Research Article

A Delay Almost Periodic Competitive System in Discrete Time

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A discrete almost periodic competitive system with delay is proposed and analyzed. The system admits a unique almost periodic solution, which is shown to be uniformly asymptotically stable by using the method of Lyapunov function. Some specific numerical examples are provided to verify our analytical results.

1. Introduction

As we know, a discrete time model governed by difference equations is more suitable than a corresponding continuous version when the populations have a short-life expectancy and nonoverlapping generations and can also provide efficient computation for numerical simulation (see [1–7]). On the other hand, the coefficients of model are changed owing to environmental variation. The assumption of almost periodicity of the coefficients is a way of incorporating the time-dependent variability of the environment if the various components of the environment are with incommensurable periods. One of the important ecological problems associated with the investigation of populations interaction in an almost periodic environment is the positive almost periodic solution which plays the significant role played by the equilibrium of the autonomous model (see [8–14]). In this contribution, we discuss the positive almost periodic solutions of a delay almost periodic competitive system in discrete time, and our motivation comes from the works of [8, 9, 15].

Firstly, let us introduce the following autonomous differential model which was proposed by Ayala et al. [15]:

$$\begin{aligned}x_1'(t) &= x_1(t) [r_1 - x_1(t) - a_1 x_2(t) - c_1 x_2^2(t)], \\x_2'(t) &= x_2(t) [r_2 - x_2(t) - a_2 x_1(t) - c_2 x_1^2(t)],\end{aligned}\quad (1)$$

where x_1 and x_2 stand for the population densities of two competing species. r_1 and r_2 are the intrinsic growth rates.

a_i and c_i represent the interspecific competitive effects, $i = 1, 2$. Assume that a species needs some time to mature and the competition occurs after some time lag required for maturity of the species; a revised version was introduced by Gopalsamy [16]

$$\begin{aligned}x_1'(t) &= x_1(t) [r_1 - x_1(t) - a_1 x_2(t - \tau_2) - c_1 x_2^2(t - \tau_2)], \\x_2'(t) &= x_2(t) [r_2 - x_2(t) - a_2 x_1(t - \tau_1) - c_2 x_1^2(t - \tau_1)].\end{aligned}\quad (2)$$

Furthermore, considering the biological parameters naturally being subject to almost periodic fluctuation in time and the influence of many generations on the density of species population, we establish the following model:

$$\begin{aligned}x_1(n+1) &= x_1(n) \exp [r_1(n) - x_1(n) \\&\quad - a_1(n) x_2(n - \tau_2) - c_1(n) x_2^2(n - \tau_2)], \\x_2(n+1) &= x_2(n) \exp [r_2(n) - x_2(n) \\&\quad - a_2(n) x_1(n - \tau_1) - c_2(n) x_1^2(n - \tau_1)],\end{aligned}\quad (3)$$

$n = 0, 1, 2, \dots$, and the initial conditions satisfy

$$\begin{aligned} x_1(\theta) = \phi_1(\theta) &\geq 0, & x_2(\theta) = \phi_2(\theta) &\geq 0, \\ \phi_1(0) &> 0, & \phi_2(0) &> 0, \\ \theta &\in \{-\tau, -\tau+1, \dots, 0\}, & \tau &= \max\{\tau_i, i = 1, 2\}. \end{aligned} \quad (4)$$

Here $x_i(n)$ and $r_i(n)$ are, respectively, the densities and intrinsic growth rates of species x_i at the n th generation. $a_i(n)$ and $c_i(n)$ measure the interspecific influence of the $(n - \tau_j)$ th generation of species x_j on species x_i ($i, j = 1, 2; i \neq j$). The delays τ_1 and τ_2 are positive integers and the coefficients $\{r_i(n)\}$, $\{a_i(n)\}$, and $\{c_i(n)\}$ are bounded positive almost periodic sequences, $i = 1, 2$.

The rest part of this paper is organized as follows. In Section 2, some preliminaries are given. Sufficient conditions for the uniformly asymptotic stability of a unique positive almost periodic solution for the system are established in Section 3. In final section, some specific numerical examples are carried out to illustrate the feasibilities of our theoretical results.

