Research Article

Empirical Bayes Inference for the Parameter of Power Distribution Based on Ranked Set Sampling

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This research is based on ranked set sampling. Through the analysis and proof, the empirical Bayes test rule and asymptotical property for the parameter of power distribution are obtained.

1. Introduction

Ranked set sampling (RSS) is now regarded as an effective tool in statistical inference and important alternative to simple random sampling. RSS was first applied in agriculture [1]. In recent years, it has been applied more and more in areas such as environment and ecology. It is also a potential for the method to be successfully applied in industrial statistics and sociology, for which readers can refer to the monograph by Chen et al. [2].

Empirical Bayes (EB) approach was originally proposed by Robbins [3, 4], soon after it had been studied in the literature [5–9]. Up to now, EB methods are commonly based on simple random sampling (SRS). As a statistical procedure based on the RSS would perform better than its counterpart based on simple random sampling (SRS), a natural idea is to develop EB methods based on RSS. We will construct empirical Bayes test rule for the parameter of power distribution based on RSS.

Let X have a conditional density function for given θ :

$$f(x|\theta) = (\theta x)^{-1} \exp\{\theta^{-1} \ln x\}, \quad (1)$$

where θ is an unknown parameter, $\Omega = \{x | x > 1\}$ is sample space, and $\Theta = \{\theta | \theta > 0\}$ is parameter space.

In the paper, the main motivation of studying the density of (1) is as follows: (a) it is widely used in the fields of reliability

and economy and so on; (b) it is a common distribution in reliability distribution; (c) on the basis of studying it, we intend to further consider empirical Bayes inference for some other distributions based on ranked set sampling. In the paper, we study the following test problem:

$$H_0 : \theta \leq \theta_0 \iff H_1 : \theta > \theta_0, \quad (2)$$

where θ_0 is a given positive constant.

To construct test function, we take loss function:

$$L_0(\theta, d_0) = \begin{cases} 0, & \theta \leq \theta_0 \\ a \left[1 - \left(\frac{\theta_0}{\theta} \right) \right], & \theta > \theta_0; \end{cases} \quad (3)$$

$$L_1(\theta, d_1) = \begin{cases} a \left[\left(\frac{\theta_0}{\theta} \right) - 1 \right], & \theta \leq \theta_0 \\ 0, & \theta > \theta_0, \end{cases}$$

where $a > 0$, $d = \{d_0, d_1\}$ is action space, respectively, and d_0 and d_1 imply acceptance and rejection of H_0 .

Suppose that the prior distribution $G(\theta)$ of parameter θ is unknown, we can get randomized decision function:

$$\delta(x) = P(\text{accept } H_0 | X = x). \quad (4)$$

Then, the risk function of $\delta(x)$ is given by

$$\begin{aligned} R(\delta(x), G(\theta)) &= \int_{\Theta} \int_{\Omega} [L_0(\theta, d_0) f(x|\theta) \delta(x) \\ &\quad + L_1(\theta, d_1) f(x|\theta) (1 - \delta(x))] dx dG(\theta) \\ &= a \int_{\Omega} \beta(x) \delta(x) dx + C_G, \end{aligned} \quad (5)$$

where

$$\begin{aligned} C_G &= \int_{\Theta} L_1(\theta, d_1) dG(\theta), \\ \beta(x) &= \int_{\Theta} \left[1 - \left(\frac{\theta_0}{\theta} \right) \right] f(x|\theta) dG(\theta). \end{aligned} \quad (6)$$

The marginal density function of X is shown by

$$\begin{aligned} f_G(x) &= \int_{\Theta} f(x|\theta) dG(\theta) \\ &= \int_{\Theta} (\theta x)^{-1} \exp\{\theta^{-1} \ln x\} dG(\theta). \end{aligned} \quad (7)$$

