma(\alpha-2)} \right] \\ &\quad \cdot \gamma(s) ds, \quad t \in [0, 1]. \end{aligned} \quad (15)$$

□

Now, for $s \in [0, 1]$, we denote

$$\begin{aligned} M(s) &= \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2}}{2\Gamma(\alpha-1)} + \frac{(1-s)^{\alpha-3}}{2\Gamma(\alpha-2)}, \\ \gamma(s) &= \frac{5(\alpha-1)(1-s) + 5(\alpha-1)(\alpha-2)}{18(1-s)^2 + 9(\alpha-1)(1-s) + 9(\alpha-1)(\alpha-2)}. \end{aligned} \quad (16)$$

Remark 7. From the definition of function γ , it is not difficult to get that $\gamma(s) \geq \gamma(0)$ for $s \in [0, 1]$ and $\gamma(0) \in (0, 1)$.

Lemma 8. Green's function $G(t, s)$ defined by (11) has the following properties. Consider

- (1) $G(t, s) \in C([0, 1] \times [0, 1])$ and $G(t, s) > 0$ for $(t, s) \in (0, 1] \times [0, 1]$,

(2)

$$\begin{aligned} G(t, s) &\leq M(s), \quad (t, s) \in [0, 1] \times [0, 1], \\ G(t, s) &\geq \gamma(s) M(s), \quad (t, s) \in \left[\frac{1}{3}, \frac{2}{3}\right] \times [0, 1]. \end{aligned} \quad (17)$$

Proof. Since (1) is obvious, we only need to prove (2).

First, for any $(t, s) \in [0, 1] \times [0, 1]$, if $s < t$, then

$$\begin{aligned} G(t, s) &= \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(t-t^2/2)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \\ &\quad + \frac{(t-t^2/2)(1-s)^{\alpha-3}}{\Gamma(\alpha-2)} \\ &\leq \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2}}{2\Gamma(\alpha-1)} + \frac{(1-s)^{\alpha-3}}{2\Gamma(\alpha-2)} \\ &= M(s), \end{aligned} \quad (18)$$

and if $t \leq s$, then

$$\begin{aligned} G(t, s) &= \frac{(t-t^2/2)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{(t-t^2/2)(1-s)^{\alpha-3}}{\Gamma(\alpha-2)} \\ &\leq \frac{(s-s^2/2)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \\ &\quad + \frac{(s-s^2/2)(1-s)^{\alpha-3}}{\Gamma(\alpha-2)} \\ &< \frac{(1-s)^{\alpha-2}}{2\Gamma(\alpha-1)} + \frac{(1-s)^{\alpha-3}}{2\Gamma(\alpha-2)} < M(s). \end{aligned} \quad (19)$$

Next, for any $(t, s) \in [1/3, 2/3] \times [0, 1]$, if $s < t$, then

$$\begin{aligned} G(t, s) &= \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(t-t^2/2)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \\ &\quad + \frac{(t-t^2/2)(1-s)^{\alpha-3}}{\Gamma(\alpha-2)} \\ &> \frac{(t-t^2/2)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \\ &\quad + \frac{(t-t^2/2)(1-s)^{\alpha-3}}{\Gamma(\alpha-2)} \\ &\geq \frac{5}{18} \left[\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{(1-s)^{\alpha-3}}{\Gamma(\alpha-2)} \right] \\ &= \gamma(s) M(s), \end{aligned} \quad (20)$$

and if $t \leq s$, then

$$\begin{aligned} G(t, s) &= \frac{(t-t^2/2)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{(t-t^2/2)(1-s)^{\alpha-3}}{\Gamma(\alpha-2)} \\ &\geq \frac{5}{18} \left[\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{(1-s)^{\alpha-3}}{\Gamma(\alpha-2)} \right] = \gamma(s) M(s). \quad \square \end{aligned} \quad (21)$$

Let $E = C[0, 1]$ be equipped with norm $\|u\| = \max_{t \in [0, 1]} |u(t)|$ and

$$K = \left\{ u \in E : u(t) \geq 0, t \in [0, 1], \min_{t \in [1/3, 2/3]} u(t) \geq \gamma(0) \|u\| \right\}. \quad (22)$$

Then it is easy to check that E is a Banach space and K is a cone in E .

