Research Article

Oscillation Properties for Systems of Higher-Order Partial Differential Equations with Distributed Deviating Arguments

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New sufficient conditions are obtained for oscillation for the solutions of systems of a class of higher-order quasilinear partial functional differential equations with distributed deviating arguments. The obtained results are illustrated by example.

1. Introduction and Preliminary

In this paper, we consider systems of higher-order quasilinear partial functional differential equations with distributed deviating arguments:

$$\begin{aligned} & \frac{\partial^n}{\partial t^n} \left[r(t) u_i(x, t) + \int_{\alpha}^{\beta} p(t, \tau) u_i(x, t - \tau) d\tau \right] \\ &= a_i(t) \Delta u_i(x, t) + \sum_{j=1}^m \sum_{k=1}^s b_{ijk}(t) \Delta u_j(x, \rho_k(t)) \\ & \quad - q_i(x, t) u_i(x, t) \\ & \quad - \sum_{j=1}^m \sum_{\gamma=1}^l \int_{c_{\gamma}}^{d_{\gamma}} f_{ij\gamma}(x, t, u_j(x, t - \sigma_{\gamma})) d\sigma_{\gamma}, \\ & (x, t) \in \Omega \times [0, \infty) \equiv G, \quad i = 1, 2, \dots, m, \end{aligned} \quad (1)$$

where n is even, Ω is a bounded domain in R^N with a piecewise smooth boundary $\partial\Omega$, and

$$\Delta u_i(x, t) = \sum_{r=1}^N \frac{\partial^2 u_i(x, t)}{\partial x_r^2}, \quad i = 1, 2, \dots, m. \quad (2)$$

Throughout this paper, we assume that the following conditions hold:

- (H1) $\beta > \alpha > 0$, $d_{\gamma} > c_{\gamma} > 0$, $r \in C^n([0, \infty), [0, \infty))$, $\lim_{t \rightarrow \infty} \int_0^t (1/r(s)) ds = \infty$; $p \in C^n([0, \infty) \times [\alpha, \beta], [0, \infty))$, $\gamma \in I_l = \{1, 2, \dots, l\}$;
- (H2) $q_i \in C(\bar{G}, [0, \infty))$, $q_i(t) = \min_{x \in \bar{G}} q_i(x, t)$, $q(t) = \min_{1 \leq i \leq m} q_i(t)$, $i \in I_m = \{1, 2, \dots, m\}$;
- (H3) $f \in C(\bar{G} \times R, R)$, $f_{ij\gamma}(t, x, u_j)/u_j \geq h_{ij\gamma}(x, t)$ if $u_j \neq 0$, where $h_{ij\gamma} \in C(\bar{G}, R)$, $h_{ii\gamma}(x, t) > 0$, $h_{ii\gamma}(t) = \min_{x \in \bar{G}} h_{ii\gamma}(x, t)$, $\bar{h}_{ij\gamma}(t) = \sup_{x \in \bar{G}} |h_{ij\gamma}(x, t)|$, and $Q_{\gamma}(t) = \min_{1 \leq i \leq m} \{h_{ii\gamma}(t) - \sum_{j=1, j \neq i}^m \bar{h}_{ij\gamma}(t)\} \geq 0$, $i, j \in I_m$, $\gamma \in I_l$;
- (H4) $a_i \in C([0, \infty), [0, \infty))$, $b_{ijk} \in C([0, \infty), (0, \infty))$, $B_k(t) = \min_{1 \leq i \leq m} \{b_{iik}(t) - \sum_{j=1, j \neq i}^m |b_{ijk}(t)|\} \geq 0$, $i, j \in I_m$, $k \in I_s = \{1, 2, \dots, s\}$;
- (H5) $\rho_k \in C([0, \infty), R)$, $\rho_k(t) \leq t$, ρ_k is nondecreasing functions and $\lim_{t \rightarrow \infty} \rho_k(t) = \infty$, $k \in I_s$.

We consider two kinds of boundary conditions:

$$\begin{aligned} & \frac{\partial u_i(x, t)}{\partial \nu} + g_i(x, t) u_i(x, t) = 0, \\ & (x, t) \in \partial\Omega \times [0, \infty), \quad i \in I_m, \end{aligned} \quad (3)$$

where ν is the unit exterior normal vector to $\partial\Omega$, $g_i(x, t)$ is a nonnegative continuous function on $\partial\Omega \times [0, \infty)$, and

$$u_i(x, t) = 0, \quad (x, t) \in \partial\Omega \times [0, \infty), \quad i \in I_m. \quad (4)$$

During the past three decades, the investigation of oscillatory theory for partial functional differential equations has attracted attention of numerous researchers due to their significance in theory and applications. In 2000, Li [1] has investigated systems of second-order partial differential equations:

$$\begin{aligned} & \frac{\partial^2}{\partial t^2} \left[p(t) u_i(x, t) + \sum_{r=1}^d \lambda_r(t) u_i(x, t - \tau_r) \right] \\ &= a_i(t) \Delta u_i(x, t) + \sum_{j=1}^m \sum_{k=1}^s a_{ijk}(t) \Delta u_j(x, \rho_k(t)) \\ & \quad - q_i(x, t) u_i(x, t) \\ & \quad - \sum_{j=1}^m \sum_{h=1}^l q_{ijh}(x, t) u_j(x, \sigma_h(t)), \\ & (x, t) \in \Omega \times [0, \infty) \equiv G, \quad i = 1, 2, \dots, m, \end{aligned} \quad (5)$$

and obtained sufficient conditions for the oscillation of solutions. In 2004, Lin [2] has studied higher-order nonlinear differential equation:

