Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2015, Article ID 867206, 13 pages
http://dx.doi.org/10.1155/2015/867206

Research Article

H_∞ Fault Detection for Linear Discrete Time-Varying Descriptor Systems with Missing Measurements

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Received 18 October 2014; Accepted 19 November 2014

Academic Editor: Zidong Wang

This paper deals with the problem of H_∞ fault detection for a class of linear discrete time-varying descriptor systems with missing measurements, and the missing measurements are described by a Bernoulli random binary switching sequence. We first translate the H_∞ fault detection problem into an indefinite quadratic form problem. Then, a sufficient and necessary condition on the existence of the minimum is derived. Finally, an observer-based H_∞ fault detection filter is obtained such that the minimum is positive and its parameter matrices are calculated recursively by solving a matrix differential equation. A numerical example is given to demonstrate the efficiency of the proposed method.

1. Introduction

During the last four decades, the fault diagnosis theory has received considerable attention, and many remarkable achievements have been obtained [1–6]. As mentioned in [2], it is well recognized that the model-based fault diagnosis techniques can be classified into three classical approaches: observer-based methods, parity space methods, and parameter identification based methods. Most of the achievements on fault diagnosis always assume that the observations contain the signal to be detected. However, in practice, the observation may contain the signal in a random manner. In this paper, the H_∞ fault detection problem for a class of linear discrete time-varying descriptor systems with random missing measurements is investigated.

In practice, time variability is the inherent characteristic of most systems. Recently, research on fault diagnosis of linear systems with time-varying parameters has attracted more and more attention; see, for example, [7–22]. By dividing the filter gain matrix into two sections, an unknown input decoupling optimal filter for linear stochastic systems has been designed in [7], and its application on fault diagnosis has also been addressed. By employing the invariant subspace method and game theory, a game fault detection filter has been proposed in [8]. Based on parity space approach and stochastic signal processing methods, the fault detection and isolation have been studied for a class of linear discrete systems with stochastic inputs and deterministic disturbances and faults in [15, 21]. By applying adaptive observer method, a residual generator has been provided for linear discrete time-varying systems in [19, 22], and the residual adaptive threshold is derived by set-membership computations based on zonotopes. By assuming that the mean and variance of the fault and disturbances are known, the optimal fault detection filter for a class of stochastic systems has been developed in [10, 11, 13, 16]; however, in general, the prior information of the fault cannot be obtained. By solving a min-max problem with a generalized least-squares cost criterion, a generalized least-squares fault detection filter has been designed in [9, 12]. By using an adaptive observer method, a fault diagnosis technique for linear time-varying systems has been developed in [14, 17, 18]. However, to the best of authors' knowledge, few reports on fault diagnosis problem of linear discrete time-varying descriptor systems have been published, which motivates the present study.

In classical fault diagnosis theories, all residual signals are obtained in the case that the measured output contains valid information. However, the data packet dropout is inevitable

in navigation and guidance system, industrial control system, and network control system. Thus, in recent years, increasing attention has been paid to fault diagnosis problems for systems with missing measurements [23–25]. By employing the linear matrix inequality technique, both full-order and reduced-order fault detection filters have been considered for a class of linear discrete time-invariant systems with missing measurements and parameter uncertainty in [23]. In finite frequency domain, the fault detection problem has been concerned with systems with missing measurements in [24], and the fault detection scheme has been utilized to an aircraft model. In [25], the missing measurements are described by Markov random process, and the residual generator is presented as a discrete-time Markovian jump linear system. Note that the existing results mainly focus on the nondescriptor systems; there are few achievements on fault detection problem for descriptor systems with missing measurements. Therefore, in this paper, we aim to design the H_∞ fault detection filter and residual evaluation scheme for a class of linear discrete time-varying descriptor systems with random missing measurements.

In this paper, based on the estimation method proposed in [26], a new fault detection filter design approach is developed for a class of linear discrete time-varying descriptor systems with random missing measurements. First, the H_∞ fault detection problem is converted to the problem in which a certain indefinite quadratic form has a minimum and the fault detection filter parameter matrices are such that the minimum is positive. Then, by applying matrix analysis method, a necessary and sufficient condition for the indefinite quadratic form is analyzed. And by guaranteeing the positivity of the minimum, the parameter matrices of the fault detection filter are obtained. Moreover, the residual evaluation function and threshold are designed for the fault detection. Finally, a numerical example is provided to illustrate the performance of the H_∞ fault detection filter and the residual evaluation scheme.

2. Problem Statement

Consider a discrete time-varying descriptor system described by the following model:

$$\begin{aligned} M(i+1)\,x(i+1) &= A(i)\,x(i) + B_f(i)\,f(i) + w(i), \\ y(i) &= \lambda(i)\left(C(i)\,x(i) + D_f(i)\,f(i)\right) + v(i), \end{aligned} \tag{1}$$

where $x(i) \in R^n$, $w(i) \in R^n$, $y(i) \in R^m$, $v(i) \in R^m$, and $f(i) \in R^p$ are the state, external disturbance input, uncertain measurement output, measurement noise, and fault to be detected, respectively; $f(i)$, $w(i)$, and $v(i)$ are bounded signals belonging to $l_2[0, N]$; N is a positive integer; $M(i)$, $A(i)$, $B_f(i)$, $C(i)$, and $D_f(i)$ are known real-time-varying matrices with appropriate dimensions; and $M(i)$ is a singular matrix with $\operatorname{rank}[M(i)] = n_1$, $0 < n_1 < n$; the random variable $\lambda(i) \in R$ is a Bernoulli distributed white sequence taking the values of 0 and 1 with

$$\begin{gathered} \operatorname{Prob}\{\lambda(i) = 1\} = E\{\lambda(i)\} = \rho, \\ \operatorname{Prob}\{\lambda(i) = 0\} = 1 - E\{\lambda(i)\} = 1 - \rho, \\ E\left\{\lambda^2(i)\right\} = \rho, \qquad E\left\{(1-\lambda(i))^2\right\} = 1 - \rho, \\ E\{(1-\lambda(i))\,\lambda(i)\} = 0, \end{gathered} \tag{2}$$

where $E(\cdot)$ denotes the mathematical expectation, $\operatorname{Prob}(\cdot)$ denotes the probability distribution, and $\rho \in R$ is a known positive scalar.

Hypothesis 1. The initial matrices of system (1) satisfy the condition that $M(0) = A(-1) = B_f(-1) = I$.

The H_∞ fault detection problem under investigation in this paper can be stated as follows. Given a disturbance attenuation level $\gamma > 0$, based on the measurement output sequence $\{y(i)\}_{i=1}^k$, find a residual signal $r(k)$. If it exists, the following inequality is satisfied:

$$\begin{aligned} &\sup_{x(0),\{d(i)\}_{i=0}^k} \left(\left(E\left\{\sum_{i=1}^k (r(i) - f(i-1))^T (r(i) - f(i-1))\right\}\right)\right. \\ &\quad \times \left(E\left\{(M(0)\,x(0) - A(-1)\,\check{x}(0))^T P_0^{-1}\right.\right. \\ &\qquad \times (M(0)\,x(0) - A(-1)\,\check{x}(0)) \\ &\qquad \left.\left. + \sum_{i=0}^k d^T(i)\,d(i)\right\}\right)^{-1}\Bigg) \\ &< \gamma^2, \end{aligned} \tag{3}$$

where

$$d(i) = \begin{cases} \left[f^T(i) \;\; w^T(i) \;\; v^T(i)\right]^T, & i = 0, 1, \ldots, k-1, \\ \left[f^T(i) \;\; 0 \;\; v^T(i)\right]^T, & i = k. \end{cases} \tag{4}$$

P_0 is a given positive definite matrix function which reflects the relative uncertainty of the initial state $x(0)$ about the initial state estimate $\check{x}(0)$. Without loss of generality, let $\check{x}(0) = 0$.

Even if the measurement data are fully available, the valid information of fault $f(i)$ is not contained in the measurement output $\{y(j)\}_{j=0}^i$ when $D_f(i) = 0$. Thus, an H_∞ one-step lag fault detection issue is defined as (3) to overcome this problem.