Now, we define an operator T on K by

$$(Tu)(t) = \int_0^1 G(t, s) f(s, u(s)) ds, \quad (23)$$

$$u \in K, t \in [0, 1].$$

Since $G(t, s)$ is singular at $s = 1$, we need to prove that operator T is well defined. In fact, for any fixed $u \in K$, we have $0 \leq u(s) \leq \|u\|$, $s \in [0, 1]$. If we let

$$L = \max_{(s, x) \in [0, 1] \times [0, \|u\|]} f(s, x), \quad (24)$$

then

$$f(s, u(s)) \leq L, \quad s \in [0, 1], \quad (25)$$

which together with Lemma 8 implies that

$$G(t, s) f(s, u(s)) \leq LM(s), \quad (t, s) \in [0, 1] \times [0, 1]. \quad (26)$$

Since $\int_0^1 M(s)ds$ is convergent, we obtain that $\int_0^1 G(t, s)f(s, u(s))ds$ is uniformly convergent on $[0, 1]$. This shows that operator T is well defined. Furthermore, we know that if $u \in K$, then $Tu \in E$ and if u is a fixed point of T in K , then u is a nonnegative solution of BVP (3).

Lemma 9. $T : K \rightarrow K$ is completely continuous.

Proof. First, for any $u \in K$, in view of Lemma 8, we have

$$\begin{aligned} 0 \leq (Tu)(t) &= \int_0^1 G(t, s) f(s, u(s)) ds \\ &\leq \int_0^1 M(s) f(s, u(s)) ds, \quad t \in [0, 1]. \end{aligned} \quad (27)$$

So,

$$\|Tu\| \leq \int_0^1 M(s) f(s, u(s)) ds, \quad (28)$$

which together with Lemma 8 implies that

$$\begin{aligned} (Tu)(t) &= \int_0^1 G(t, s) f(s, u(s)) ds \\ &\geq \int_0^1 \gamma(s) M(s) f(s, u(s)) ds \\ &\geq \gamma(0) \int_0^1 M(s) f(s, u(s)) ds \geq \gamma(0) \|Tu\|, \end{aligned} \quad (29)$$

$$t \in \left[\frac{1}{3}, \frac{2}{3} \right],$$

which shows that

$$\min_{t \in [1/3, 2/3]} (Tu)(t) \geq \gamma(0) \|Tu\|. \quad (30)$$

Therefore, $Tu \in K$.

Next, we prove that T is continuous. Suppose that u_n ($n = 1, 2, \dots$), $u_0 \in K$ and $\|u_n - u_0\| \rightarrow 0$ ($n \rightarrow \infty$). Then there exists $D^* > 0$ such that for any n , $0 \leq u_n(s) \leq D^*$, $s \in [0, 1]$. If we let

$$D = \max_{(s, x) \in [0, 1] \times [0, D^*]} f(s, x), \quad (31)$$

then for any n and $t \in [0, 1]$, in view of Lemma 8, we have

$$G(t, s) f(s, u_n(s)) \leq DM(s), \quad s \in [0, 1]. \quad (32)$$

By applying Lebesgue dominated convergence theorem, we get

$$\begin{aligned} \lim_{n \rightarrow \infty} (Tu_n)(t) &= \lim_{n \rightarrow \infty} \int_0^1 G(t, s) f(s, u_n(s)) ds \\ &= \int_0^1 G(t, s) f(s, u_0(s)) ds \\ &= (Tu_0)(t), \quad t \in [0, 1], \end{aligned} \quad (33)$$

which indicates that T is continuous.