$$\begin{aligned} & \frac{\partial^n}{\partial t^n} \left[p(t) u_i(x, t) + \sum_{r=1}^d \lambda_r(t) u_i(x, t - \tau_r(t)) \right] \\ &= a_i(t) \Delta u_i(x, t) + \sum_{j=1}^m \sum_{k=1}^s a_{ijk}(t) \Delta u_j(x, \rho_k(t)) \\ & \quad - q_i(x, t) u_i(x, t) \\ & \quad - \sum_{j=1}^m \sum_{h=1}^l q_{ijh}(x, t) u_j(x, \sigma_h(t)), \\ & (x, t) \in \Omega \times [0, \infty) \equiv G, \quad i = 1, 2, \dots, m, \end{aligned} \quad (6)$$

and obtained sufficient conditions for the oscillation of solutions. In 2004, Li et al. [3] have investigated a class of vector hyperbolic differential equations with deviating arguments:

$$\begin{aligned} & \frac{\partial^2}{\partial t^2} U(x, t) = a(t) \Delta U(x, t) \\ & \quad + \sum_{k=1}^m b_k(t) \Delta U(x, \tau_k(t)) - p(x, t) U(x, t) \\ & \quad - \sum_{h=1}^l \int_a^b q_h(x, t, \zeta) U(x, g_h(t, \zeta)) d\sigma(\zeta) \\ & \quad + F(x, t), \quad (x, t) \in \Omega \times [0, \infty) \equiv G, \end{aligned} \quad (7)$$

and obtained sufficient conditions for H -oscillation of solutions. In 2009, Gui and Xu [4] have investigated even-order neutral type partial differential equations with distributed deviating arguments:

$$\begin{aligned} & \frac{\partial}{\partial t} \left(p(t) \frac{\partial^{(m-1)}}{\partial t^{(m-1)}} [u(x, t) + c(t) u_i(x, t - \tau)] \right) \\ &= a_0(t) \Delta u(x, t) + a_1(t) \Delta u(x, t - \nu) \\ & \quad - \int_a^b F(t, \xi, u(x, g(t, \xi))) d\mu(\xi), \\ & (x, t) \in \Omega \times [0, \infty) \equiv G. \end{aligned} \quad (8)$$

For related work, we refer the reader to [5–10]. However, to the best of our knowledge, few investigations have been reported on the oscillation theory of systems of partial functional differential equations with distributed deviating arguments. Thus, the research presents its significance.

Definition 1. The vector function $u(x, t) = \{u_1(x, t), u_2(x, t), \dots, u_m(x, t)\}^T$ is said to be a solution of problem (1), (3) (or (1), (4)) if it satisfies (1) in $G = \Omega \times [0, \infty)$ and boundary condition (3) (or (4)).

Definition 2. A nontrivial component $u_i(x, t)$ of the vector function $u(x, t) = \{u_1(x, t), u_2(x, t), \dots, u_m(x, t)\}^T$ is said to oscillate in $\Omega \times [\mu_0, \infty)$ if, for each $\mu > \mu_0$, there is a point $(x_0, t_0) \in \Omega \times [\mu, \infty)$ such that $u_i(x_0, t_0) = 0$.

Definition 3. The vector solution $u(x, t) = \{u_1(x, t), u_2(x, t), \dots, u_m(x, t)\}^T$ of problem (1), (3) (or (1), (4)) is said to be oscillatory in $G = \Omega \times [0, \infty)$ if at least one of its nontrivial components is oscillatory in G . Otherwise, the vector solution $u(x, t)$ is said to be nonoscillatory.

2. The Main Results

Theorem 4. Assume that $r(t) \geq 1$ is a monotone decreasing function, and

$$\int_{\alpha}^{\beta} p(t, \tau) d\tau < 1. \quad (9)$$

If there exist some $\gamma_0 \in I_l$ such that

$$\begin{aligned} & \int_{t_1}^{\infty} \left\{ Q_{\gamma_0}(t) \int_{\zeta_0}^{d_{\gamma_0}} \left[1 - \int_{\alpha}^{\beta} p(t - \sigma_{\gamma_0}, \tau) d\tau \right] d\sigma_{\gamma_0} \right\} dt = \infty, \\ & t_0 > 0, \end{aligned} \quad (10)$$

then every solution of boundary value system (1), (3) is oscillatory.

Proof. Suppose (1) and (3) have a nonoscillatory solution $\{u_1, u_2, \dots, u_m\}$. We assume that $|u_i(x, t)| > 0$ for $t \geq t_0 \geq 0$, $i \in I_m$. Let $\delta_i = \text{sgn } u_i(x, t)$, $Z_i(x, t) = \delta_i u_i(x, t)$; then $Z_i(x, t) > 0$, $(x, t) \in \Omega \times [t_0, \infty)$, $i \in I_m$. From (H1) and

(H5), there exists a number $t_1 \geq t_0$ such that $Z_i(x, t) > 0$, $Z_i(x, t - \tau) > 0$, $Z_i(x, \rho_k(t)) > 0$, $Z_i(x, t - \sigma_\gamma) > 0$, in $\Omega \times [t_0, \infty)$, $i \in I_m$, $k \in I_s$, $\gamma \in I_l$. Multiplying both sides of (1) by δ_i and integrating with respect to x over the domain Ω , we get

$$\begin{aligned} & \frac{d^n}{dt^n} \left[r(t) \int_{\Omega} \delta_i u_i(x, t) dx + \int_{\alpha}^{\beta} p(t, \tau) \int_{\Omega} \delta_i u_i(x, t - \tau) dx d\tau \right] \\ &= a_i(t) \int_{\Omega} \delta_i \Delta u_i(x, t) dx \\ &+ \sum_{j=1}^m \sum_{k=1}^s b_{ijk}(t) \int_{\Omega} \delta_i \Delta u_j(x, \rho_k(t)) dx \\ &- \int_{\Omega} q_i(x, t) \delta_i u_i(x, t) dx \\ &- \sum_{j=1}^m \sum_{\gamma=1}^l \int_{\Omega} \delta_i \int_{c_\gamma}^{d_\gamma} f_{ij\gamma}(x, t, u_j(x, t - \sigma_\gamma)) d\sigma_\gamma dx, \\ &t \geq t_1, \quad i \in I_m. \end{aligned} \quad (11)$$