Define the following new variables:

$$\overline{x}(i) = \begin{bmatrix} x(i) \\ f(i) \\ x(i-1) \\ f(i-1) \end{bmatrix}, \qquad \overline{w}(i) = \begin{bmatrix} w(i) \\ f(i+1) \\ 0 \\ 0 \end{bmatrix}. \tag{5}$$

Thus, system (1) can be described as the following augmented model:

$$\overline{M}(i+1)\overline{x}(i+1) = \overline{A}(i)\overline{x}(i) + \overline{w}(i),$$
$$y(i) = \lambda(i)\overline{C}(i)\overline{x}(i) + v(i), \tag{6}$$

where

$$\overline{M}(i+1) = \begin{bmatrix} M(i+1) & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix},$$
$$\overline{A}(i) = \begin{bmatrix} A(i) & B_f(i) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix}, \tag{7}$$
$$\overline{C}(i) = \begin{bmatrix} C(i) & D_f(i) & 0 & 0 \end{bmatrix}.$$

For the purpose of fault detection, the following observer-based fault detection filter is proposed as a residual generator:

$$\hat{\overline{x}}(k+1) = V_1(k)\hat{\overline{x}}(k) + V_2(k)y(k),$$
$$r(k) = L(k)\hat{\overline{x}}(k), \tag{8}$$
$$\hat{\overline{x}}(0) = 0,$$

where $L(i) = [0 \;\; 0 \;\; 0 \;\; I]$. Thus, the fault estimation problem can come down to the following: find the parameter matrices $V_1(k)$ and $V_2(k)$; if they exist, the performance index (3) is satisfied.

According to system (6), the performance index (3) is reexpressed as

$$\begin{aligned} \sup_{x(0),\{\overline{w}(i),v(i)\}_{i=0}^{k}} & \left(\left(E\left\{ \sum_{i=1}^{k} (r(i) - L(i)\overline{x}(i))^T \right.\right.\right. \\ & \left.\left. \times (r(i) - L(i)\overline{x}(i)) \right\} \right) \\ & \times \left(E\left\{ \left(\overline{M}(0)\overline{x}(0) - \overline{A}(-1)\breve{\overline{x}}(0)\right)^T \overline{P}_0^{-1} \right.\right. \\ & \times \left(\overline{M}(0)\overline{x}(0) - \overline{A}(-1)\breve{\overline{x}}(0)\right) \\ & + \sum_{i=0}^{k-1} \overline{w}^T(i)\overline{w}(i) \\ & \left.\left. + \sum_{i=0}^{k} v^T(i)v(i) \right\} \right)^{-1} \Bigg) \\ & < \gamma^2, \end{aligned} \tag{9}$$

where $\overline{P}_0 = \mathrm{diag}\{P_0, I_p, I_n, I_p\}$ and $\breve{\overline{x}}(0) = [\breve{x}^T(0) \;\; 0 \;\; 0 \;\; 0]^T$, and assume that $x(-1) = 0$ and $f(-1) = 0$.

Define

$$\begin{aligned} J_k = E & \left\{ \left(\overline{M}(0)\overline{x}(0) - \overline{A}(-1)\breve{\overline{x}}(0)\right)^T \overline{P}_0^{-1} \right. \\ & \times \left(\overline{M}(0)\overline{x}(0) - \overline{A}(-1)\breve{\overline{x}}(0)\right) \\ & \left. + \sum_{i=0}^{k-1} \overline{w}^T(i)\overline{w}(i) + \sum_{i=0}^{k} v^T(i)v(i) \right\} \\ & - \gamma^{-2} E\left\{ \sum_{i=1}^{k} (r(i) - L(i)\overline{x}(i))^T (r(i) - L(i)\overline{x}(i)) \right\}. \end{aligned} \tag{10}$$

Then, the H_∞ fault detection problem is equivalent to the following:

(1) J_k of (10) has minimum with respect to $\{\overline{x}(i)\}_{i=0}^{k}$ and $\{\overline{w}(i)\}_{i=0}^{k}$;

(2) $\{r(i)\}_{i=0}^{k}$ can be chosen such that the value of J_k at its minimum is positive.

In the following, we will first discuss the mathematical expectation of J_k based on system (6). Then, the existence of the minimum J_k over $\{\overline{x}(i)\}_{i=0}^{k}$ and $\{\overline{w}(i)\}_{i=0}^{k}$ can be derived. Finally, a solution to parameter matrices $V_1(k)$ and $V_2(k)$ will be obtained such that $J_k > 0$.

3. Main Results

From (6) and (10), we have

$$\begin{aligned} J_k = E & \left\{ \left(\overline{M}(0)\overline{x}(0) - \overline{A}(-1)\breve{\overline{x}}(0)\right)^T \overline{P}_0^{-1} \right. \\ & \left. \times \left(\overline{M}(0)\overline{x}(0) - \overline{A}(-1)\breve{\overline{x}}(0)\right) \right\} \\ & + E\left\{ \sum_{i=0}^{k-1} \left(\overline{M}(i+1)\overline{x}(i+1) - \overline{A}(i)\overline{x}(i)\right)^T \right. \\ & \left. \times \left(\overline{M}(i+1)\overline{x}(i+1) - \overline{A}(i)\overline{x}(i)\right) \right\} \\ & + E\left\{ \sum_{i=0}^{k} \left(y(i) - \lambda(i)\overline{C}(i)\overline{x}(i)\right)^T \right. \\ & \left. \times \left(y(i) - \lambda(i)\overline{C}(i)\overline{x}(i)\right) \right\} \\ & - \gamma^{-2} E\left\{ \sum_{i=1}^{k} (r(i) - L(i)\overline{x}(i))^T (r(i) - L(i)\overline{x}(i)) \right\}. \end{aligned} \tag{11}$$

Notice that

$$\begin{aligned}
&E\left\{\left(y(i)-\lambda(i)\overline{C}(i)\overline{x}(i)\right)^T\left(y(i)-\lambda(i)\overline{C}(i)\overline{x}(i)\right)\right\}\\
&\quad=E\left\{y^T(i)y(i)-\lambda(i)y^T(i)\overline{C}(i)\overline{x}(i)-\lambda(i)\overline{x}^T(i)\right.\\
&\qquad\left.\times\overline{C}^T(i)y(i)+\lambda^2(i)\overline{x}^T(i)\overline{C}^T(i)\overline{C}(i)\overline{x}(i)\right\}\\
&\quad=y^T(i)y(i)-\rho y^T(i)\overline{C}(i)\overline{x}(i)-\rho\overline{x}^T(i)\overline{C}^T(i)y(i)\\
&\qquad+\rho\overline{x}^T(i)\overline{C}^T(i)\overline{C}(i)\overline{x}(i)\\
&\quad=\left(y(i)-\rho\overline{C}(i)\overline{x}(i)\right)^T\left(y(i)-\rho\overline{C}(i)\overline{x}(i)\right)\\
&\qquad+\left(\rho-\rho^2\right)\overline{x}^T(i)\overline{C}^T(i)\overline{C}(i)\overline{x}(i)\\
&\quad=\left(y(i)-\widetilde{C}(i)\overline{x}(i)\right)^T\left(y(i)-\widetilde{C}(i)\overline{x}(i)\right)\\
&\qquad+\left(F(i)\overline{x}(i)\right)^T\left(F(i)\overline{x}(i)\right),
\end{aligned}\tag{12}$$

where

$$\widetilde{C}(i)=\rho\overline{C}(i),\qquad F(i)=\sqrt{\rho-\rho^2}\overline{C}(i).\tag{13}$$

Then, we have

$$\begin{aligned}
J_k&=\left(\overline{M}(0)\overline{x}(0)-\overline{A}(-1)\breve{\overline{x}}(0)\right)^T\overline{P}_0^{-1}\\
&\quad\times\left(\overline{M}(0)\overline{x}(0)-\overline{A}(-1)\breve{\overline{x}}(0)\right)\\
&\quad+\sum_{i=0}^{k-1}\left(\overline{M}(i+1)\overline{x}(i+1)-\overline{A}(i)\overline{x}(i)\right)^T\\
&\qquad\times\left(\overline{M}(i+1)\overline{x}(i+1)-\overline{A}(i)\overline{x}(i)\right)\\
&\quad+\sum_{i=0}^{k}\left(y(i)-\widetilde{C}(i)\overline{x}(i)\right)^T\left(y(i)-\widetilde{C}(i)\overline{x}(i)\right)\\
&\quad+\sum_{i=0}^{k}\left(F(i)\overline{x}(i)\right)^T\left(F(i)\overline{x}(i)\right)\\
&\quad-\gamma^{-2}\sum_{i=1}^{k}\left(r(i)-L(i)\overline{x}(i)\right)^T\left(r(i)-L(i)\overline{x}(i)\right).
\end{aligned}\tag{14}$$