2. Preliminaries

This section is concerned with some notations, definitions, and lemmas which will be used for our main results.

Let R , R^+ , Z , and Z^+ be, respectively, the sets of real numbers, nonnegative real numbers, integers, and nonnegative integers. Assign R^2 and R^k which denote the cone of 2-dimensional and k -dimensional real Euclidean space, respectively. Denote $[a, b]_Z = [a, b] \cap Z$, $a, b \in Z$, and then $K = [-\tau, +\infty)_Z$, where τ is defined in (4). For simplicity in the following discussion, we use the notations $\mathcal{A}^U = \sup_{n \in Z^+} \{\mathcal{A}(n)\}$, $\mathcal{A}^L = \inf_{n \in Z^+} \{\mathcal{A}(n)\}$, where $\{\mathcal{A}(n)\}$ is an almost periodic sequence.

Definition 1 (see [12]). A sequence $x : Z \rightarrow R^k$ is called an almost periodic sequence if the ε -translation set of x

$$E\{\varepsilon, x\} := \{\tau \in Z : |x(n + \tau) - x(n)| < \varepsilon, \forall n \in Z\} \quad (5)$$

is a relatively dense set in Z for all $\varepsilon > 0$, that is, for any given $\varepsilon > 0$, there exists an integer $l(\varepsilon) > 0$ such that each discrete interval of length $l(\varepsilon)$ contains an integer $\tau = \tau(\varepsilon) \in E\{\varepsilon, x\}$ such that $|x(n + \tau) - x(n)| < \varepsilon, \forall n \in Z$. τ is called the ε -translation number of $x(n)$.

Definition 2 (see [12]). Let $f : Z \times D \rightarrow R^k$, where D is an open set in $C : \{\varphi : [-\tau, 0]_Z \rightarrow R^k\}$. $f(n, \varphi)$ is said to be almost periodic in n uniformly for $\varphi \in D$, if for any $\varepsilon > 0$ and any compact set S in D , there exists a positive integer $l(\varepsilon, S)$ such that any interval of length $l(\varepsilon, S)$ contains an integer τ for which

$$|f(n + \tau, \varphi) - f(n, \varphi)| < \varepsilon, \quad \forall n \in Z, \varphi \in S, \quad (6)$$

and τ is called the ε -translation number of $f(n, \varphi)$.

Lemma 3 (see [17]). $\{x(n)\}$ is an almost periodic sequence if and only if for any sequence $\{t'_k\} \subset Z$ there exists a subsequence

$\{t_k\} \subset \{t'_k\}$ such that $x(n + t_k)$ converges uniformly on $n \in Z$ as $k \rightarrow +\infty$.

Zhang and Zheng [12] considered the following almost periodic delay difference system:

$$x(n + 1) = f(n, x_n), \quad n \in Z^+, \quad (7)$$

where $f : Z^+ \times C_B \rightarrow R^k$, $C_B = \{\varphi \in C : \|\varphi\| < B\}$, $C = \{\varphi : [-\tau, 0]_Z \rightarrow R^k\}$ with $\|\varphi\| = \sup_{s \in [-\tau, 0]_Z} |\varphi(s)|$, $f(n, \varphi)$ is almost periodic in n uniformly for $\varphi \in C_B$ and is continuous in φ , while $x_n \in C_B$ is defined as $x_n(s) = x(n + s)$ for all $s \in [-\tau, 0]_Z$. The product system of (7) is as follows:

$$x(n + 1) = f(n, x_n), \quad y(n + 1) = f(n, y_n). \quad (8)$$

A discrete Lyapunov function of (8) is a function $V : Z^+ \times C_B \times C_B \rightarrow R^+$ which is continuous in its second and third variables. Define the difference of V along the solution of system (8) by

$$\begin{aligned} \Delta V_{(8)}(n, \varphi, \psi) \\ = V(n + 1, x_{n+1}(n, \varphi), y_{n+1}(n, \psi)) - V(n, \varphi, \psi), \end{aligned} \quad (9)$$

where $(x(n, \varphi), y(n, \psi))$ is a solution of system (8) through $(n, (\varphi, \psi))$, $\varphi, \psi \in C_B$.