Therefore,

$$\begin{aligned} & \frac{d^n}{dt^n} \left[r(t) \int_{\Omega} Z_i(x, t) dx + \int_{\alpha}^{\beta} p(t, \tau) \int_{\Omega} Z_i(x, t - \tau) dx d\tau \right] \\ &\leq a_i(t) \int_{\Omega} \Delta Z_i(x, t) dx \\ &+ \sum_{j=1}^m \sum_{k=1}^s b_{ijk}(t) \int_{\Omega} \frac{\delta_j}{\delta_i} \Delta Z_j(x, \rho_k(t)) dx \\ &- \int_{\Omega} q_i(x, t) Z_i(x, t) dx \\ &- \sum_{j=1}^m \sum_{\gamma=1}^l \frac{\delta_j}{\delta_i} \int_{c_\gamma}^{d_\gamma} \int_{\Omega} h_{ij\gamma}(x, t) Z_j(x, t - \sigma_\gamma) dx d\sigma_\gamma, \\ &t \geq t_1, \quad i \in I_m. \end{aligned} \quad (12)$$

From Green's formula and boundary condition (3), it follows that

$$\begin{aligned} \int_{\Omega} \Delta Z_i(x, t) dx &= \int_{\partial\Omega} \frac{\partial Z_i(x, t)}{\partial \nu} dS \\ &= - \int_{\partial\Omega} g_i(x, t) Z_i(x, t) dS \leq 0, \\ \int_{\Omega} \Delta Z_i(x, \rho_k(t)) dx &= \int_{\partial\Omega} \frac{\partial Z_i(x, \rho_k(t))}{\partial \nu} dS \\ &= - \int_{\partial\Omega} g_i(x, \rho_k(t)) Z_i(x, \rho_k(t)) dS \\ &\leq 0, \end{aligned} \quad (13)$$

where dS is the surface element on $\partial\Omega$. Combining (12)-(13), we find

$$\begin{aligned} & \frac{d^n}{dt^n} \left[r(t) \int_{\Omega} Z_i(x, t) dx + \int_{\alpha}^{\beta} p(t, \tau) \int_{\Omega} Z_i(x, t - \tau) dx d\tau \right] \\ &+ \sum_{j=1}^m \sum_{k=1}^s b_{ijk}(t) \frac{\delta_j}{\delta_i} \int_{\partial\Omega} g_i(x, \rho_k(t)) Z_i(x, \rho_k(t)) dS \\ &+ q_i(t) \int_{\Omega} Z_i(x, t) dx \\ &+ \sum_{\gamma=1}^l h_{ii\gamma}(t) \int_{c_\gamma}^{d_\gamma} \int_{\Omega} Z_i(x, t - \sigma_\gamma) dx d\sigma_\gamma \\ &- \sum_{j=1}^m \sum_{j=1, j \neq i}^m \bar{h}_{ij\gamma}(t) \int_{c_\gamma}^{d_\gamma} \int_{\Omega} Z_j(x, t - \sigma_\gamma) dx d\sigma_\gamma \leq 0. \end{aligned} \quad (14)$$

Therefore,

$$\begin{aligned} & \frac{d^n}{dt^n} \left[r(t) \int_{\Omega} Z_i(x, t) dx + \int_{\alpha}^{\beta} p(t, \tau) \int_{\Omega} Z_i(x, t - \tau) dx d\tau \right] \\ &+ \sum_{k=1}^s b_{iik}(t) \int_{\partial\Omega} g_i(x, \rho_k(t)) Z_i(x, \rho_k(t)) dS \\ &- \sum_{j=1, j \neq i}^m \sum_{k=1}^s |b_{iik}(t)| \int_{\partial\Omega} g_i(x, \rho_k(t)) Z_i(x, \rho_k(t)) dS \\ &+ q_i(t) \int_{\Omega} Z_i(x, t) dx \\ &+ \sum_{\gamma=1}^l h_{ii\gamma}(t) \int_{c_\gamma}^{d_\gamma} \int_{\Omega} Z_i(x, t - \sigma_\gamma) d\sigma_\gamma dx \\ &- \sum_{j=1}^m \sum_{j=1, j \neq i}^m \bar{h}_{ij\gamma}(t) \int_{c_\gamma}^{d_\gamma} \int_{\Omega} Z_j(x, t - \sigma_\gamma) d\sigma_\gamma dx \leq 0. \end{aligned} \quad (15)$$

Set

$$\begin{aligned} V_i &= \int_{\Omega} Z_i(x, t) dx, \\ W_i &= \int_{\partial\Omega} g_i(x, \rho_k(t)) Z_i(x, \rho_k(t)) dS, \\ &t \geq t_1, \quad i \in I_m. \end{aligned} \quad (16)$$

Then (15) yields

$$\begin{aligned} & \frac{d^n}{dt^n} \left[r(t) V_i(t) + \int_{\alpha}^{\beta} p(t, \tau) V_i(t - \tau) d\tau \right] \\ &+ \sum_{k=1}^s [b_{iik}(t) W_i(\rho_k(t)) \\ &- \sum_{j=1, j \neq i}^m |b_{ijk}(t)| W_j(\rho_k(t))] + q_i(t) V_i(t) \end{aligned}$$

$$\begin{aligned}
& + \sum_{\gamma=1}^l \left[h_{ii\gamma}(t) \int_{c_\gamma}^{d_\gamma} V_i(t - \sigma_\gamma) d\sigma_\gamma \right. \\
& \quad \left. - \sum_{j=1, j \neq i}^m \bar{h}_{ij\gamma}(t) \int_{c_\gamma}^{d_\gamma} V_j(t - \sigma_\gamma) d\sigma_\gamma \right] \leq 0.
\end{aligned} \tag{17}$$