Thus, J_k can be further expressed as

$$J_k=\left(\Sigma_kX_k-Y_k\right)^T\Pi_k\left(\Sigma_kX_k-Y_k\right),\tag{15}$$

where

$$X_k=\begin{bmatrix}\overline{x}(k)\\\overline{x}(k-1)\\\overline{x}(k-2)\\\vdots\\\overline{x}(2)\\\overline{x}(1)\\\overline{x}(0)\end{bmatrix},\qquad Y_k=\begin{bmatrix}\overline{y}(k)\\\overline{y}(k-1)\\\overline{y}(k-2)\\\vdots\\\overline{y}(2)\\\overline{y}(1)\\\overline{y}(0)\end{bmatrix},$$

$$\Sigma_k=\begin{bmatrix}\Omega(k)&\Psi(k-1)&0&0&0&0&0\\0&\Omega(k-1)&\Psi(k-2)&0&0&0&0\\0&0&\Omega(k-2)&\ddots&0&0&0\\0&0&0&\ddots&\Psi(2)&0&0\\0&0&0&0&\Omega(2)&\Psi(1)&0\\0&0&0&0&0&\Omega(1)&\Psi(0)\\0&0&0&0&0&0&\Omega(0)\end{bmatrix},$$

$$\Pi_k=\begin{bmatrix}\Phi(k)&0&0&0\\0&\Phi(k-1)&0&0\\0&0&\ddots&0\\0&0&0&\Phi(0)\end{bmatrix},\qquad\Omega(i)=\begin{bmatrix}\overline{M}(i)\\F(i)\\\widetilde{C}(i)\\L(i)\end{bmatrix},$$

$$\Omega(0)=\begin{bmatrix}\overline{M}(0)\\F(0)\\\widetilde{C}(0)\end{bmatrix},\qquad\Psi(i-1)=\begin{bmatrix}-\overline{A}(i-1)\\0\\0\\0\end{bmatrix},$$

$$\Phi(i)=\begin{bmatrix}I&0&0&0\\0&I&0&0\\0&0&I&0\\0&0&0&-\gamma^{-2}I\end{bmatrix},\qquad\Phi(0)=\begin{bmatrix}\overline{P}_0^{-1}&0&0\\0&I&0\\0&0&I\end{bmatrix},$$

$$\overline{y}(i)=\begin{bmatrix}0\\0\\y(i)\\r(i)\end{bmatrix},\qquad\overline{y}(0)=\begin{bmatrix}\breve{\overline{x}}(0)\\0\\y(0)\end{bmatrix},$$

$$i=1,2,\ldots,k.\tag{16}$$

In virtue of the above variables, for all $k>0$, we have

$$\begin{gathered}
X_k=\begin{bmatrix}\overline{x}(k)\\X_{k-1}\end{bmatrix},\qquad Y_k=\begin{bmatrix}\overline{y}(k)\\Y_{k-1}\end{bmatrix},\\
\Sigma_k=\begin{bmatrix}\Omega(k)&\alpha(k-1)\\0&\Sigma_{k-1}\end{bmatrix},\qquad\Pi_k=\begin{bmatrix}\Phi(k)&0\\0&\Pi_{k-1}\end{bmatrix},\\
\alpha(k-1)=\begin{bmatrix}\Psi(k-1)&0&\cdots&0\end{bmatrix}.
\end{gathered}\tag{17}$$

Lemma 1 (see [26]). *Consider matrices α, β, R, and x of appropriate dimensions, and R is symmetric. If and only if $\alpha^T R\alpha \geq 0$ and $\mathrm{Ker}(\alpha^T R\alpha) \subset \mathrm{Ker}(R\alpha)$, for any β, we have*

$$\inf_x (\alpha x - \beta)^T R (\alpha x - \beta) > -\infty. \tag{18}$$

If the minimum is attained, it is unique if and only if $\alpha^T R\alpha > 0$. Moreover, the optimal solution is derived by $\hat{x} = (\alpha^T R\alpha)^{-1}\alpha^T R\beta$.

3.1. Existence Conditions of the Minimum. According to Lemma 1, J_k has the minimum if and only if $\Sigma_k^T \Pi_k \Sigma_k > 0$. When $k = 0$, let

$$\begin{aligned} P(0) &= \Sigma_0^T \Pi_0 \Sigma_0 \\ &= \begin{bmatrix} \overline{M}(0) \\ F(0) \\ \widetilde{C}(0) \end{bmatrix}^T \begin{bmatrix} \overline{P}_0^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \overline{M}(0) \\ F(0) \\ \widetilde{C}(0) \end{bmatrix} \\ &= \overline{M}^T(0)\overline{P}_0^{-1}\overline{M}(0) + \widetilde{C}^T(0)\widetilde{C}(0) + F^T(0)F(0). \end{aligned} \tag{19}$$

Therefore, when $k = 0$, J_0 has the minimum if and only if $P(0) > 0$.

Furthermore, for all $k > 0$, we can obtain the following equation from (17):

$$\Sigma_k^T \Pi_k \Sigma_k = \begin{bmatrix} \Omega^T(k)\Phi(k)\Omega(k) & \Omega^T(k)\Phi(k)\alpha(k-1) \\ \alpha^T(k-1)\Phi(k)\Omega(k) & \Sigma_{k-1}^T\Pi_{k-1}\Sigma_{k-1} + \alpha^T(k-1)\Phi(k)\alpha(k-1) \end{bmatrix}. \tag{20}$$

To ensure that (20) is positive definite, $\Sigma_{k-1}^T\Pi_{k-1}\Sigma_{k-1} + \alpha^T(k-1)\Phi(k)\alpha(k-1)$ must be positive definite. Assume that $\Sigma_{k-1}^T\Pi_{k-1}\Sigma_{k-1} > 0$ is satisfied, and note that

$$\begin{aligned} &\alpha^T(k-1)\Phi(k)\alpha(k-1) \\ &= \begin{bmatrix} \Psi^T(k-1) \\ 0 \end{bmatrix} \Phi(k) \begin{bmatrix} \Psi(k-1) & 0 \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} -\overline{A}(k-1) \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & -\gamma^{-2}I \end{bmatrix} \begin{bmatrix} -\overline{A}(k-1) \\ 0 \\ 0 \\ 0 \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \overline{A}^T(k-1)\overline{A}(k-1) & 0 \\ 0 & 0 \end{bmatrix} \\ &\geq 0. \end{aligned} \tag{21}$$

Therefore, $\Sigma_{k-1}^T\Pi_{k-1}\Sigma_{k-1} + \alpha^T(k-1)\Phi(k)\alpha(k-1) > 0$. If and only if the Schur complement of $\Sigma_{k-1}^T\Pi_{k-1}\Sigma_{k-1} + \alpha^T(k-1)\Phi(k)\alpha(k-1)$ in (20) is positive definite, we have $\Sigma_k^T\Pi_k\Sigma_k > 0$.