Lemma 4 (see [12]). Suppose that there exists a Lyapunov function $V(n, \varphi, \psi)$ satisfying the following conditions.

- (I) $a(|\varphi(0) - \psi(0)|) \leq V(n, \varphi, \psi) \leq b(\|\varphi - \psi\|)$, where $a, b \in P$ with $P = \{a : [0, +\infty) \rightarrow [0, +\infty) \mid a(0) = 0 \text{ and } a(u) \text{ is continuous, increasing in } u\}$.
- (II) $|V(n, \varphi_1, \psi_1) - V(n, \varphi_2, \psi_2)| \leq L(\|\varphi_1 - \varphi_2\| + \|\psi_1 - \psi_2\|)$, where $L > 0$ is a constant.
- (III) $\Delta V(n, \varphi, \psi) \leq -\alpha V(n, \varphi, \psi)$, where $0 < \alpha < 1$ is a constant.

Moreover, if there exists a solution $x(n)$ of system (7) such that $\|x_n\| \leq B^* \leq B$ for all $n \in Z^+$, then there exists a unique uniformly asymptotically stable almost periodic solution $q(n)$ of system (7) which satisfies $|q(n)| \leq B^*$ for all $n \in K$. In particular, if $f(n, \varphi)$ is periodic of period ω , then system (7) has a unique uniformly asymptotically stable periodic solution of period ω .

Remark 5 (see [9]). Condition (III) of Lemma 4 can be replaced by

(III)' $\Delta V(n, \varphi, \psi) \leq -\beta(|\varphi(0) - \psi(0)|)$, where $\beta \in \{c : [0, +\infty) \rightarrow [0, +\infty) \mid c \text{ is continuous, } c(0) = 0 \text{ and } c(s) > 0 \text{ for } s > 0\}$.

Lemma 6 (see [18]). Assume that $x(n) > 0$ and

$$\begin{aligned} x(n + 1) &\leq x(n) \exp(r(n)(1 - ax(n))) \\ &\text{for } n \in [n_1, +\infty), \end{aligned} \quad (10)$$

where $r(n)$ is a bounded positive sequence and a is a positive constant. Then

$$\limsup_{n \rightarrow +\infty} x(n) \leq \frac{1}{ar^U} \exp(r^U - 1). \quad (11)$$

Lemma 7 (see [18]). Assume that $\{x(n)\}$ satisfies

$$x(n+1) \geq x(n) \exp(r(n)(1-ax(n))), \quad n \geq N_0, \quad (12)$$

$\limsup_{n \rightarrow +\infty} x(n) \leq \mathcal{M}$ and $x(N_0) > 0$, where $r(n)$ is a bounded positive sequence and a is a positive constant such that $a\mathcal{M} > 1$ and $N_0 \in \mathbb{Z}^+$. Then

$$\liminf_{n \rightarrow +\infty} x(n) \geq \frac{1}{a} \exp(r^U(1-a\mathcal{M})). \quad (13)$$

Denote

$$H_i = \exp(r_i^U - 1), \quad \Delta_i = r_i^L - a_i^U H_j - c_i^U H_j^2, \quad (14)$$

$$h_i = \Delta_i \exp(\Delta_i - H_i), \quad i, j = 1, 2; i \neq j.$$

Lemma 8. If $\Delta_i > 0$, then any positive solution $(x_1(n), x_2(n))$ of system (3) satisfies $h_i \leq \liminf_{n \rightarrow +\infty} x_i(n) \leq \limsup_{n \rightarrow +\infty} x_i(n) \leq H_i$, $i = 1, 2$.

Proof. According to the first equation of system (3), one has

$$\begin{aligned} x_1(n+1) &\leq x_1(n) \exp[r_1(n) - x_1(n)] \\ &\leq x_1(n) \left[r_1(n) \left(1 - \frac{1}{r_1^U} x_1(n) \right) \right]. \end{aligned} \quad (15)$$

It follows from Lemma 6 that

$$\limsup_{n \rightarrow +\infty} x_1(n) \leq \exp(r_1^U - 1) = H_1. \quad (16)$$

Analogously, from the second equation of system (3) we obtain that

$$\limsup_{n \rightarrow +\infty} x_2(n) \leq \exp(r_2^U - 1) = H_2. \quad (17)$$