Now, let $V(t) = \sum_{i=1}^m V_i(t)$, $W(t) = \sum_{i=1}^m W_i(t)$, $t \geq t_1$; from (17) we have

$$\begin{aligned}
& \frac{d^n}{dt^n} \left[r(t) V(t) + \int_\alpha^\beta p(t, \tau) V(t - \tau) d\tau \right] \\
& + \sum_{k=1}^s \left\{ \sum_{i=1}^m \left[b_{iik}(t) W_i(\rho_k(t)) - \sum_{j=1, j \neq i}^m |b_{ijk}(t)| W_j(\rho_k(t)) \right] \right\} \\
& + q(t) V(t) \\
& + \sum_{\gamma=1}^l \left\{ \sum_{i=1}^m \left[h_{ii\gamma}(t) \int_{c_\gamma}^{d_\gamma} V_i(t - \sigma_\gamma) d\sigma_\gamma \right. \right. \\
& \quad \left. \left. - \sum_{j=1, j \neq i}^m \bar{h}_{ij\gamma}(t) \int_{c_\gamma}^{d_\gamma} V_j(t - \sigma_\gamma) d\sigma_\gamma \right] \right\} \leq 0.
\end{aligned} \tag{18}$$

Note that

$$\begin{aligned}
& \sum_{i=1}^m \left\{ h_{ii\gamma}(t) \int_{c_\gamma}^{d_\gamma} V_i(t - \sigma_\gamma) d\sigma_\gamma \right. \\
& \quad \left. - \sum_{j=1, j \neq i}^m \bar{h}_{ij\gamma}(t) \int_{c_\gamma}^{d_\gamma} V_j(t - \sigma_\gamma) d\sigma_\gamma \right\} \\
& = \left[h_{11\gamma}(t) \int_{c_\gamma}^{d_\gamma} V_1(t - \sigma_\gamma) d\sigma_\gamma \right. \\
& \quad \left. - \sum_{j=1, j \neq 1}^m \bar{h}_{1j\gamma}(t) \int_{c_\gamma}^{d_\gamma} V_j(t - \sigma_\gamma) d\sigma_\gamma \right] \\
& + \left[h_{22\gamma}(t) \int_{c_\gamma}^{d_\gamma} V_2(t - \sigma_\gamma) d\sigma_\gamma \right. \\
& \quad \left. - \sum_{j=1, j \neq 2}^m \bar{h}_{2j\gamma}(t) \int_{c_\gamma}^{d_\gamma} V_j(t - \sigma_\gamma) d\sigma_\gamma \right] \\
& + \cdots + \left[h_{mm\gamma}(t) \int_{c_\gamma}^{d_\gamma} V_m(t - \sigma_\gamma) d\sigma_\gamma \right. \\
& \quad \left. - \sum_{j=1, j \neq m}^m \bar{h}_{mj\gamma}(t) \int_{c_\gamma}^{d_\gamma} V_j(t - \sigma_\gamma) d\sigma_\gamma \right]
\end{aligned}$$

$$\begin{aligned}
& = \left[h_{11\gamma}(t) - \sum_{j=1, j \neq 1}^m \bar{h}_{j1\gamma}(t) \right] \int_{c_\gamma}^{d_\gamma} V_1(t - \sigma_\gamma) d\sigma_\gamma \\
& + \left[h_{22\gamma}(t) - \sum_{j=1, j \neq 1}^m \bar{h}_{j2\gamma}(t) \right] \int_{c_\gamma}^{d_\gamma} V_2(t - \sigma_\gamma) d\sigma_\gamma \\
& + \cdots + \left[h_{mm\gamma}(t) - \sum_{j=1, j \neq 1}^m \bar{h}_{jm\gamma}(t) \right] \int_{c_\gamma}^{d_\gamma} V_m(t - \sigma_\gamma) d\sigma_\gamma \\
& \geq \min_{1 \leq i \leq m} \left\{ h_{ii\gamma}(t) - \sum_{j=1, j \neq i}^m \bar{h}_{ji\gamma}(t) \right\} \sum_{i=1}^m \int_{c_\gamma}^{d_\gamma} V_i(t - \sigma_\gamma) d\sigma_\gamma \\
& = Q_\gamma(t) \int_{c_\gamma}^{d_\gamma} V(t - \sigma_\gamma) d\sigma_\gamma.
\end{aligned} \tag{19}$$

Similarly, we have

$$\begin{aligned}
& \sum_{k=1}^s \left\{ \sum_{i=1}^m \left[b_{iik}(t) W_i(\rho_k(t)) - \sum_{j=1, j \neq i}^m |b_{ijk}(t)| W_j(\rho_k(t)) \right] \right\} \\
& \geq \min_{1 \leq i \leq m} \left\{ b_{iik}(t) - \sum_{j=1, j \neq i}^m |b_{jik}(t)| \right\} \sum_{i=1}^m W_i(\rho_k(t)) \\
& = B_k(t) W(\rho_k(t)).
\end{aligned} \tag{20}$$

Thus, from (18) for $t \geq t_1$, we find

$$\begin{aligned}
& \left[r(t) V(t) + \int_\alpha^\beta p(t, \tau) V(t - \tau) d\tau \right]^{(n)} \\
& + \sum_{k=1}^s B_k(t) W(\rho_k(t)) + q(t) V(t) \\
& + \sum_{\gamma=1}^l Q_\gamma(t) \int_{c_\gamma}^{d_\gamma} V(t - \sigma_\gamma) d\sigma_\gamma \leq 0, \quad t \geq t_1.
\end{aligned} \tag{21}$$

It is clear that

$$W(\rho_k(t)) = \sum_{i=1}^m W_i(\rho_k(t)) \geq 0, \quad t \geq t_1, \quad k \in I_s. \tag{22}$$

Therefore, it follows that

$$\begin{aligned}
& \left[r(t) V(t) + \int_\alpha^\beta p(t, \tau) V(t - \tau) d\tau \right]^{(n)} \\
& + q(t) V(t) + \sum_{\gamma=1}^l Q_\gamma(t) \int_{c_\gamma}^{d_\gamma} V(t - \sigma_\gamma) d\sigma_\gamma \leq 0, \\
& \quad t \geq t_1.
\end{aligned} \tag{23}$$