Define

$$\begin{aligned} P(k) &= \Omega^T(k)\Phi(k)\Omega(k) - \Omega^T(k)\Phi(k)\alpha(k-1) \\ &\quad \times \left(\Sigma_{k-1}^T\Pi_{k-1}\Sigma_{k-1} + \alpha^T(k-1)\Phi(k)\alpha(k-1)\right)^{-1} \\ &\quad \times \alpha^T(k-1)\Phi(k)\Omega(k) \\ &= \Omega^T(k)\Phi(k) \\ &\quad \times \Big(I - \alpha(k-1) \\ &\qquad \times \left(\Sigma_{k-1}^T\Pi_{k-1}\Sigma_{k-1} + \alpha^T(k-1)\Phi(k)\alpha(k-1)\right)^{-1} \\ &\qquad \times \alpha^T(k-1)\Phi(k)\Big)\Omega(k) \\ &= \Omega^T(k)\Phi(k) \\ &\quad \times \left(I + \alpha(k-1)\left(\Sigma_{k-1}^T\Pi_{k-1}\Sigma_{k-1}\right)^{-1}\alpha^T(k-1)\Phi(k)\right)^{-1} \\ &\quad \times \Omega(k). \end{aligned} \tag{22}$$

And notice that

$$\begin{aligned}
\Sigma_{k-1}^{T}\Pi_{k-1}\Sigma_{k-1} &= \begin{bmatrix} \Omega^{T}(k-1)\Phi(k-1)\Omega(k-1) & \Omega^{T}(k-1)\Phi(k-1)\alpha(k-2) \\ \alpha^{T}(k-2)\Phi(k-1)\Omega(k-1) & \Sigma_{k-2}^{T}\Pi_{k-2}\Sigma_{k-2}+\alpha^{T}(k-2)\Phi(k-1)\alpha(k-2) \end{bmatrix} \\
&= \begin{bmatrix} I & \Omega^{T}(k-1)\Phi(k-1)\alpha(k-2)\left(\Sigma_{k-2}^{T}\Pi_{k-2}\Sigma_{k-2}+\alpha^{T}(k-2)\Phi(k-1)\alpha(k-2)\right)^{-1} \\ 0 & I \end{bmatrix} \\
&\quad \times \begin{bmatrix} P(k-1) & 0 \\ 0 & \Sigma_{k-2}^{T}\Pi_{k-2}\Sigma_{k-2}+\alpha^{T}(k-2)\Phi(k-1)\alpha(k-2) \end{bmatrix} \\
&\quad \times \begin{bmatrix} I & 0 \\ \left(\Sigma_{k-2}^{T}\Pi_{k-2}\Sigma_{k-2}+\alpha^{T}(k-2)\Phi(k-1)\alpha(k-2)\right)^{-1}\alpha^{T}(k-2)\Phi(k-1)\Omega(k-1) & I \end{bmatrix}.
\end{aligned} \tag{23}$$

Then,

$$\begin{aligned}
&\alpha(k-1)\left(\Sigma_{k-1}^{T}\Pi_{k-1}\Sigma_{k-1}\right)^{-1}\alpha^{T}(k-1) \\
&= \begin{bmatrix} \Psi(k-1) & 0 \end{bmatrix} \\
&\quad \times \begin{bmatrix} P(k-1) & 0 \\ 0 & \Sigma_{k-2}^{T}\Pi_{k-2}\Sigma_{k-2}+\alpha^{T}(k-2)\Phi(k-1)\alpha(k-2) \end{bmatrix}^{-1} \\
&\quad \times \begin{bmatrix} \Psi^{T}(k-1) \\ 0 \end{bmatrix} \\
&= \Psi(k-1)P^{-1}(k-1)\Psi^{T}(k-1).
\end{aligned} \tag{24}$$

Moreover, we obtain

$$\begin{aligned}
&P(k) \\
&= \Omega^{T}(k)\Phi(k) \\
&\quad \times \left(I+\alpha(k-1)\left(\Sigma_{k-1}^{T}\Pi_{k-1}\Sigma_{k-1}\right)^{-1}\alpha^{T}(k-1)\Phi(k)\right)^{-1} \\
&\quad \times \Omega(k) \\
&= \Omega^{T}(k)\Phi(k) \\
&\quad \times \left(I+\Psi(k-1)P^{-1}(k-1)\Psi^{T}(k-1)\Phi(k)\right)^{-1}\Omega(k) \\
&= \begin{bmatrix} \overline{M}(k) \\ F(k) \\ \widetilde{C}(k) \\ L(k) \end{bmatrix}^{T} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & -\gamma^{-2}I \end{bmatrix} \\
&\quad \times \left(I + \begin{bmatrix} -\overline{A}(k-1) \\ 0 \\ 0 \\ 0 \end{bmatrix} P^{-1}(k-1) \begin{bmatrix} -\overline{A}(k-1) \\ 0 \\ 0 \\ 0 \end{bmatrix}^{T} \right. \\
&\quad \left. \times \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & -\gamma^{-2}I \end{bmatrix} \right)^{-1} \begin{bmatrix} \overline{M}(k) \\ F(k) \\ \widetilde{C}(k) \\ L(k) \end{bmatrix} \\
&= \begin{bmatrix} \overline{M}^{T}(k) & F^{T}(k) & \widetilde{C}^{T}(k) & -\gamma^{-2}L^{T}(k) \end{bmatrix} \\
&\quad \times \begin{bmatrix} \left(I+\overline{A}(k-1)P^{-1}(k-1)\overline{A}^{T}(k-1)\right)^{-1} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \\
&\quad \times \begin{bmatrix} \overline{M}(k) \\ F(k) \\ \widetilde{C}(k) \\ L(k) \end{bmatrix} \\
&= \overline{M}^{T}(k)\left(I+\overline{A}(k-1)P^{-1}(k-1)\overline{A}^{T}(k-1)\right)^{-1}\overline{M}(k) \\
&\quad + \widetilde{C}^{T}(k)\widetilde{C}(k)+F^{T}(k)F(k)-\gamma^{-2}L^{T}(k)L(k).
\end{aligned} \tag{25}$$

Thus, it is readily known that J_k has a minimum if and only if $P(k) > 0$.

In light of the above discussion, we have the following results.

Theorem 2. *Consider the linear discrete time-varying descriptor system (1), given a scalar $\gamma > 0$; then J_k has a minimum over*

$\{\overline{x}(i)\}_{i=0}^{k}$ and $\{\overline{w}(i)\}_{i=0}^{k}$ if and only if $P(k) > 0$ $(k = 0, 1, \ldots, k)$, where

$$\begin{aligned} P(k) &= \overline{M}^T(k)\left(I + \overline{A}(k-1)P^{-1}(k-1)\overline{A}^T(k-1)\right)^{-1} \\ &\quad \times \overline{M}(k) + \widetilde{C}^T(k)\widetilde{C}(k) + F^T(k)F(k) \\ &\quad - \gamma^{-2}L^T(k)L(k), \\ P(0) &= \overline{M}^T(0)\overline{P}_0^{-1}\overline{M}(0) + \widetilde{C}^T(0)\widetilde{C}(0) + F^T(0)F(0). \end{aligned} \tag{26}$$

3.2. Design of the H_∞ Fault Detection Filter. According to Lemma 1, it is known that if J_k has a minimum over $\{\overline{x}(i)\}_{i=0}^{k}$ and $\{\overline{w}(i)\}_{i=0}^{k}$, the optimal solution is

$$\widehat{X}_{k|k} = \left(\Sigma_k^T\Pi_k\Sigma_k\right)^{-1}\Sigma_k^T\Pi_k Y_k. \tag{27}$$

When $k = 0$, we have

$$\begin{aligned} \widehat{\overline{x}}(0 \mid 0) &= \left(\Sigma_0^T\Pi_0\Sigma_0\right)^{-1}\Sigma_0^T\Pi_0 Y_0 \\ &= \left(\Omega^T(0)\Phi(0)\Omega(0)\right)^{-1}\Omega^T(0)\Phi(0)\begin{bmatrix}\breve{\overline{x}}(0)\\ 0\\ y(0)\end{bmatrix} \\ &= P^{-1}(0)\begin{bmatrix}\overline{M}(0)\\ F(0)\\ \widetilde{C}(0)\end{bmatrix}^T\begin{bmatrix}\overline{P}_0^{-1} & 0 & 0\\ 0 & I & 0\\ 0 & 0 & I\end{bmatrix}\begin{bmatrix}\breve{\overline{x}}(0)\\ 0\\ y(0)\end{bmatrix} \\ &= P^{-1}(0)\overline{M}^T(0)\overline{P}_0^{-1}\breve{\overline{x}}(0) + P^{-1}(0)\widetilde{C}^T(0)y(0). \end{aligned} \tag{28}$$