Assigning a positive constant ε arbitrarily small, it follows from (16) and (17) that there exists a large enough $n_1 > 0$ such that for all $n \geq n_1$,

$$x_i(n) \leq H_i + \varepsilon, \quad i = 1, 2. \quad (18)$$

From the first equation of system (3), for $n \geq n_1 + \tau$,

$$\begin{aligned} x_1(n+1) &\geq x_1(n) \exp[r_1^L - x_1(n) - a_1^U(H_2 + \varepsilon) - c_1^U(H_2 + \varepsilon)^2] \\ &= x_1(n) \exp\left[(r_1^L - a_1^U(H_2 + \varepsilon) - c_1^U(H_2 + \varepsilon)^2) \right. \\ &\quad \left. \times \left(1 - \frac{x_1(n)}{r_1^L - a_1^U(H_2 + \varepsilon) - c_1^U(H_2 + \varepsilon)^2} \right) \right]. \end{aligned} \quad (19)$$

Using Lemma 7, for $n \geq n_1 + \tau$, one has

$$\begin{aligned} \liminf_{n \rightarrow +\infty} x_1(n) &\geq (r_1^L - a_1^U(H_2 + \varepsilon) - c_1^U(H_2 + \varepsilon)^2) \\ &\quad \times \exp[r_1^L - a_1^U(H_2 + \varepsilon) - c_1^U(H_2 + \varepsilon)^2 - H_1]. \end{aligned} \quad (20)$$

Setting $\varepsilon \rightarrow 0$, it follows that

$$\begin{aligned} \liminf_{n \rightarrow +\infty} x_1(n) &\geq (r_1^L - a_1^U H_2 - c_1^U H_2^2) \\ &\quad \times \exp[r_1^L - a_1^U H_2 - c_1^U H_2^2 - H_1] = h_1. \end{aligned} \quad (21)$$

Similar to the above argument, from the second equation of system (3) we can obtain that

$$\begin{aligned} \liminf_{n \rightarrow +\infty} x_2(n) &\geq (r_2^L - a_2^U H_1 - c_2^U H_1^2) \\ &\quad \times \exp[r_2^L - a_2^U H_1 - c_2^U H_1^2 - H_2] = h_2. \end{aligned} \quad (22)$$

This completes the proof. $\square$

Denote Θ by

$$\begin{aligned} \Theta &= \{\text{every solution of system (3) satisfying} \\ &\quad h_i \leq x_i(n) \leq H_i, \quad i = 1, 2 \quad \forall n \in \mathbb{K}\}. \end{aligned} \quad (23)$$

Lemma 9. If $\Delta_i > 0$, then $\Theta \neq \emptyset$, where Δ_i are defined in (14).

Proof. Since $\{r_i(n)\}$, $\{a_i(n)\}$, and $\{c_i(n)\}$, $i = 1, 2$ are almost periodic sequences, there exists a positive integer sequence $\{t_k\}$ with $t_k \rightarrow +\infty$ as $k \rightarrow +\infty$ such that

$$\begin{aligned} r_i(n+t_k) &\rightarrow r_i(n), \quad a_i(n+t_k) \rightarrow a_i(n), \\ c_i(n+t_k) &\rightarrow c_i(n), \quad i = 1, 2, \end{aligned} \quad (24)$$

as $k \rightarrow +\infty$ for $n \in \mathbb{Z}^+$. By (16), (17), (21), and (22), for any sufficiently small $\varepsilon > 0$, there exists a positive integer n_0 such that for all $n > n_0$,

$$h_i - \varepsilon \leq x_i(n) \leq H_i + \varepsilon. \quad (25)$$

Assign

$$x_{ik}(n) = x_i(n+t_k) \quad \text{for } n \geq n_0 + \tau - t_k, \quad k = 1, 2, \dots \quad (26)$$