Let

$$Y(t) = r(t)V(t) + \int_{\alpha}^{\beta} p(t, \tau)V(t - \tau) d\tau. \quad (24)$$

We have $Y(t) > 0$, $Y^{(n)}(t) < 0$, for $t \geq t_1$. Hence, $Y^{(n-1)}(t)$ is a monotone decreasing function in the interval $[t_1, \infty)$. We can claim that $Y^{(n-1)}(t) > 0$, for $t \geq t_1$. In fact, if $Y^{(n-1)}(t) < 0$, for $t \geq t_1$, then there exists a $T > t_1$, such that $r(T)Y^{(n-1)}(T) < 0$. This implies that

$$Y^{(k)}(t) < 0, \quad k = 0, 1, 2, \dots, n-2, \quad (25)$$

which contradicts the fact that $Y(t) > 0$. Now, we claim that $Y'(t) \geq 0$, $t \geq t_1$. Otherwise $Y(t) < 0$. But from the well-known lemma of Kiguradze [5] and the fact that n is even, the inequalities imply that higher derivatives of that function are also negative. This contradicts the fact that $Y^{(n-1)}(T) \geq 0$. Thus we have proved that $Y(t)$ is the increasing function of t .

From (23), we find that there exists a $\gamma_0 \in I_l$ such that

$$Y^{(n)}(t) + Q_{\gamma_0}(t) \int_{c_{\gamma_0}}^{d_{\gamma_0}} V(t - \sigma_{\gamma_0}) d\sigma_{\gamma_0} \leq 0, \quad t \geq t_1. \quad (26)$$

Thus we obtain

$$\begin{aligned} & Y^{(n)}(t) + Q_{\gamma_0}(t) \\ & \times \int_{c_{\gamma_0}}^{d_{\gamma_0}} \left(\left[Y(t - \sigma_{\gamma_0}) \right. \right. \\ & \quad \left. \left. - \int_{\alpha}^{\beta} p(t - \sigma_{\gamma_0}, \tau) V(t - \sigma_{\gamma_0} - \tau) d\tau \right] \right. \\ & \quad \left. \times \left(r(t - \sigma_{\gamma_0}) \right)^{-1} \right) d\sigma_{\gamma_0} \\ & \leq 0, \quad t \geq t_1. \end{aligned} \quad (27)$$

Since $Y(t) \geq r(t)V(t) \geq V(t)$, $Y'(t) \geq 0$, from (27), we have

$$\begin{aligned} & Y^{(n)}(t) + Q_{\gamma_0}(t) \\ & \times \int_{c_{\gamma_0}}^{d_{\gamma_0}} \left[1 - \int_{\alpha}^{\beta} p(t - \sigma_{\gamma_0}, \tau) d\tau \right] \frac{Y(t - \sigma_{\gamma_0})}{r(t - \sigma_{\gamma_0})} d\sigma_{\gamma_0} \leq 0, \\ & t \geq t_1. \end{aligned} \quad (28)$$

Integrating inequality (28) from t_1 to t ,

$$\begin{aligned} & Y^{(n-1)}(t) - Y^{(n-1)}(t_1) \\ & + \int_{t_1}^t \left\{ Q_{\gamma_0}(s) \int_{c_{\gamma_0}}^{d_{\gamma_0}} \left[1 - \int_{\alpha}^{\beta} p(s - \sigma_{\gamma_0}, \tau) d\tau \right] \right. \\ & \quad \left. \times \frac{Y(t - \sigma_{\gamma_0})}{r(t - \sigma_{\gamma_0})} d\sigma_{\gamma_0} \right\} ds \leq 0, \quad t \geq t_1. \end{aligned} \quad (29)$$

Furthermore, we have

$$\begin{aligned} & \int_{t_1}^t \left\{ Q_{\gamma_0}(s) \int_{c_{\gamma_0}}^{d_{\gamma_0}} \left[1 - \int_{\alpha}^{\beta} p(s - \sigma_{\gamma_0}, \tau) d\tau \right] d\sigma_{\gamma_0} \right\} ds \\ & \leq \frac{r(t_1 - d_{\gamma_0})}{Y(t_1 - d_{\gamma_0})} \left(-Y^{(n-1)}(t) + Y^{(n-1)}(t_1) \right), \\ & \leq \frac{r(t_1 - d_{\gamma_0}) Y^{(n-1)}(t_1)}{Y(t_1 - d_{\gamma_0})}, \quad t \geq t_1, \end{aligned} \quad (30)$$

which contradicts condition (10). This proof is complete. $\square$

Theorem 5. Assume that $r(t) \geq 1$ is a monotone decreasing function and (9) holds. And

$$\int_{t_0}^{\infty} \left\{ q(s) \left[1 - \int_{\alpha}^{\beta} p(s, \tau) d\tau \right] \right\} ds = \infty, \quad t_0 > 0. \quad (31)$$

Then every solution of boundary value system (1), (3) is oscillatory.

Proof. As in the proof of Theorem 4, we obtain (23). Therefore,

$$Y^{(n)}(t) + q(t)V(t) \leq 0, \quad t \geq t_0. \quad (32)$$

The remaining part of the proof is similar to that of Theorem 4; therefore it is omitted. The proof is complete. $\square$

Corollary 6. If inequality (23) has no eventually positive solution, then every solution $u(x, t)$ of system (1), (3) is oscillatory in G .

Theorem 7. Suppose that there exists a positive constant M such that $0 < M \leq r(t) < 1$, and

$$\int_{\alpha}^{\beta} p(t, \tau) d\tau < M. \quad (33)$$

If there exist some $\gamma_0 \in I_l$ such that

$$\begin{aligned} & \int_{t_1}^{\infty} \left\{ Q_{\gamma_0}(t) \int_{c_{\gamma_0}}^{d_{\gamma_0}} \left[1 - M^{-1} \int_{\alpha}^{\beta} p(t - \sigma_{\gamma_0}, \tau) d\tau \right] d\sigma_{\gamma_0} \right\} dt \\ & = \infty, \quad t_0 > 0, \end{aligned} \quad (34)$$

then every solution of boundary value system (1), (3) is oscillatory.