Furthermore, notice that

$$\begin{aligned} &\begin{bmatrix}\widehat{\overline{x}}(k \mid k)\\ \widehat{X}_{k-1|k}\end{bmatrix} \\ &= \begin{bmatrix}\Omega^T(k)\Phi(k)\Omega(k) & \Omega^T(k)\Phi(k)\alpha(k-1)\\ \alpha^T(k-1)\Phi(k)\Omega(k) & \Sigma_{k-1}^T\Pi_{k-1}\Sigma_{k-1} + \alpha^T(k-1)\Phi(k)\alpha(k-1)\end{bmatrix}^{-1} \\ &\quad \times \begin{bmatrix}\Omega(k) & \alpha(k-1)\\ 0 & \Sigma_{k-1}\end{bmatrix}^T\begin{bmatrix}\Phi(k) & 0\\ 0 & \Pi_{k-1}\end{bmatrix}\begin{bmatrix}\overline{y}(k)\\ Y_{k-1}\end{bmatrix} \\ &= \begin{bmatrix}\Omega^T(k)\Phi(k)\Omega(k) & \Omega^T(k)\Phi(k)\alpha(k-1)\\ \alpha^T(k-1)\Phi(k)\Omega(k) & \Sigma_{k-1}^T\Pi_{k-1}\Sigma_{k-1} + \alpha^T(k-1)\Phi(k)\alpha(k-1)\end{bmatrix}^{-1} \\ &\quad \times \begin{bmatrix}\Omega^T(k)\Phi(k)\overline{y}(k)\\ \alpha^T(k-1)\Phi(k)\overline{y}(k) + \Sigma_{k-1}^T\Pi_{k-1}Y_{k-1}\end{bmatrix} \\ &= \begin{bmatrix}H_{11,k} & H_{12,k}\\ H_{21,k} & H_{22,k}\end{bmatrix} \\ &\quad \times \left(\begin{bmatrix}\Omega^T(k)\\ \alpha^T(k-1)\end{bmatrix}\Phi(k)\overline{y}(k) + \begin{bmatrix}0\\ \Sigma_{k-1}^T\Pi_{k-1}\Sigma_{k-1}\widehat{X}_{k-1|k-1}\end{bmatrix}\right). \end{aligned} \tag{29}$$

Then,

$$\begin{aligned} \widehat{\overline{x}}(k \mid k) &= \left(H_{11,k}\Omega^T(k) + H_{12,k}\alpha^T(k-1)\right)\Phi(k)\overline{y}(k) \\ &\quad + H_{12,k}\Sigma_{k-1}^T\Pi_{k-1}\Sigma_{k-1}\widehat{X}_{k-1|k-1}, \\ \left(\Sigma_k^T\Pi_k\Sigma_k\right)^{-1} &= \begin{bmatrix}\Omega^T(k)\Phi(k)\Omega(k) & \Omega^T(k)\Phi(k)\alpha(k-1)\\ \alpha^T(k-1)\Phi(k)\Omega(k) & \Sigma_{k-1}^T\Pi_{k-1}\Sigma_{k-1} + \alpha^T(k-1)\Phi(k)\alpha(k-1)\end{bmatrix}^{-1} \\ &= \left(\begin{bmatrix}I & \Omega^T(k)\Phi(k)\alpha(k-1)\left(\Sigma_{k-1}^T\Pi_{k-1}\Sigma_{k-1} + \alpha^T(k-1)\Phi(k)\alpha(k-1)\right)^{-1}\\ 0 & I\end{bmatrix}\right. \\ &\quad \times \begin{bmatrix}P(k) & 0\\ 0 & \Sigma_{k-1}^T\Pi_{k-1}\Sigma_{k-1} + \alpha^T(k-1)\Phi(k)\alpha(k-1)\end{bmatrix} \\ &\quad \left.\times \begin{bmatrix}I & 0\\ \left(\Sigma_{k-1}^T\Pi_{k-1}\Sigma_{k-1} + \alpha^T(k-1)\Phi(k)\alpha(k-1)\right)^{-1}\alpha^T(k-1)\Phi(k)\Omega(k) & I\end{bmatrix}\right)^{-1} \\ &= \begin{bmatrix}I & 0\\ -\left(\Sigma_{k-1}^T\Pi_{k-1}\Sigma_{k-1} + \alpha^T(k-1)\Phi(k)\alpha(k-1)\right)^{-1}\alpha^T(k-1)\Phi(k)\Omega(k) & I\end{bmatrix} \\ &\quad \times \begin{bmatrix}P^{-1}(k) & 0\\ 0 & \left(\Sigma_{k-1}^T\Pi_{k-1}\Sigma_{k-1} + \alpha^T(k-1)\Phi(k)\alpha(k-1)\right)^{-1}\end{bmatrix} \\ &\quad \times \begin{bmatrix}I & -\Omega^T(k)\Phi(k)\alpha(k-1)\left(\Sigma_{k-1}^T\Pi_{k-1}\Sigma_{k-1} + \alpha^T(k-1)\Phi(k)\alpha(k-1)\right)^{-1}\\ 0 & I\end{bmatrix} \\ &= \begin{bmatrix}H_{11,k} & H_{12,k}\\ H_{21,k} & H_{22,k}\end{bmatrix}. \end{aligned} \tag{30}$$

Hence,

$$\begin{aligned}
H_{11,k} &= P^{-1}(k), \\
H_{12,k} &= -P^{-1}(k)\,\Omega^T(k)\,\Phi(k)\,\alpha(k-1) \\
&\quad \times \left(\Sigma_{k-1}^T \Pi_{k-1} \Sigma_{k-1} + \alpha^T(k-1)\,\Phi(k)\,\alpha(k-1)\right)^{-1} \\
&= -P^{-1}(k)\,\Omega^T(k) \\
&\quad \times \Big(\Phi(k) - \Phi(k)\,\alpha(k-1)\left(\Sigma_{k-1}^T \Pi_{k-1} \Sigma_{k-1}\right)^{-1} \\
&\qquad \times \alpha^T(k-1) \\
&\qquad \times \Big(\Phi^{-1}(k) + \alpha(k-1) \\
&\qquad\quad \times \left(\Sigma_{k-1}^T \Pi_{k-1} \Sigma_{k-1}\right)^{-1} \alpha^T(k-1)\Big)^{-1}\Big) \\
&\quad \times \alpha(k-1)\left(\Sigma_{k-1}^T \Pi_{k-1} \Sigma_{k-1}\right)^{-1} \\
&= -P^{-1}(k)\,\Omega^T(k) \\
&\quad \times \Big(\Phi^{-1}(k) + \alpha(k-1)\left(\Sigma_{k-1}^T \Pi_{k-1} \Sigma_{k-1}\right)^{-1} \\
&\qquad \times \alpha^T(k-1)\Big)^{-1} \\
&\quad \times \alpha(k-1)\left(\Sigma_{k-1}^T \Pi_{k-1} \Sigma_{k-1}\right)^{-1} \\
&= -P^{-1}(k)\,\Omega^T(k) \\
&\quad \times \left(\Phi^{-1}(k) + \Psi(k-1)\,P^{-1}(k-1)\,\Psi^T(k-1)\right)^{-1} \\
&\quad \times \alpha(k-1)\left(\Sigma_{k-1}^T \Pi_{k-1} \Sigma_{k-1}\right)^{-1}.
\end{aligned} \tag{31}$$

From the above analysis, we obtain

$$\begin{aligned}
&H_{11,k}\Omega^T(k)\,\Phi(k) + H_{12,k}\alpha^T(k-1)\,\Phi(k) \\
&\quad = P^{-1}(k)\,\Omega^T(k)\,\Phi(k) - P^{-1}(k)\,\Omega^T(k) \\
&\qquad \times \left(\Phi^{-1}(k) + \Psi(k-1)\,P^{-1}(k-1)\,\Psi^T(k-1)\right)^{-1} \\
&\qquad \times \alpha(k-1)\left(\Sigma_{k-1}^T \Pi_{k-1} \Sigma_{k-1}\right)^{-1} \alpha^T(k-1)\,\Phi(k) \\
&\quad = P^{-1}(k)\,\Omega^T(k) \\
&\qquad \times \Big(\Phi(k) \\
&\qquad\quad - \left(\Phi^{-1}(k) + \Psi(k-1)\,P^{-1}(k-1)\,\Psi^T(k-1)\right)^{-1} \\
&\qquad\quad \times \Psi(k-1)\,P^{-1}(k-1)\,\Psi^T(k-1)\,\Phi(k)\Big) \\
&\quad = P^{-1}(k)\,\Omega^T(k) \\
&\qquad \times \left(\Phi^{-1}(k) + \Psi(k-1)\,P^{-1}(k-1)\,\Psi^T(k-1)\right)^{-1}.
\end{aligned} \tag{32}$$

As such,

$$\begin{aligned}
\hat{\hat{x}}(k \mid k) &= P^{-1}(k)\,\Omega^T(k) \\
&\quad \times \left(\Phi^{-1}(k) + \Psi(k-1)\,P^{-1}(k-1)\,\Psi^T(k-1)\right)^{-1} \\
&\quad \times \left(\overline{y}(k) - \Psi(k-1)\,\hat{\hat{x}}(k-1 \mid k-1)\right).
\end{aligned} \tag{33}$$