Applying (6), we get

$$\begin{aligned} \beta(x) &= \int_{\Theta} \left(1 - \frac{\theta_0}{\theta} \right) f(x|\theta) dG(\theta) \\ &= \int_{\Theta} \left(1 - \theta_0 + \theta_0 - \frac{\theta_0}{\theta} \right) f(x|\theta) dG(\theta) \\ &= \int_{\Theta} (1 - \theta_0) f(x|\theta) dG(\theta) \\ &\quad + \int_{\Theta} \left(\theta_0 - \frac{\theta_0}{\theta} \right) f(x|\theta) dG(\theta) \\ &= (1 - \theta_0) f_G(x) - \theta_0 x \\ &\quad \cdot \int_{\Theta} \left(-1 + \frac{1}{\theta} \right) \frac{1}{x} f(x|\theta) dG(\theta) \\ &= (1 - \theta_0) f_G(x) - \theta_0 x \\ &\quad \cdot \int_{\Theta} \left(-\frac{1}{x} \right) f(x|\theta) dG(\theta) - \theta_0 x \\ &\quad \cdot \int_{\Theta} \frac{1}{\theta x} f(x|\theta) dG(\theta). \end{aligned} \quad (8)$$

Since $f_G(x) = \int_{\Theta} (\theta x)^{-1} \exp\{\theta^{-1} \ln x\} dG(\theta)$, we get

$$\begin{aligned} f_G^{(1)}(x) &= \int_{\Theta} \left(-\frac{1}{x} \right) f(x|\theta) dG(\theta) \\ &\quad + \frac{1}{\theta x} \int_{\Theta} f(x|\theta) dG(\theta). \end{aligned} \quad (9)$$

Thus, we have

$$\begin{aligned} \beta(x) &= (1 - \theta_0) f_G(x) - \theta_0 x \cdot f_G^{(1)}(x) \\ &= u(x) f_G(x) + v(x) f_G^{(1)}(x), \end{aligned} \quad (10)$$

where $f_G^{(1)}(x)$ is derivative of $f_G(x)$, $u(x) = 1 - \theta_0$, and $v(x) = -\theta_0 x$.

By (5), Bayes test function is obtained as follows:

$$\delta_G(x) = \begin{cases} 1, & \text{if } \beta(x) \leq 0, \\ 0, & \text{if } \beta(x) > 0. \end{cases} \quad (11)$$

Hence, we can get minimum Bayes risk:

$$\begin{aligned} R(G) &= \inf_{\delta} R(\delta, G) = R(\delta_G, G) \\ &= a \int_{\Omega} \beta(x) \delta_G(x) dx + C_G. \end{aligned} \quad (12)$$

If the prior distribution of $G(\theta)$ is known and $\delta(x) = \delta_G(x)$, $R(G)$ is achieved. While $G(\theta)$ is unknown, we cannot make use of $\delta_G(x)$ and need to introduce EB method.

2. Construction of EB Test Based on Ranked Set Sampling

A balanced RSS procedure can be described by Chen et al. [2]. Under the following conditions, we need to construct EB test function. Let $X_{(1)1}, X_{(1)2}, \dots, X_{(1)m}, X_{(2)1}, X_{(2)2}, \dots, X_{(2)m}, \dots, X_{(k)1}, X_{(k)2}, \dots, X_{(k)m}$ be a balanced ranked set sample from population which has the common marginal density function $f_G(x)$. We assume perfect ranking. Denote by $X_{(1)1}, X_{(1)2}, \dots, X_{(1)m}, X_{(2)1}, X_{(2)2}, \dots, X_{(2)m}, \dots, X_{(k)1}, X_{(k)2}, \dots, X_{(k)m}$ the historical samples, and X is present sample. Assume $f_G(x) \in C_{s,\alpha}$, $x \in R^1$, where $C_{s,\alpha} = \{g(x) \mid g(x) \text{ is a probability density function and has continuous } s \text{th order derivative } g^{(s)}(x) \text{ with } |g^{(s)}(x)| \leq \alpha, s \geq 3, \alpha > 0\}$. First construct estimator of $\beta(x)$.