Proof. As in the proof of Theorem 4, we obtain (27). Using the fact that $r(t) < 1$, from (27), we have

$$\begin{aligned} & Y^{(n)}(t) + Q_{\gamma_0}(t) \\ & \times \int_{c_{\gamma_0}}^{d_{\gamma_0}} \left[Y(t - \sigma_{\gamma_0}) \right. \\ & \quad \left. - \int_{\alpha}^{\beta} p(t - \sigma_{\gamma_0}, \tau) V(t - \sigma_{\gamma_0} - \tau) d\tau \right] d\sigma_{\gamma_0} \\ & \leq 0, \quad t \geq t_1. \end{aligned} \quad (35)$$

Since $Y(t) \geq r(t)V(t) \geq MV(t)$, $Y'(t) \geq 0$, from (35) we have

$$\begin{aligned} & Y^{(n)}(t) + Q_{\gamma_0}(t) \\ & \times \int_{c_{\gamma_0}}^{d_{\gamma_0}} \left[1 - M^{-1} \int_{\alpha}^{\beta} p(t - \sigma_{\gamma_0}, \tau) d\tau \right] Y(t - \sigma_{\gamma_0}) d\sigma_{\gamma_0} \\ & \leq 0, \quad t \geq t_1. \end{aligned} \quad (36)$$

Integrating inequality (36) from t_1 to t ,

$$\begin{aligned} & Y^{(n-1)}(t) - Y^{(n-1)}(t_1) \\ & + \int_{t_1}^t \left\{ Q_{\gamma_0}(s) \int_{c_{\gamma_0}}^{d_{\gamma_0}} \left[1 - M^{-1} \int_{\alpha}^{\beta} p(s - \sigma_{\gamma_0}, \tau) d\tau \right] \right. \\ & \quad \left. \times Y(t - \sigma_{\gamma_0}) d\sigma_{\gamma_0} \right\} ds \leq 0, \\ & \quad t \geq t_1. \end{aligned} \quad (37)$$

Furthermore, we have

$$\begin{aligned} & \int_{t_1}^t \left\{ Q_{\gamma_0}(s) \int_{c_{\gamma_0}}^{d_{\gamma_0}} \left[1 - M^{-1} \int_{\alpha}^{\beta} p(s - \sigma_{\gamma_0}, \tau) d\tau \right] d\sigma_{\gamma_0} \right\} ds \\ & \leq \frac{(-Y^{(n-1)}(t) + Y^{(n-1)}(t_1))}{Y(t_1 - d_{\gamma_0})}, \\ & \leq \frac{Y^{(n-1)}(t_1)}{Y(t_1 - d_{\gamma_0})}, \quad t \geq t_1, \end{aligned} \quad (38)$$

which contradicts condition (10). This proof is complete. $\square$

Theorem 8. Suppose that there exists a positive constant M such that $0 < M \leq r(t) < 1$, and (33) holds. If

$$\int_{t_0}^{\infty} \left\{ q(s) \left[1 - M^{-1} \int_{\alpha}^{\beta} p(s, \tau) d\tau \right] \right\} ds = \infty, \quad t_0 > 0, \quad (39)$$

then every solution of boundary value system (1), (3) is oscillatory.

Proof. As in the proof of Theorem 5, we obtain (32). The remaining part of the proof is similar to that of Theorem 7 and we omit it.

Consider the smallest eigenvalue λ_0 of the Dirichlet problem:

$$\begin{aligned} \Delta \omega(x) + \lambda \omega(x) &= 0, \quad x \in \Omega, \\ \omega(x) &= 0, \quad x \in \partial \Omega, \end{aligned} \quad (40)$$

where λ is a positive constant and the corresponding eigenfunction $\phi(x)$ is positive in Ω . $\square$

Theorem 9. Assume all conditions of Theorem 4 hold. Then every solution $u(x, t)$ of system (1), (4) is oscillatory in G .

Proof. Suppose (1) and (4) have a nonoscillatory solution $\{u_1, u_2, \dots, u_m\}^T$. We assume that $|u_i(x, t)| > 0$ for $t \geq t_0 \geq 0$, $i \in I_m$. Let $\delta_i = \text{sgn } u_i(x, t)$, $Z_i(x, t) = \delta_i u_i(x, t)$; then $Z_i(x, t) > 0$, $(x, t) \in \Omega \times [t_0, \infty)$, $i \in I_m$. From (H1) and (H5), there exists a number $t_1 \geq t_0$ such that $Z_i(x, t) > 0$, $Z_i(x, t - \tau) > 0$, $Z_i(x, \rho_k(t)) > 0$, $Z_i(x, t - \sigma_{\gamma}) > 0$, in $\Omega \times [t_0, \infty)$, $i \in I_m$, $k \in I_s$, $\gamma \in I_l$. Multiplying both sides of (1) by $\delta_i \phi(x)$, and integrating with respect to x over the domain Ω , we have

$$\begin{aligned} & \frac{d^n}{dt^n} \left[r(t) \int_{\Omega} \delta_i u_i(x, t) \phi(x) dx \right. \\ & \quad \left. + \int_{\alpha}^{\beta} p(t, \tau) \int_{\Omega} \delta_i u_i(x, t - \tau) \phi(x) dx d\tau \right] \\ & = a_i(t) \int_{\Omega} \delta_i \phi(x) \Delta u_i(x, t) dx \\ & \quad + \sum_{j=1}^m \sum_{k=1}^s b_{ijk}(t) \int_{\Omega} \delta_i \phi(x) \Delta u_j(x, \rho_k(t)) dx \\ & \quad - \int_{\Omega} q_i(x, t) \delta_i u_i(x, t) \phi(x) dx \\ & \quad - \sum_{\gamma=1}^l \int_{\Omega} \delta_i \phi(x) \int_{c_{\gamma}}^{d_{\gamma}} f_{ij\gamma}(x, t, u_j(x, t - \sigma_{\gamma})) d\sigma_{\gamma} dx, \\ & \quad t \geq t_1, \quad i \in I_m. \end{aligned} \quad (41)$$