For any positive integer q , it is obvious that there exists a sequence $\{x_{ik}(n) : k \geq q\}$ such that the sequence $\{x_{ik}(n)\}$ has a subsequence, also denoted by $\{x_{ik}(n)\}$, converging on any definite interval of K as $k \rightarrow +\infty$. Thus, there is a sequence $\{w_i(n)\}$ satisfying

$$x_{ik}(n) \rightarrow w_i(n), \quad \text{for } n \in K \text{ as } k \rightarrow +\infty, \quad (27)$$

which, together with (24), yields that

$$\begin{aligned} x_{1k}(n+1) &= x_{1k}(n) \exp[r_1(n+t_k) - x_{1k}(n) \\ &\quad - a_1(n+t_k)x_{2k}(n-\tau_2) \\ &\quad - c_1(n+t_k)x_{2k}^2(n-\tau_2)], \\ x_{2k}(n+1) &= x_{2k}(n) \exp[r_2(n+t_k) - x_{2k}(n) \\ &\quad - a_2(n+t_k)x_{1k}(n-\tau_1) \\ &\quad - c_2(n+t_k)x_{1k}^2(n-\tau_1)], \end{aligned} \quad (28)$$

we have from (24), (27), and (28) that

$$\begin{aligned} w_1(n+1) &= w_1(n) \exp \left[r_1(n) - w_1(n) \right. \\ &\quad \left. - a_1(n) w_2(n - \tau_2) \right. \\ &\quad \left. - c_1(n) w_2^2(n - \tau_2) \right], \\ w_2(n+1) &= w_2(n) \exp \left[r_2(n) - w_2(n) \right. \\ &\quad \left. - a_2(n) w_1(n - \tau_1) \right. \\ &\quad \left. - c_2(n) w_1^2(n - \tau_1) \right]. \end{aligned} \quad (29)$$

It is easy to see that $(w_1(n), w_2(n))$ is a solution of system (3) and $h_i - \varepsilon \leq w_i(n) \leq H_i + \varepsilon$ for $n \in K$. Since ε is small enough, we derive that $h_i \leq w_i(n) \leq H_i$, $i = 1, 2$ for $n \in K$. This completes the proof. $\square$

3. Main Result

In this section, we focus on the result of the uniformly asymptotic stability of positive almost periodic solutions of system (3).

Denote

$$\begin{aligned} \lambda_i &= 1 - \max(|1 - h_i|, |1 - H_i|) \\ &\quad - (a_j^U H_i + 2c_j^U H_i^2), \quad i, j = 1, 2; \quad i \neq j. \end{aligned} \quad (30)$$

Theorem 10. *If $\Delta_i > 0$ and $\lambda_i > 0$, system (3) has a unique positive almost periodic solution which is uniformly asymptotically stable, where Δ_i are defined in (14).*

Proof. We make the change of variables

$$y_1(n) = \ln x_1(n), \quad y_2(n) = \ln x_2(n), \quad (31)$$

and then system (3) can be rewritten as

$$\begin{aligned} y_1(n+1) &= y_1(n) + r_1(n) - e^{y_1(n)} \\ &\quad - a_1(n) e^{y_2(n-\tau_2)} - c_1(n) e^{2y_2(n-\tau_2)}, \\ y_2(n+1) &= y_2(n) + r_2(n) - e^{y_2(n)} \\ &\quad - a_2(n) e^{y_1(n-\tau_1)} - c_2(n) e^{2y_1(n-\tau_1)}. \end{aligned} \quad (32)$$

It follows from Lemma 9 that there exists a bounded solution $(y_1(n), y_2(n))$ of system (32) satisfying

$$\begin{aligned} \ln h_1 \leq y_1(n) \leq \ln H_1, \quad \ln h_2 \leq y_2(n) \leq \ln H_2, \\ \forall n \in K, \end{aligned} \quad (33)$$

which implies that $|y_1(n)| \leq B_1$, $|y_2(n)| \leq B_2$, where $B_i = \max\{|\ln h_i|, |\ln H_i|\}$, $i = 1, 2$. Assign

$$\begin{aligned} Y_n(s) &= (y_1(n+s), y_2(n+s)), \\ Z_n(s) &= (z_1(n+s), z_2(n+s)), \\ n \in \mathbb{Z}^+, \quad s \in [-\tau, 0]_{\mathbb{Z}} \end{aligned} \quad (34)$$

are two solutions of system (32) defined on D , where

$$\begin{aligned} D &= \{(y_1(n), y_2(n)) \mid \\ &\quad \ln h_i \leq y_i(n) \leq \ln H_i, \quad i = 1, 2, \quad n \in K\}. \end{aligned} \quad (35)$$