Let $K_r(x)$ be a Borel measurable bounded function vanishing off $(0, 1)$ such that

$$\begin{aligned} (C1): \quad &\frac{1}{t!} \int_0^1 y^t K_r(y) dy \\ &= \begin{cases} (-1)^t, & \text{when } t = r, \\ 0, & \text{when } t \neq r, t = 0, 1, 2, \dots, s-1. \end{cases} \end{aligned} \quad (13)$$

Kernel estimator of $f_G(x)$ is defined by

$$f_n^{(r)}(x) = \frac{1}{m k h_n^{(1+r)}} \sum_{i=1}^k \sum_{j=1}^m K_r \left(\frac{x - X_{(i)j}}{h_n} \right), \quad (14)$$

where h_n is a positive and smoothing bandwidth and $\lim_{n \rightarrow \infty} h_n = 0$.

Denote by $f_n^{(0)}(x) = f_n(x)$, $f_n^{(r)}(x)$ the r th order derivative of $f_n(x)$, for $r = 0, 1$.

The estimator of $\beta(x)$ is obtained by

$$\beta_n(x) = u(x) f_n(x) + v(x) f_n^{(1)}(x). \quad (15)$$

Hence, EB test function is defined by

$$\delta_n(x) = \begin{cases} 1, & \beta_n(x) \leq 0, \\ 0, & \beta_n(x) > 0. \end{cases} \quad (16)$$

Let E stand for mathematical expectation with respect to the joint distribution of $X_{(1)1}, X_{(1)2}, \dots, X_{(1)m}, X_{(2)1}, X_{(2)2}, \dots, X_{(2)m}, \dots, X_{(k)1}, X_{(k)2}, \dots, X_{(k)m}$. Hence, we get the over-all Bayes risk of $\delta_n(x)$:

$$R(\delta_n(x), G) = a \int_{\Omega} \beta(x) E[\delta_n(x)] dx + C_G. \quad (17)$$

If $\lim_{n \rightarrow \infty} R(\delta_n, G) = R(\delta_G, G)$, $\{\delta_n(x)\}$ is called asymptotical optimality of EB test function. If $R(\delta_n, G) - R(\delta_G, G) = O(n^{-q})$, where $q > 0$, $O(n^{-q})$ is asymptotically optimal convergence rate of EB test function $\{\delta_n(x)\}$. Before proving the theorems, we need the following lemmas.

Let c, c_1, c_2, c_3 be different constants in different cases even in the same expression.

Lemma 1. Let $X_{(1)1}, X_{(1)2}, \dots, X_{(1)m}, X_{(2)1}, X_{(2)2}, \dots, X_{(2)m}, \dots, X_{(k)1}, X_{(k)2}, \dots, X_{(k)m}$ be balanced ranked set samples. Suppose that (C1) holds, $\forall x \in \Omega$.

(I) When $f_G^{(r)}(x)$ is continuous function, $\lim_{n \rightarrow \infty} h_n = 0$, and $\lim_{n \rightarrow \infty} n h_n^{2r+1} = \infty$, one has

$$\lim_{n \rightarrow \infty} E \left| f_n^{(r)}(x) - f_G^{(r)}(x) \right|^2 = 0. \quad (18)$$

(II) When $f_G(x) \in C_{s,\alpha}$, putting $h_n = n^{-1/(2+s)}$, for $0 < \lambda \leq 1$, one has

$$E \left| f_n^{(r)}(x) - f_G^{(r)}(x) \right|^{2\lambda} \leq c \cdot n^{-\lambda(s-2r+1)/(2+s)}. \quad (19)$$

Proof. Consider the following.