Therefore,

$$\begin{aligned} & \frac{d^n}{dt^n} \left[r(t) \int_{\Omega} Z_i(x, t) \phi(x) dx \right. \\ & \quad \left. + \int_{\alpha}^{\beta} p(t, \tau) \int_{\Omega} Z_i(x, t - \tau) \phi(x) dx d\tau \right] \\ & \leq a_i(t) \int_{\Omega} \Delta Z_i(x, t) \phi(x) dx \\ & \quad + \sum_{j=1}^m \sum_{k=1}^s b_{ijk}(t) \int_{\Omega} \frac{\delta_j}{\delta_i} \Delta Z_j(x, \rho_k(t)) \phi(x) dx \end{aligned}$$

$$\begin{aligned}
& - \int_{\Omega} q_i(x, t) Z_i(x, t) \phi(x) dx \\
& - \sum_{j=1}^m \sum_{\gamma=1}^l \frac{\delta_j}{\delta_i} \int_{\Omega} \int_{c_{\gamma}}^{d_{\gamma}} h_{ij\gamma}(x, t) Z_j(x, t - \sigma_{\gamma}) d\sigma_{\gamma} \phi(x) dx, \\
& t \geq t_1, \quad i \in I_m.
\end{aligned} \tag{42}$$

Green's formula and boundary condition (4) yield

$$\begin{aligned}
\int_{\Omega} \Delta Z_i(x, t) \phi(x) dx &= \int_{\Omega} Z_i(x, t) \Delta \phi(x) dx \\
&= -\lambda_0 \int_{\Omega} Z_i(x, t) \phi(x) dx \leq 0, \\
\int_{\Omega} \Delta Z_i(x, \rho_k(t)) \phi(x) dx & \\
&= \int_{\Omega} Z_i(x, \rho_k(t)) \Delta \phi(x) dx \\
&= -\lambda_0 \int_{\Omega} Z_i(x, \rho_k(t)) \Delta \phi(x) dx \leq 0.
\end{aligned} \tag{43}$$

Combining (42)-(43), we get

$$\begin{aligned}
& \frac{d^n}{dt^n} \left[r(t) \int_{\Omega} Z_i(x, t) \phi(x) dx \right. \\
& \quad \left. + \int_{\alpha}^{\beta} p(t, \tau) \int_{\Omega} Z_i(x, t - \tau) \phi(x) dx d\tau \right] \\
& \leq -\lambda_0 \sum_{k=1}^s b_{iik}(t) \int_{\Omega} \Delta Z_i(x, \rho_k(t)) \phi(x) dx \\
& \quad + \lambda_0 \sum_{j=1, j \neq i}^m \sum_{k=1}^s |b_{ijk}(t)| \int_{\Omega} \Delta Z_j(x, \rho_k(t)) \phi(x) dx \\
& \quad - q_i(t) \int_{\Omega} Z_i(x, t) \phi(x) dx \\
& \quad - \sum_{\gamma=1}^l h_{ii\gamma}(t) \int_{c_{\gamma}}^{d_{\gamma}} \int_{\Omega} Z_i(x, t - \sigma_{\gamma}) d\sigma_{\gamma} \phi(x) dx \\
& \quad + \sum_{j=1, j \neq i}^m \sum_{\gamma=1}^l \bar{h}_{ij\gamma}(t) \int_{c_{\gamma}}^{d_{\gamma}} \int_{\Omega} Z_j(x, t - \sigma_{\gamma}) \phi(x) dx d\sigma_{\gamma} \leq 0.
\end{aligned} \tag{44}$$

Set

$$V_i = \int_{\Omega} Z_i(x, t) \phi(x) dx, \quad t \geq t_1, \quad i \in I_m. \tag{45}$$

Then (44) yields

$$\begin{aligned}
& \left[r(t) V_i(t) + \int_{\alpha}^{\beta} p(t, \tau) V_i(t - \tau) d\tau \right]^{(n)} \\
& + \lambda_0 \sum_{k=1}^s \left[b_{iik}(t) \Delta V_i(x, \rho_k(t)) \right. \\
& \quad \left. - \sum_{j=1, j \neq i}^m |b_{ijk}(t)| \Delta V_j(x, \rho_k(t)) \right] + q_i(t) V_i(x, t) \\
& + \sum_{\gamma=1}^l \left[h_{ii\gamma}(t) \int_{c_{\gamma}}^{d_{\gamma}} V_i(x, t - \sigma_{\gamma}) d\sigma_{\gamma} \right. \\
& \quad \left. - \sum_{j=1, j \neq i}^m \bar{h}_{ij\gamma}(t) \int_{c_{\gamma}}^{d_{\gamma}} V_j(x, t - \sigma_{\gamma}) d\sigma_{\gamma} \right] \leq 0.
\end{aligned} \tag{46}$$

Let $V(t) = \sum_{i=1}^m V_i(t)$, $t \geq t_1$; from (46) we have

$$\begin{aligned}
& \left[r(t) V(t) + \int_{\alpha}^{\beta} p(t, \tau) V(t - \tau) d\tau \right]^{(n)} \\
& + \lambda_0 \sum_{k=1}^s \left\{ \sum_{i=1}^m \left[b_{iik}(t) \Delta V_i(x, \rho_k(t)) \right. \right. \\
& \quad \left. \left. - \sum_{j=1, j \neq i}^m |b_{ijk}(t)| \Delta V_j(x, \rho_k(t)) \right] \right\} \\
& + q_i(t) V(x, t) \\
& + \sum_{\gamma=1}^l \left\{ \sum_{i=1}^m \left[h_{ii\gamma}(t) \int_{c_{\gamma}}^{d_{\gamma}} V_i(x, t - \sigma_{\gamma}) d\sigma_{\gamma} \right. \right. \\
& \quad \left. \left. - \sum_{j=1, j \neq i}^m \bar{h}_{ij\gamma}(t) \int_{c_{\gamma}}^{d_{\gamma}} V_j(x, t - \sigma_{\gamma}) d\sigma_{\gamma} \right] \right\} \leq 0.
\end{aligned} \tag{47}$$