Defining

$$\begin{aligned} \|Y_n(s)\| &= \|(y_1(n+s), y_2(n+s))\| \\ &= \sup_{s \in [-\tau, 0]_{\mathbb{Z}}} \{|y_1(n+s)| + |y_2(n+s)|\}, \end{aligned} \quad (36)$$

where $(y_1(n+s), y_2(n+s)) \in \mathbb{R}^2$, we have

$$\|Y_n\| \leq A, \quad \|Z_n\| \leq A, \quad (37)$$

where $A = B_1 + B_2$.

Let us consider the associate product system of system (32)

$$\begin{aligned} y_1(n+1) &= y_1(n) + r_1(n) - e^{y_1(n)} \\ &\quad - a_1(n) e^{y_2(n-\tau_2)} - c_1(n) e^{2y_2(n-\tau_2)}, \\ y_2(n+1) &= y_2(n) + r_2(n) - e^{y_2(n)} \\ &\quad - a_2(n) e^{y_1(n-\tau_1)} - c_2(n) e^{2y_1(n-\tau_1)}, \\ z_1(n+1) &= z_1(n) + r_1(n) - e^{z_1(n)} \\ &\quad - a_1(n) e^{z_2(n-\tau_2)} - c_1(n) e^{2z_2(n-\tau_2)}, \\ z_2(n+1) &= z_2(n) + r_2(n) - e^{z_2(n)} \\ &\quad - a_2(n) e^{z_1(n-\tau_1)} - c_2(n) e^{2z_1(n-\tau_1)}. \end{aligned} \quad (38)$$

Construct the following Lyapunov function $V(n) = V(n, Y_n, Z_n)$ defined on $\mathbb{Z}^+ \times D \times D$

$$\begin{aligned} V(n) &= V(n, Y_n, Z_n) = |y_1(n) - z_1(n)| + |y_2(n) - z_2(n)| \\ &\quad + \sum_{\zeta=n-\tau_1}^{n-1} (a_2^U H_1 + 2c_2^U H_1^2) |y_1(\zeta) - z_1(\zeta)| \\ &\quad + \sum_{\zeta=n-\tau_2}^{n-1} (a_1^U H_2 + 2c_1^U H_2^2) |y_2(\zeta) - z_2(\zeta)|. \end{aligned} \quad (39)$$

Apparently,

$$\begin{aligned}
& |Y_n(0) - Z_n(0)| \\
& \leq V(n) \leq |y_1(n) - z_1(n)| + |y_2(n) - z_2(n)| \\
& \quad + \sum_{\zeta=n-\tau}^{n-1} (a_2^U H_1 + 2c_2^U H_1^2) |y_1(\zeta) - z_1(\zeta)| \\
& \quad + \sum_{\zeta=n-\tau}^{n-1} (a_1^U H_2 + 2c_1^U H_2^2) |y_2(\zeta) - z_2(\zeta)| \\
& \leq |y_1(n) - z_1(n)| + |y_2(n) - z_2(n)| \\
& \quad + \sum_{\zeta=n-\tau}^{n-1} \sigma \{|y_1(\zeta) - z_1(\zeta)| + |y_2(\zeta) - z_2(\zeta)|\} \\
& \leq (1 + \tau\sigma) \sup_{s \in [-\tau, 0]_{\mathbb{Z}}} \{|y_1(n+s) - z_1(n+s)| \\
& \quad + |y_2(n+s) - z_2(n+s)|\} \\
& = \rho \|Y_n - Z_n\|,
\end{aligned} \tag{40}$$

where

$$\begin{aligned}
& |Y_n(0) - Z_n(0)| \\
& = \left[(y_1(n) - z_1(n))^2 + (y_2(n) - z_2(n))^2 \right]^{1/2} \\
& \leq |y_1(n) - z_1(n)| + |y_2(n) - z_2(n)|,
\end{aligned} \tag{41}$$

$$\rho = 1 + \tau\sigma, \quad \sigma = \max \{a_2^U H_1 + 2c_2^U H_1^2, a_1^U H_2 + 2c_1^U H_2^2\}. \tag{42}$$

Denote

$$a(x) = x, \quad b = \rho x, \quad a, b \in C(R^+, R^+), \tag{43}$$

and thus condition (I) in Lemma 4 is satisfied.