Proof of (I). Using C_r inequation, we obtain

$$\begin{aligned} & E \left| f_n^{(r)}(x) - f_G^{(r)}(x) \right|^2 \\ & \leq 2 \left| E f_n^{(r)}(x) - f_G^{(r)}(x) \right|^2 + 2 \text{Var} \left(f_n^{(r)}(x) \right) \\ & := 2 \left(A_1^2 + A_2 \right), \end{aligned} \quad (20)$$

where

$$\begin{aligned} E f_n^{(r)}(x) &= \sum_{i=1}^k \sum_{j=1}^m n^{-1} h_n^{-(r+1)} E \left[K_r \left(\frac{x - X_{(i)j}}{h_n} \right) \right] \\ &= h_n^{-(1+r)} E \left[K_r \left(\frac{x - X}{h_n} \right) \right] \\ &= h_n^{-(1+r)} \int_0^\infty K_r \left(\frac{x-y}{h_n} \right) f_G(y) dy \\ &= h_n^{-r} \int_0^1 K_r(u) f_G(x - h_n u) du. \end{aligned} \quad (21)$$

Using Taylor expansion, we get

$$\begin{aligned} & f_G(x - h_n u) - f_G(u) \\ &= \frac{f'_G(x)}{1!} (-h_n u) + \frac{f''_G(x)}{2!} (-h_n u)^2 + \dots \\ &+ \frac{f_G^{(s)}(x - \xi h_n u)}{s!} (-h_n u)^s. \end{aligned} \quad (22)$$

Because $f_G^{(r)}(x)$ is continuous in x and condition (C1), it follows that

$$\begin{aligned} 0 &\leq \lim_{n \rightarrow \infty} \left| E f_n^{(r)}(x) - f_G^{(r)}(x) \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{1}{h_n^r} \int_0^1 K_r(u) f_G(x - h_n u) du - f_G^{(r)}(x) \right| \\ &\leq \frac{1}{r!} \int_0^1 u^r |K_r(u)| \\ &\cdot \lim_{n \rightarrow \infty} \left| f_G^{(r)}(x - \xi h_n u) - f_G^{(r)}(x) \right| du = 0; \end{aligned} \quad (23)$$

further, we have

$$\lim_{n \rightarrow \infty} A_1^2 = \lim_{n \rightarrow \infty} \left| E f_n^{(r)}(x) - f_G^{(r)}(x) \right|^2 = 0. \quad (24)$$

It is easy to see that

$$\begin{aligned} A_2 &= 2 \text{Var} \left[f_n^{(r)}(x) \right] \\ &= 2 \sum_{i=1}^k \sum_{j=1}^m n^{-2} h_n^{-2(r+1)} \text{Var} \left[K_r \left(\frac{x - X_{(i)j}}{h_n} \right) \right] \\ &\leq 2 n^{-2} h_n^{-2(r+1)} \sum_{i=1}^k \sum_{j=1}^m E \left[K_r \left(\frac{x - X_{(i)j}}{h_n} \right) \right]^2 \\ &\leq c \cdot (n h_n^{2r+1})^{-1}. \end{aligned} \quad (25)$$

When $h_n \rightarrow 0$ and $n h_n^{2r+1} \rightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} A_2 = \lim_{n \rightarrow \infty} \text{Var} \left(f_n^{(r)}(x) \right) = 0. \quad (26)$$

Substituting (24) and (26) into (20), proof of (I) is finished.

Proof of (II). Similarly to (20), we can show that

$$\begin{aligned} & E \left| f_n^{(r)}(x) - f_G^{(r)}(x) \right|^{2\lambda} \\ & \leq 2 \left[E f_n^{(r)}(x) - f_G^{(r)}(x) \right]^{2\lambda} + 2 \left[\text{Var} f_n^{(r)}(x) \right]^\lambda \\ & := 2 \left(B_1^{2\lambda} + B_2^\lambda \right). \end{aligned} \quad (27)$$