Thus,

$$\begin{aligned}
&\hat{\hat{x}}(k \mid k) \\
&\quad = P^{-1}(k)\,\Omega^T(k) \\
&\qquad \times \left(\Phi^{-1}(k) + \Psi(k-1)\,P^{-1}(k-1)\,\Psi^T(k-1)\right)^{-1} \\
&\qquad \times \left(\overline{y}(k) - \Psi(k-1)\,\hat{\hat{x}}(k-1 \mid k-1)\right) \\
&\quad = P^{-1}(k) \begin{bmatrix} \overline{M}(k) \\ F(k) \\ \widetilde{C}(k) \\ L(k) \end{bmatrix}^T \\
&\qquad \times \left(\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & -\gamma^{-2} I \end{bmatrix}^{-1} + \begin{bmatrix} -\overline{A}(k-1) \\ 0 \\ 0 \\ 0 \end{bmatrix} \right. \\
&\qquad\qquad \left. \times P^{-1}(k-1) \begin{bmatrix} -\overline{A}(k-1) \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \right)^{-1} \\
&\qquad \times \left(\begin{bmatrix} 0 \\ 0 \\ y(k) \\ r(k) \end{bmatrix} - \begin{bmatrix} -\overline{A}(k-1) \\ 0 \\ 0 \\ 0 \end{bmatrix} \hat{\hat{x}}(k-1 \mid k-1) \right) \\
&\quad = P^{-1}(k) \begin{bmatrix} \overline{M}(k) \\ F(k) \\ \widetilde{C}(k) \\ L(k) \end{bmatrix}^T \\
&\qquad \times \begin{bmatrix} \left(I + \overline{A}(k-1)\,P^{-1}(k-1)\,\overline{A}^T(k-1)\right)^{-1} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & -\gamma^{-2} I \end{bmatrix} \\
&\qquad \times \begin{bmatrix} \overline{A}(k-1)\,\hat{\hat{x}}(k-1 \mid k-1) \\ 0 \\ y(k) \\ r(k) \end{bmatrix} \\
&\quad = P^{-1}(k)\,\overline{M}^T(k) \\
&\qquad \times \left(I + \overline{A}(k-1)\,P^{-1}(k-1)\,\overline{A}^T(k-1)\right)^{-1}
\end{aligned}$$

$$\times \overline{A}(k-1)\hat{\bar{x}}(k-1 \mid k-1) + P^{-1}(k)\widetilde{C}^T(k)y(k) - \gamma^{-2}P^{-1}(k)L^T(k)r(k). \tag{34}$$

Based on the above discussion, we present the main results of this paper.

Theorem 3. *Consider system (1), given a scalar $\gamma > 0$; then the H_∞ fault detection filter (8) that achieves (3) exists if, and only if, $P(k) > 0$ $(k = 0, 1, \ldots, k)$ and*

$$\begin{aligned} V_1(k) &= \left(P(k) + \gamma^{-2}L^T(k)L(k)\right)^{-1}\overline{M}^T(k) \\ &\quad \times \left(I + \overline{A}(k-1)P^{-1}(k-1)\overline{A}^T(k-1)\right)^{-1}\overline{A}(k-1) \\ V_2(k) &= \left(P(k) + \gamma^{-2}L^T(k)L(k)\right)^{-1}\widetilde{C}^T(k), \end{aligned} \tag{35}$$

where $P(k)$ is calculated by (26).

Proof. Note that J_k has a minimum over $\{\overline{x}(i)\}_{i=0}^k$ and $\{\overline{w}(i)\}_{i=0}^k$ if and only if $P(k) > 0$ $(k = 0, 1, \ldots, k)$, and the minimum is

$$\begin{aligned} \min_{\{x(i)\}_{i=0}^k} J_k &= \left(\overline{M}(0)\hat{\bar{x}}(0 \mid k) - \overline{A}(-1)\check{\bar{x}}(0)\right)^T \overline{P}_0^{-1} \\ &\quad \times \left(\overline{M}(0)\hat{\bar{x}}(0 \mid k) - \overline{A}(-1)\check{\bar{x}}(0)\right) \\ &\quad + \sum_{i=0}^{k-1}\left(\overline{M}(i+1)\hat{\bar{x}}(i+1 \mid k) - \overline{A}(i)\hat{\bar{x}}(i \mid k)\right)^T \\ &\quad \times \left(\overline{M}(i+1)\hat{\bar{x}}(i+1 \mid k) - \overline{A}(i)\hat{\bar{x}}(i \mid k)\right) \\ &\quad + \sum_{i=0}^{k}\left(y(i) - \widetilde{C}(i)\hat{\bar{x}}(i \mid k)\right)^T \\ &\quad \times \left(y(i) - \widetilde{C}(i)\hat{\bar{x}}(i \mid k)\right) \\ &\quad + \sum_{i=0}^{k}\left(F(i)\hat{\bar{x}}(i \mid k)\right)^T\left(F(i)\hat{\bar{x}}(i \mid k)\right) \\ &\quad - \gamma^{-2}\sum_{i=1}^{k}\left(r(i) - L(i)\hat{\bar{x}}(i \mid k)\right)^T \\ &\quad \times \left(r(i) - L(i)\hat{\bar{x}}(i \mid k)\right). \end{aligned} \tag{36}$$

It is readily seen that a positive minimum of J_k is guaranteed by setting $r(i) = L(i)\hat{\bar{x}}(i \mid k)$, $i = 0, 1, \ldots, k$. Furthermore, substituting $r(k) = L(k)\hat{\bar{x}}(k \mid k)$ into (34), we obtain

$$\begin{aligned} &\hat{\bar{x}}(k \mid k) \\ &\quad = \left(P(k) + \gamma^{-2}L^T(k)L(k)\right)^{-1} \\ &\quad \times \left(\overline{M}^T(k)\left(I + \overline{A}(k-1)P^{-1}(k-1)\overline{A}^T(k-1)\right)^{-1}\right. \\ &\quad \left.\times \overline{A}(k-1)\hat{\bar{x}}(k-1 \mid k-1) + \widetilde{C}^T(k)y(k)\right). \end{aligned} \tag{37}$$

In the light of the above equation, the parameter matrices of (8) can be given by (35). Hence, the theorem is proven. □

From Theorem 3 in this section, the H_∞ fault detection filter $r(k)$ can be computed in the following steps.

Step 1. Set $\gamma > 0$, $P_0 > 0$, $\check{x}(0) = 0$, and $k = 0$; calculate $\overline{P}_0$ and $\check{\bar{x}}(0)$ in (9).

Step 2. Calculate $P(0)$ and $\hat{\bar{x}}(0 \mid 0)$ using (26) and (28).

Step 3. If $P(0) > 0$, let $r(0) = L(0)\hat{\bar{x}}(0 \mid 0)$, and go to Step 4; otherwise, exit.

Step 4. Let $k = k + 1$; compute $P(k)$ using (26).

Step 5. If $P(k) > 0$, compute $V_1(k)$, $V_2(k)$, and $r(k)$ using (35) and (8), and go to Step 4; otherwise, exit.

Step 6. Repeat Steps 4 to 5 till $k = N$.

Remark 4. Note that the system augmentation has been applied to design the H_∞ fault detection filter (8); it may lead to more expensive computational cost. Fortunately, an H_∞ simultaneous state and unknown input estimator for descriptor system have been proposed in [27], and an H_∞ fixed-lag smoother for missing measurements system has been given in [28], so it is possible, in the future, to design a new H_∞ fault detection filter with lower computational cost by using the algorithm given in [27, 28].