Finally, we show that T is compact. Assume that $P \subset K$ is a bounded set. Then there exists a constant $H^* > 0$ such that $0 \leq u(s) \leq H^*$, $s \in [0, 1]$ for any $u \in P$. In what follows, we will prove that $T(P)$ is relatively compact. Let

$$H = \max_{(s, x) \in [0, 1] \times [0, H^*]} f(s, x). \quad (34)$$

On the one hand, for any $y \in T(P)$, there exists $u \in P$ such that $y = Tu$, and so, it follows from (34) and Lemma 8 that

$$\begin{aligned} 0 \leq y(t) &= (Tu)(t) = \int_0^1 G(t, s) f(s, u(s)) ds \\ &\leq H \int_0^1 M(s) ds, \quad t \in [0, 1], \end{aligned} \quad (35)$$

which together with the fact that $\int_0^1 M(s)ds$ is convergent implies that $T(P)$ is uniformly bounded. On the other hand, for any $\varepsilon > 0$, since $\int_0^1 M(s)ds$ is convergent, we may choose $\xi \in (0, 1)$ such that

$$\int_\xi^1 M(s) ds < \frac{\varepsilon}{4H}; \quad (36)$$

since $G(t, s)$ is uniformly continuous on $[0, 1] \times [0, \xi]$, there exists $\delta > 0$ such that for any $t_1, t_2 \in [0, 1]$ with $|t_1 - t_2| < \delta$,

$$|G(t_1, s) - G(t_2, s)| < \frac{\varepsilon}{2H\xi}, \quad s \in [0, \xi]. \quad (37)$$

For any $y \in T(P)$, there exists $u \in P$ such that $y = Tu$, and so, for any $t_1, t_2 \in [0, 1]$ with $|t_1 - t_2| < \delta$, it follows from (36), (37), and Lemma 8 that

$$\begin{aligned}
 |y(t_1) - y(t_2)| &= |(Tu)(t_1) - (Tu)(t_2)| \\
 &= \left| \int_0^1 [G(t_1, s) - G(t_2, s)] f(s, u(s)) ds \right| \\
 &\leq \int_0^1 |G(t_1, s) - G(t_2, s)| f(s, u(s)) ds \\
 &\leq H \int_0^1 |G(t_1, s) - G(t_2, s)| ds \\
 &= H \left[\int_0^\xi |G(t_1, s) - G(t_2, s)| ds \right. \\
 &\quad \left. + \int_\xi^1 |G(t_1, s) - G(t_2, s)| ds \right] \\
 &\leq H \left[\int_0^\xi |G(t_1, s) - G(t_2, s)| ds + 2 \int_\xi^1 M(s) ds \right] \\
 &< H \left(\frac{\varepsilon}{2H} + \frac{\varepsilon}{2H} \right) = \varepsilon,
 \end{aligned} \tag{38}$$

which indicates that $T(P)$ is equicontinuous. By Arzela-Ascoli theorem, we know that $T(P)$ is relatively compact. Therefore, $T : K \rightarrow K$ is completely continuous. $\square$

For convenience, we denote

$$\begin{aligned}
 A &= \left[\int_0^1 M(s) ds \right]^{-1}, \\
 B &= \left[\int_{1/3}^{2/3} \gamma(s) M(s) ds \right]^{-1}.
 \end{aligned} \tag{39}$$

Theorem 10. Assume that there exist two different positive constants r_1 and r_2 such that the following conditions are fulfilled:

$$f(t, x) \leq Ar_1, \quad (t, x) \in [0, 1] \times [0, r_1], \tag{40}$$

$$f(t, x) \geq Br_2, \quad (t, x) \in \left[\frac{1}{3}, \frac{2}{3} \right] \times [\gamma(0)r_2, r_2]. \tag{41}$$

Then BVP (3) has a positive solution u with $\min\{r_1, r_2\} \leq \|u\| \leq \max\{r_1, r_2\}$.