For any $y, z, \bar{y}, \bar{z}$, we obtain that

$$\begin{aligned}
& ||y - z| - |\bar{y} - \bar{z}|| \\
& = \begin{cases} |y - z| - |\bar{y} - \bar{z}|, & \text{if } |y - z| \geq |\bar{y} - \bar{z}|, \\ |\bar{y} - \bar{z}| - |y - z|, & \text{if } |\bar{y} - \bar{z}| > |y - z| \end{cases} \\
& \leq \begin{cases} |(y - \bar{y}) + (\bar{z} - z)|, & \text{if } |y - z| \geq |\bar{y} - \bar{z}|, \\ |(\bar{y} - y) + (z - \bar{z})|, & \text{if } |\bar{y} - \bar{z}| > |y - z| \end{cases} \\
& \leq \begin{cases} |y - \bar{y}| + |\bar{z} - z|, & \text{if } |y - z| \geq |\bar{y} - \bar{z}|, \\ |\bar{y} - y| + |z - \bar{z}|, & \text{if } |\bar{y} - \bar{z}| > |y - z| \end{cases} \\
& = |y - \bar{y}| + |z - \bar{z}|.
\end{aligned} \tag{44}$$

Hence, for any $Y_n, Z_n, \bar{Y}_n, \bar{Z}_n \in D$, we obtain, by (44), that

$$\begin{aligned}
& |V(n, Y_n, Z_n) - V(n, \bar{Y}_n, \bar{Z}_n)| \\
& = |y_1(n) - z_1(n)| + |y_2(n) - z_2(n)| \\
& \quad + \sum_{\zeta=n-\tau_1}^{n-1} (a_2^U H_1 + 2c_2^U H_1^2) |y_1(\zeta) - z_1(\zeta)| \\
& \quad + \sum_{\zeta=n-\tau_2}^{n-1} (a_1^U H_2 + 2c_1^U H_2^2) |y_2(\zeta) - z_2(\zeta)| \\
& \quad - |\bar{y}_1(n) - \bar{z}_1(n)| - |\bar{y}_2(n) - \bar{z}_2(n)| \\
& \quad - \sum_{\zeta=n-\tau_1}^{n-1} (a_2^U H_1 + 2c_2^U H_1^2) |\bar{y}_1(\zeta) - \bar{z}_1(\zeta)| \\
& \quad - \sum_{\zeta=n-\tau_2}^{n-1} (a_1^U H_2 + 2c_1^U H_2^2) |\bar{y}_2(\zeta) - \bar{z}_2(\zeta)| \\
& \leq ||y_1(n) - z_1(n)| + |y_2(n) - z_2(n)| \\
& \quad - |\bar{y}_1(n) - \bar{z}_1(n)| - |\bar{y}_2(n) - \bar{z}_2(n)|| \\
& \quad + \sum_{\zeta=n-\tau_1}^{n-1} (a_2^U H_1 + 2c_2^U H_1^2) \\
& \quad \quad \times ||y_1(\zeta) - z_1(\zeta)| - |\bar{y}_1(\zeta) - \bar{z}_1(\zeta)|| \\
& \quad + \sum_{\zeta=n-\tau_2}^{n-1} (a_1^U H_2 + 2c_1^U H_2^2) \\
& \quad \quad \times ||y_2(\zeta) - z_2(\zeta)| - |\bar{y}_2(\zeta) - \bar{z}_2(\zeta)|| \\
& \leq \{|y_1(n) - \bar{y}_1(n)| + |y_2(n) - \bar{y}_2(n)| \\
& \quad + |z_1(n) - \bar{z}_1(n)| + |z_2(n) - \bar{z}_2(n)|\} \\
& \quad + \sum_{\zeta=n-\tau}^{n-1} (a_2^U H_1 + 2c_2^U H_1^2) \\
& \quad \quad \times \{|y_1(\zeta) - \bar{y}_1(\zeta)| + |z_1(\zeta) - \bar{z}_1(\zeta)|\} \\
& \quad + \sum_{\zeta=n-\tau}^{n-1} (a_1^U H_2 + 2c_1^U H_2^2) \\
& \quad \quad \times \{|y_2(\zeta) - \bar{y}_2(\zeta)| + |z_2(\zeta) - \bar{z}_2(\zeta)|\} \\
& \leq \{|y_1(n) - \bar{y}_1(n)| + |y_2(n) - \bar{y}_2(n)| \\
& \quad + |z_1(n) - \bar{z}_1(n)| + |z_2(n) - \bar{z}_2(n)|\} \\
& \quad + \sum_{\zeta=n-\tau}^{n-1} \sigma \{|y_1(\zeta) - \bar{y}_1(\zeta)| + |z_1(\zeta) - \bar{z}_1(\zeta)| \\
& \quad \quad + |y_2(\zeta) - \bar{y}_2(\zeta)| + |z_2(\zeta) - \bar{z}_2(\zeta)|\}
\end{aligned}$$