Now, as in the proof of Theorem 4, from (47) we have

$$\begin{aligned}
& \left[r(t) V(t) + \int_{\alpha}^{\beta} p(t, \tau) V(t - \tau) d\tau \right]^{(n)} \\
& + \lambda_0 \sum_{k=1}^s B_k(t) V(\rho_k(t)) + q(t) V(t) \\
& + \sum_{\gamma=1}^l Q_{\gamma}(t) \int_{c_{\gamma}}^{d_{\gamma}} V(t - \sigma_{\gamma}) d\sigma_{\gamma} \leq 0, \quad t \geq t_1.
\end{aligned} \tag{48}$$

The remaining part of the proof is similar to that of Theorem 4; therefore it is omitted. The proof is complete. $\square$

Corollary 10. *If inequality (48) has no eventually positive solution, then every solution $u(x, t)$ of system (1), (4) is oscillatory in G .*

It is not difficult to see that the following theorems are true.

Theorem 11. *Assume that $r(t) \geq 1$ is a monotone decreasing function, and (9) holds. If there exists some $k_0 \in I_s$ such that*

$$\int_{t_0}^{\infty} \left\{ \lambda_0 B_{k_0}(t) \left[1 - \int_{\alpha}^{\beta} p(\rho_0(t), \tau) d\tau \right] \right\} dt = \infty, \quad t_0 > 0, \quad (49)$$

then every solution of boundary value system (1), (4) is oscillatory in G .

Theorem 12. *Suppose that there exists a positive constant M such that $0 < M \leq r(t) < 1$, and (33) holds. If there exists some $k_0 \in I_s$ such that*

$$\int_{t_0}^{\infty} \left\{ \lambda_0 B_{k_0}(t) \left[1 - M^{-1} \int_{\alpha}^{\beta} p(\rho_0(t), \tau) d\tau \right] \right\} dt = \infty, \quad t_0 > 0, \quad (50)$$

then every solution of boundary value system (1), (4) is oscillatory in G .

Theorem 13. *Assume all conditions of Theorem 5 hold. Then every solution $u(x, t)$ of boundary value system (1), (4) is oscillatory in G .*

Theorem 14. *Assume all conditions of Theorem 7 hold. Then every solution $u(x, t)$ of boundary value system (1), (4) is oscillatory in G .*

Theorem 15. *Assume all conditions of Theorem 8 hold. Then every solution $u(x, t)$ of boundary value system (1), (4) is oscillatory in G .*

3. Example

Example 1. Consider the systems of second-order partial differential equations

$$\begin{aligned} & \frac{\partial^2}{\partial t^2} \left((1 + e^{-t}) u_1(x, t) + \frac{1}{2} \int_{\pi/2}^{\pi} u_1(x, t - \tau) d\tau \right) \\ &= \Delta u_1(x, t) + \left(\frac{3}{2} + 3e^{-t} \right) \Delta u_1 \left(x, t - \frac{3\pi}{2} \right) - \frac{3}{2} u_1(x, t) \\ & \quad - \frac{e^{-t} + 4}{2} \int_{\pi/2}^{\pi} u_1(x, t - \sigma_1) d\sigma_1 \\ & \quad - \frac{e^{-t}}{2} \int_{\pi/2}^{\pi} u_2(x, t - \sigma_1) d\sigma_1 \end{aligned} \quad (51)$$

$$\begin{aligned} & \frac{\partial^2}{\partial t^2} \left((1 + e^{-t}) u_2(x, t) + \frac{1}{2} \int_{\pi/2}^{\pi} u_2(x, t - \tau) d\tau \right) \\ &= e^{-t} \Delta u_2(x, t) + \left(\frac{3}{2} + 2e^{-t} \right) \Delta u_2 \left(x, t - \frac{3\pi}{2} \right) - \frac{5}{2} u_2(x, t) \\ & \quad - \frac{e^{-t}}{2} \int_{\pi/2}^{\pi} u_1(x, t - \sigma_1) d\sigma_1 \\ & \quad - \frac{e^{-t} + 4}{2} \int_{\pi/2}^{\pi} u_2(x, t - \sigma_1) d\sigma_1 \\ & \quad (x, t) \in (0, \pi) \times [0, \infty) \end{aligned} \quad (52)$$

with boundary condition

$$\frac{\partial}{\partial x} u_i(0, t) = \frac{\partial}{\partial x} u_i(\pi, t) = 0, \quad t \geq 0, \quad i = 1, 2. \quad (53)$$

Here, $N = 1$, $n = 2$, $m = 2$, $\alpha = \pi/2$, $\beta = \pi$, $k = 1$, $c_1 = \pi/2$, $d_1 = \pi$, $i = 2$, $a_1(t) = 1$, $a_2(t) = e^{-t}$, $q_1(x, t) = 3/2$, $q_2(x, t) = 5/2$, $q_{111}(t) = (e^{-t} + 4)/2$, $q_{121}(t) = e^{-t}/2$, $q_{212}(t) = e^{-t}/2$, $q_{221}(t) = (e^{-t} + 4)/2$, $f_{ij\gamma}(x, t, u_j) = u_j$. It is easy to see that all conditions of Theorem 4 are fulfilled. Then every solution of system (52), (53) oscillates in $(0, \pi) \times [0, \infty)$. In fact, $u_1(x, t) = \cos x \sin t$, $u_2(x, t) = \cos x \cos t$ is such a solution.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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