By Taylor expansion, we obtain

$$\begin{aligned} f_G(x - h_n v) &= f_G(x) + \frac{f'_G(x)}{1!} (-h_n v) \\ &+ \frac{f''_G(x)}{2!} (-h_n v)^2 + \dots \\ &+ \frac{f_G^{(s)}(x - \xi h_n v)}{s!} (-h_n v)^s, \end{aligned} \quad (28)$$

where $0 < \xi < 1$, due to condition (C1) and $f_G(x) \in C_{s,\alpha}$, we have

$$\begin{aligned} & \left| E f_n^{(r)}(x) - f_G^{(r)}(x) \right| \\ & \leq \int_0^1 |K_r(v)| h_n^{s-r} v^s \left| \frac{f_G^{(s)}(x - \xi h_n v)}{s!} \right| dv \leq c \cdot h_n^{s-r}. \end{aligned} \quad (29)$$

Therefore, taking $h_n = n^{-1/(2+s)}$, we get

$$B_1^{2\lambda} = \left| E f_n^{(r)}(x) - f_G^{(r)}(x) \right|^{2\lambda} \leq c \cdot n^{-2\lambda(s-r)/(s+2)}. \quad (30)$$

By (26), choosing $h_n = n^{-1/(2+s)}$, we can get

$$B_2^\lambda \leq \left[c_1 \left(n h_n^{2r+1} \right)^{-1} \right]^\lambda \leq c \cdot n^{-\lambda(s-2r+1)/(2+s)}. \quad (31)$$

Substituting (30) and (31) into (27), proof of (II) is finished. $\square$

Lemma 2 (see [8]). $R(\delta_G, G)$ and $R(\delta_n, G)$ are defined by (12) and (17); then

$$\begin{aligned} 0 &\leq R(\delta_n, G) - R(\delta_G, G) \\ &\leq a \int_{\Omega} |\beta(x)| P(|\beta_n(x) - \beta(x)| \geq |\beta(x)|) dx. \end{aligned} \quad (32)$$

3. Asymptotic Optimality and Convergence Rates Based on Ranked Set Sampling

Theorem 3. Assume (C1) and the following regularity conditions hold:

- (i) $\lim_{n \rightarrow \infty} n h_n^3 = \infty$;
- (ii) $\int_{\Theta} \theta^{-2} dG(\theta) < +\infty$;
- (iii) $f_G^{(1)}(x)$ is continuous function, and one has

$$\lim_{n \rightarrow \infty} R(\delta_n, G) = R(\delta_G, G). \quad (33)$$

Proof of Theorem 3. Applying Lemma 2, we can get

$$\begin{aligned} 0 &\leq R(\delta_n, G) - R(\delta_G, G) \\ &\leq a \int_{\Omega} |\beta(x)| P(|\beta_n(x) - \beta(x)| \geq |\beta(x)|) dx. \end{aligned} \quad (34)$$

Put $M_n(x) = |\beta(x)| P(|\beta_n(x) - \beta(x)| \geq |\beta(x)|)$.

It is easy to see that $M_n(x) \leq |\beta(x)|$.

Again using (10) and Fubini theorem, we obtain

$$\begin{aligned} &\int_{\Omega} |\beta(x)| dx \\ &= \int_{\Omega} \int_{\Theta} \left[1 - \left(\frac{\theta_0}{\theta} \right)^2 \right] f(x | \theta) dG(\theta) dx \\ &= 1 + \theta_0^2 \int_{\Omega} \int_{\Theta} \theta^{-2} f(x | \theta) dG(\theta) dx \\ &= 1 + \theta_0^2 \int_{\Theta} \theta^{-2} dG(\theta) < +\infty. \end{aligned} \quad (35)$$

By domain convergence theorem, then

$$\begin{aligned} 0 &\leq \lim_{n \rightarrow \infty} R(\delta_n, G) - R(\delta_G, G) \\ &\leq \int_{\Omega} \left[\lim_{n \rightarrow \infty} M_n(x) \right] dx. \end{aligned} \quad (36)$$

If Theorem 3 holds, we only need to prove $\lim_{n \rightarrow \infty} M_n(x) = 0$ a.s. x .