4. Residual Evaluation

When the design of the H_∞ fault detection filter has been completed, the next task is residual evaluation. First, the following residual evaluation function and the threshold are introduced to facilitate fault detection:

$$\begin{aligned} J(r,k) &= \sum_{i=0}^{k} r^T(i)r(i), \\ J_{\text{th}}(k) &= \sup_{w(i),\, v(i)\in l_2,\, f(i)=0} E\left[J(r,k)\right]. \end{aligned} \tag{38}$$

The following strategy is applied for fault detection:

$$\begin{aligned} J(r,k) > J_{\text{th}}(k) &\Longrightarrow \text{There is fault} \Longrightarrow \text{alarm} \\ J(r,k) \le J_{\text{th}}(k) &\Longrightarrow \text{There is no fault.} \end{aligned} \tag{39}$$

If, for all $0 \le i \le N$, $f(i) \equiv 0$, the system (1) can be redescribed as follows:

$$\begin{aligned} M(i+1)x(i+1) &= A(i)x(i) + w(i), \\ y(i) &= \lambda(i)C(i)x(i) + v(i). \end{aligned} \tag{40}$$

Employing a similar technical line of Section 3 in this paper, we can obtain the residual signal of system (40). For a given scalar $\gamma_c > 0$, by employing performance index (3) and Theorem 3, we can judge whether the residual evaluation function with zero initial conditions achieves the following inequality:

$$J(r,k) < \gamma_c^2 E\left\{\sum_{i=0}^{k} w^T(i)\,w(i) + \sum_{i=0}^{k} v^T(i)\,v(i)\right\}. \quad (41)$$

Note that $w(i)$ and $v(i)$ are bounded signals, so there exist $m_1(k)$ and $m_2(k)$ such that

$$\sum_{i=0}^{k} w^T(i)\,w(i) \le m_1(k), \qquad \sum_{i=0}^{k} v^T(i)\,v(i) \le m_2(k). \quad (42)$$

Suppose that the minimum γ_c achieving (41) is $\gamma_{c\min}$; then we have

$$J(r,k) < \gamma_{c\min}^2\left(m_1(k) + m_2(k)\right). \quad (43)$$

Thus, the residual threshold can be defined as

$$J_{\text{th}}(k) = \gamma_{c\min}^2\left(m_1(k) + m_2(k)\right). \quad (44)$$

Finally, it can be judged based on the strategy given in (39).

5. A Numerical Example

Consider the discrete system (1) with the following parameters:

$$M(k) = \begin{bmatrix} 2.9 & 0 & 0 \\ 0 & 2.5 + 0.3\sin(k) & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$A(k) = \begin{bmatrix} 2.1 & e^{-k} & 0 \\ 0 & 1.6 & 0 \\ 0 & 0 & -1.9 \end{bmatrix},$$
$$B_f(k) = \begin{bmatrix} 3.2 \\ 2.7 \\ 4.5 \end{bmatrix},$$
$$C(k) = \begin{bmatrix} 2.6 & 0 & 1.8 \end{bmatrix}. \quad (45)$$

Set $\gamma = 0.25$ and $P_0 = I$. The unknown signals $w(k)$, $v(k)$, and $f(k)$ are supposed to be

$$w(k) = \begin{bmatrix} 0.3\sin(k) \\ 0.2\cos(3k) \\ 0.4\cos(2k) \end{bmatrix},$$
$$f(k) = \begin{cases} 0 & 0 \le k < 40, \\ 1, & 40 \le k \le 60, \\ 0, & k > 60. \end{cases} \quad v(k) = 0.5\sin(4k), \quad (46)$$

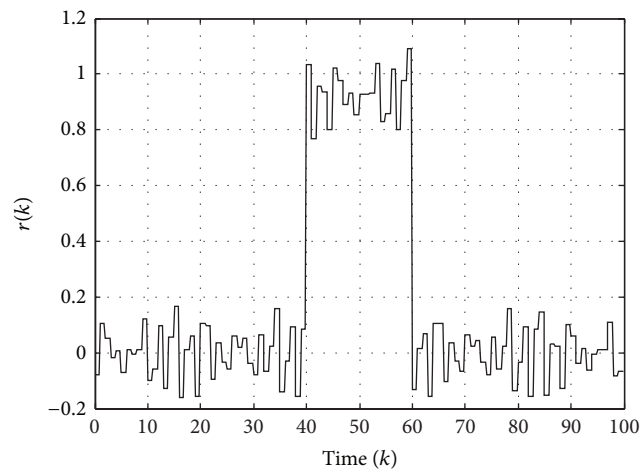

Figure 1: Residual signal $r(k)$ when $\rho = 1$ and $D_f(k) = 1.8$.

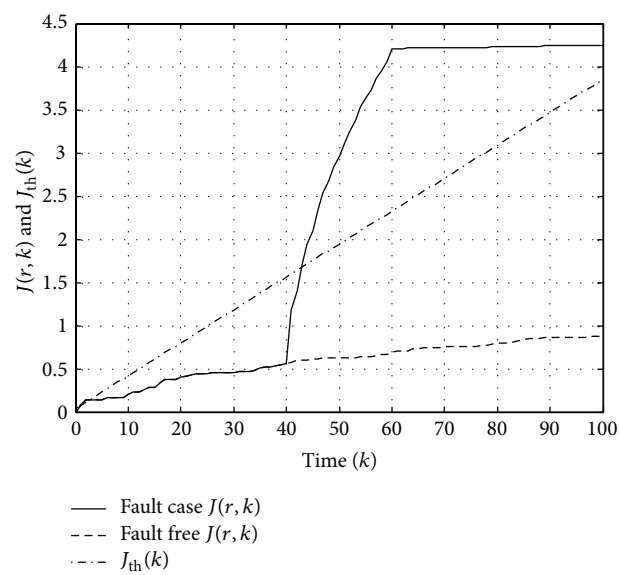

Figure 2: Residual evaluation function $J(r,k)$ and threshold $J_{\text{th}}(k)$ when $\rho = 1$ and $D_f(k) = 1.8$.

By applying Theorem 3, the H_∞ fault detection filter is designed. Figure 1 shows the residual signal $r(k)$ when $\rho = 1$ and $D_f(k) = 1.8$, and Figure 2 shows the residual evaluation function $J(r,k)$ and threshold $J_{\text{th}}(k)$ when $\rho = 1$ and $D_f(k) = 1.8$. Figure 3 shows the residual signal $r(k)$ when $\rho = 1$ and $D_f(k) = 0$, and Figure 4 shows the residual evaluation function $J(r,k)$ and threshold $J_{\text{th}}(k)$ when $\rho = 1$ and $D_f(k) = 0$. It is shown that the tracking performance of H_∞ fault detection filtering is good in the above two cases. Figure 5 shows the variation law of random parameter $\lambda(k)$ when $\rho = 0.9$. Figure 6 shows the residual signal $r(k)$ when $\rho = 0.9$ and $D_f(k) = 1.8$, and Figure 7 shows the residual evaluation function $J(r,k)$ and threshold $J_{\text{th}}(k)$ when $\rho = 0.9$ and $D_f(k) = 1.8$. Figure 8 shows the residual signal $r(k)$ when $\rho = 0.9$ and $D_f(k) = 0$, and Figure 9 shows the residual evaluation function $J(r,k)$ and threshold $J_{\text{th}}(k)$ when $\rho = 0.9$

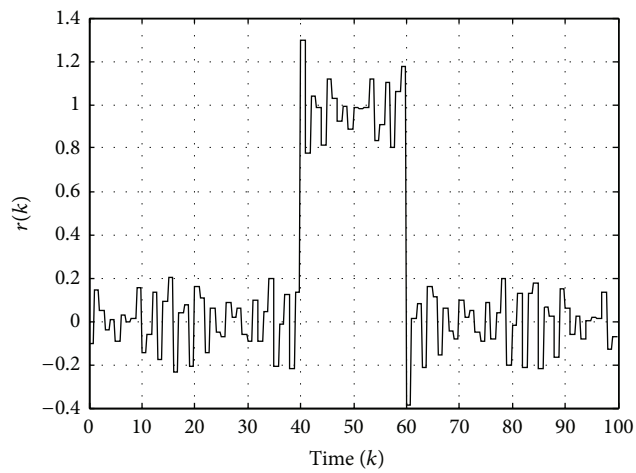

FIGURE 3: Residual signal $r(k)$ when $\rho = 1$ and $D_f(k) = 0$.