Proof. Without loss of generality, we assume that $r_1 < r_2$. Let

$$\begin{aligned}
 \Omega_1 &= \{u \in E : \|u\| < r_1\}, \\
 \Omega_2 &= \{u \in E : \|u\| < r_2\}.
 \end{aligned} \tag{42}$$

Then for any $u \in K \cap \partial\Omega_1$, we get $0 \leq u(s) \leq r_1$, $s \in [0, 1]$, which together with (40) and Lemma 8 implies that

$$\begin{aligned}
 0 &\leq (Tu)(t) = \int_0^1 G(t, s) f(s, u(s)) ds \\
 &\leq Ar_1 \int_0^1 M(s) ds = r_1 = \|u\|, \quad t \in [0, 1].
 \end{aligned} \tag{43}$$

This shows that

$$\|Tu\| \leq \|u\| \quad \text{for } u \in K \cap \partial\Omega_1. \tag{44}$$

For any $u \in K \cap \partial\Omega_2$, we get $\gamma(0)r_2 \leq u(s) \leq r_2$, $s \in [1/3, 2/3]$, which together with (41) and Lemma 8 implies that

$$\begin{aligned}
 (Tu)\left(\frac{1}{2}\right) &= \int_0^1 G\left(\frac{1}{2}, s\right) f(s, u(s)) ds \\
 &\geq \int_{1/3}^{2/3} G\left(\frac{1}{2}, s\right) f(s, u(s)) ds \\
 &\geq Br_2 \int_{1/3}^{2/3} \gamma(s) M(s) ds = r_2 = \|u\|.
 \end{aligned} \tag{45}$$

This indicates that

$$\|Tu\| \geq \|u\| \quad \text{for } u \in K \cap \partial\Omega_2. \tag{46}$$

Therefore, it follows from Theorem 1, Lemma 9, (44), and (46) that the operator T has a fixed point $u \in K \cap (\bar{\Omega}_2 \setminus \Omega_1)$, which is a desired positive solution of BVP (3). $\square$

Example 11. We consider the BVP

$$\begin{aligned}
 ({}^C D_{0+}^{5/2} u)(t) &= \frac{867\sqrt{\pi}tu^2(t)}{25}, \quad t \in [0, 1], \\
 u(0) &= 0,
 \end{aligned} \tag{47}$$

$$u'(0) + u''(0) = 0,$$

$$u'(1) + u''(1) = 0.$$

Since $\alpha = 5/2$, a direct calculation shows that

$$\begin{aligned}
 M(s) &= \frac{1}{\sqrt{\pi}} \left[\frac{4}{3} (1-s)^{3/2} + (1-s)^{1/2} + \frac{1}{2} (1-s)^{-1/2} \right], \\
 \gamma(s) &= \frac{5}{3} \left[\frac{2(1-s)+1}{8(1-s)^2+6(1-s)+3} \right].
 \end{aligned} \tag{48}$$

So,

$$\begin{aligned}
 \gamma(0) &= \frac{5}{17}, \\
 A &= \frac{5\sqrt{\pi}}{11}, \\
 B &= \frac{81\sqrt{3\pi}(13\sqrt{2}+11)}{1085}.
 \end{aligned} \tag{49}$$

Let $f(t, x) = 867\sqrt{\pi}tx^2/25$, $(t, x) \in [0, 1] \times [0, +\infty)$. If we choose $r_1 = 125/9537$ and $r_2 = 81\sqrt{3}(13\sqrt{2}+11)/1085$, then it is easy to verify that

$$\begin{aligned}
 f(t, x) &\leq f\left(1, \frac{125}{9537}\right) = Ar_1, \quad (t, x) \in [0, 1] \times \left[0, \frac{125}{9537}\right], \\
 f(t, x) &\geq f\left(\frac{1}{3}, \frac{81\sqrt{3}(13\sqrt{2}+11)}{3689}\right) = Br_2, \\
 (t, x) &\in \left[\frac{1}{3}, \frac{2}{3}\right] \times \left[\frac{81\sqrt{3}(13\sqrt{2}+11)}{3689}, \frac{81\sqrt{3}(13\sqrt{2}+11)}{1085}\right].
 \end{aligned} \tag{50}$$

So, all the hypotheses of Theorem 10 are fulfilled. Therefore, it follows from Theorem 10 that BVP (47) has a positive solution u satisfying

$$\frac{125}{9537} \leq \|u\| \leq \frac{81\sqrt{3}(13\sqrt{2} + 11)}{1085}. \quad (51)$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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