Applying Markov's and Jensen's inequations, we have

$$\begin{aligned} M_n(x) &\leq E |\beta_n(x) - \beta(x)| \\ &\leq |u(x)| E |f_n(x) - f_G(x)| \\ &\quad + |v(x)| E |f_n^{(1)}(x) - f_G^{(1)}(x)| \\ &\leq |u(x)| \left[E |f_n(x) - f_G(x)|^2 \right]^{1/2} \\ &\quad + |v(x)| \left[E |f_n^{(1)}(x) - f_G^{(1)}(x)|^2 \right]^{1/2}. \end{aligned} \quad (37)$$

Again using Lemma 1(I), for fixed $x \in \Omega$ and $r = 0, 1$, we get

$$\begin{aligned} 0 &\leq \lim_{n \rightarrow \infty} M_n(x) \\ &\leq |u(x)| \left[\lim_{n \rightarrow \infty} E |f_n(x) - f_G(x)|^2 \right]^{1/2} \\ &\quad + |v(x)| \left[\lim_{n \rightarrow \infty} E |f_n^{(1)}(x) - f_G^{(1)}(x)|^2 \right]^{1/2} = 0. \end{aligned} \quad (38)$$

Substituting (38) into (36), proof of Theorem 3 is finished. $\square$

Theorem 4. Assume (C1) and the following regularity conditions hold.

(C2): consider $\int_{\Omega} x^{-m\lambda} |\beta(x)|^{1-\lambda} dx < +\infty$, where $0 < \lambda \leq 1$, $m = 0, 1$. When $h_n = n^{-1/(2+2s)}$, where $s \geq 2$, one can obtain

$$R(\delta_n, G) - R(\delta_G, G) = O\left(n^{-\lambda(s-1)/(2(s+1))}\right), \quad (39)$$

where $s \geq 2$.

Proof of Theorem 4. Applying Lemma 2 and Markov's inequations, we have

$$\begin{aligned} 0 &\leq R(\delta_n, G) - R(\delta_G, G) \\ &\leq \int_{\Omega} |\beta(x)|^{1-\lambda} E |\beta_n(x) - \beta(x)|^\lambda dx \\ &\leq c_1 \int_{\Omega} |\beta(x)|^{1-\lambda} |u(x)| E |f_n(x) - f_G(x)| dx \\ &\quad + c_2 \int_{\Omega} |\beta(x)|^{1-\lambda} |v(x)| E |f_n^{(1)}(x) - f_G^{(1)}(x)| dx \\ &= D_n + E_n. \end{aligned} \quad (40)$$

By Lemma 1(II) and condition (C2), we get

$$\begin{aligned} D_n &\leq c_1 n^{-\lambda s/(2s+2)} \int_{\Omega} |\beta(x)|^{1-\lambda} |u(x)|^\lambda dx \\ &\leq c_3 n^{-\lambda s/(2s+2)}, \\ E_n &\leq c_2 n^{-\lambda(s-1)/(2s+2)} \int_{\Omega} |\beta(x)|^{1-\lambda} |v(x)|^\lambda dx \\ &\leq c_4 n^{-\lambda(s-1)/(2s+2)}. \end{aligned} \quad (41)$$

Substituting (41) into (40), we can obtain $R(\delta_n, G) - R(\delta_G, G) = O(n^{-\lambda(s-1)/2(s+1)})$. Proof of Theorem 4 is finished. $\square$

Remark 5. When $\lambda \rightarrow 1$ and $s \rightarrow \infty$, $O(n^{-\lambda(s-1)/2(s+1)})$ nears $O(n^{-1/2})$.

4. Conclusions

In this paper, we propose the empirical Bayes test rule for the parameter of power distribution. Based on ranked set sampling, the asymptotical optimality and convergence rates of EB test function are obtained. Another extension of this work would consider empirical Bayes inference for some other distributions based on ranked set sampling.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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