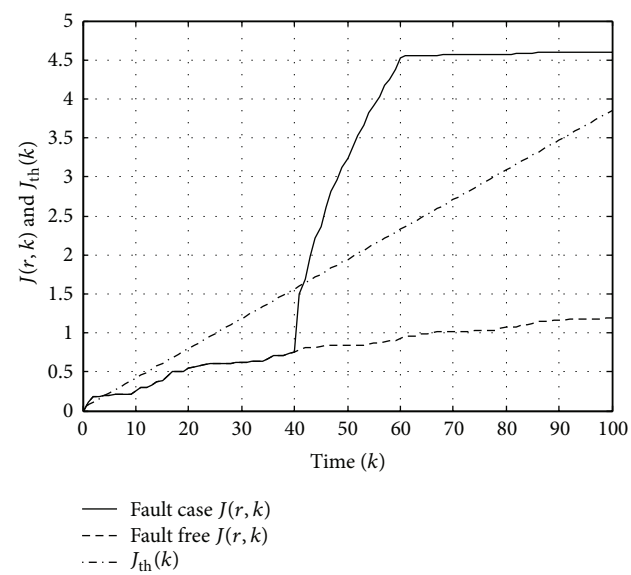

FIGURE 4: Residual evaluation function $J(r,k)$ and threshold $J_{\text{th}}(k)$ when $\rho = 1$ and $D_f(k) = 0$.

and $D_f(k) = 0$. It is shown that the tracking performance of H_∞ fault detection filtering is weakening when the system has missing measurements, but the threshold $J_{\text{th}}(k)$ in Figures 7 and 9 has good performance.

6. Conclusions

In this paper, the Bernoulli random binary switching sequence has been applied to describe missing measurements, and it has been used as a basis on the study of the H_∞ fault detection problem for linear discrete time-varying descriptor systems. The main contribution of this paper is to develop a novel approach on solving the H_∞ fault detection problem for time-varying descriptor systems with missing measurements. An augmented system has been first obtained, and a relationship has been established between the H_∞ fault

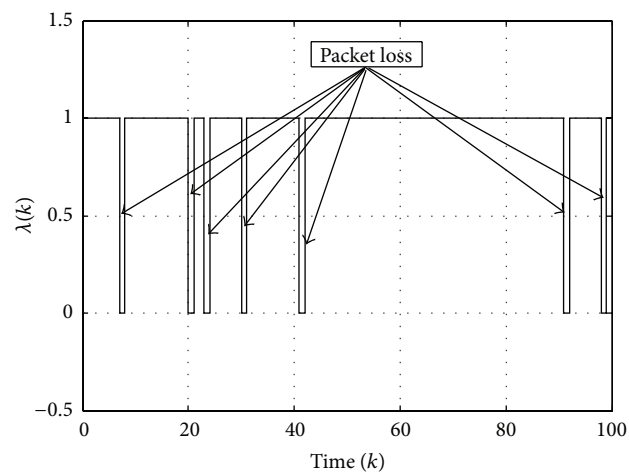

FIGURE 5: Random parameters $\lambda(k)$ when $\rho = 0.9$.

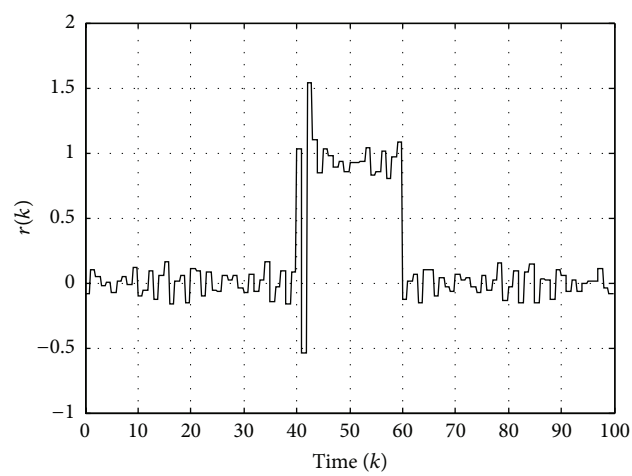

FIGURE 6: Residual signal $r(k)$ when $\rho = 0.9$ and $D_f(k) = 1.8$.

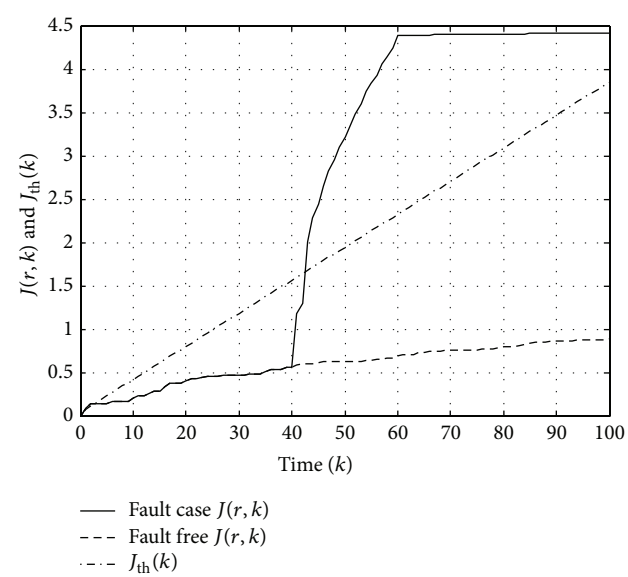

FIGURE 7: Residual evaluation function $J(r,k)$ and threshold $J_{\text{th}}(k)$ when $\rho = 0.9$ and $D_f(k) = 1.8$.

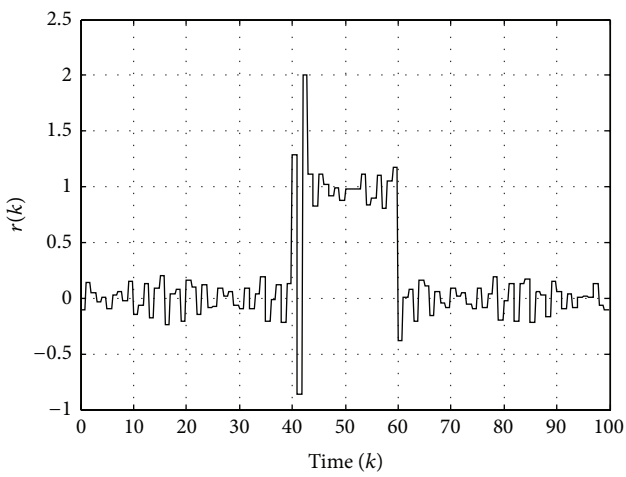

Figure 8: Residual signal $r(k)$ when $\rho = 0.9$ and $D_f(k) = 0$.

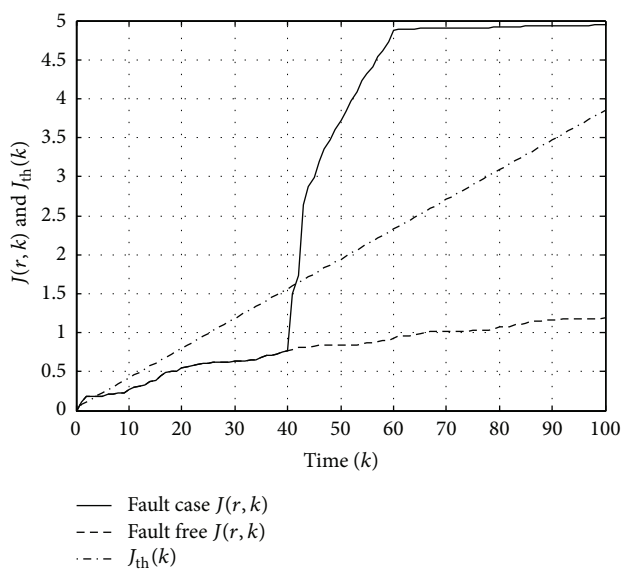

Figure 9: Residual evaluation function $J(r,k)$ and threshold $J_{\text{th}}(k)$ when $\rho = 0.9$ and $D_f(k) = 0$.

detection problem and a certain indefinite quadratic form problem. By applying matrix analysis method, a necessary and sufficient existence condition and the explicit formula of the H_∞ fault detection filter have been derived. The residual evaluation function and the threshold have been designed to facilitate fault detection. Finally, the proposed algorithm has been proved to be effective by a numerical example.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work was supported in part by the National Natural Science Foundation of China (nos. 61403061, 60774004, and 61273097), the Science and Technology Development Project of Dezhou, China (no. 2012B05), the Talent Introduction Project of Dezhou University, China (no. 311432), and the Solar Energy Special Project of Dezhou University, China (no. 14ZX03). The authors would like to thank the editor and reviewers for valuable comments and constructive suggestions aimed at improving the quality of this paper.

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