$$\begin{aligned}
&\leq (1 + \tau\sigma) \sup_{s \in [-\tau, 0]_{\mathbb{Z}}} \{ |y_1(n+s) - \bar{y}_1(n+s)| \\
&\quad + |y_2(n+s) - \bar{y}_1(n+s)| \\
&\quad + |z_1(n+s) - \bar{z}_1(n+s)| \\
&\quad + |z_2(n+s) - \bar{z}_2(n+s)| \} \\
&\leq \rho (\|Y_n - \bar{Y}_n\| + \|Z_n - \bar{Z}_n\|),
\end{aligned} \tag{45}$$

where ρ and σ are defined in (42). That is, condition (II) in Lemma 4 is also satisfied.

It follows from the mean-value theorem that we find

$$\begin{aligned}
e^{y_i(n)} - e^{z_i(n)} &= e^{\theta_i(n)} (y_i(n) - z_i(n)), \\
e^{y_i(n-\tau_i)} - e^{z_i(n-\tau_i)} &= e^{\eta_i(n-\tau_i)} (y_i(n-\tau_i) - z_i(n-\tau_i)), \\
e^{2y_i(n-\tau_i)} - e^{2z_i(n-\tau_i)} &= 2e^{2\varsigma_i(n-\tau_i)} (y_i(n-\tau_i) - z_i(n-\tau_i)), \quad i = 1, 2,
\end{aligned} \tag{46}$$

where $\theta_i(n)$ lie between $y_i(n)$ and $z_i(n)$ and $\eta_i(n-\tau_i)$, $\varsigma_i(n-\tau_i)$ all lie between $y_i(n-\tau_i)$ and $z_i(n-\tau_i)$, respectively. Then

$$\ln h_i \leq \theta_i(n), \quad \eta_i(n-\tau_i), \quad \varsigma_i(n-\tau_i) \leq \ln H_i, \tag{47}$$

$n \in \mathbb{Z}^+$.

This one together with system (38) and (46) yields that

$$\begin{aligned}
&|y_1(n+1) - z_1(n+1)| + |y_2(n+1) - z_2(n+1)| \\
&= |y_1(n) - z_1(n) - e^{\theta_1(n)} (y_1(n) - z_1(n)) \\
&\quad - a_1(n) e^{\eta_1(n-\tau_1)} (y_1(n-\tau_1) - z_1(n-\tau_1)) \\
&\quad - 2c_1(n) e^{2\varsigma_1(n-\tau_1)} (y_1(n-\tau_1) - z_1(n-\tau_1))| \\
&+ |y_2(n) - z_2(n) - e^{\theta_2(n)} (y_2(n) - z_2(n)) \\
&\quad - a_2(n) e^{\eta_2(n-\tau_2)} (y_2(n-\tau_2) - z_2(n-\tau_2)) \\
&\quad - 2c_2(n) e^{2\varsigma_2(n-\tau_2)} (y_2(n-\tau_2) - z_2(n-\tau_2))| \\
&\leq |1 - e^{\theta_1(n)}| \cdot |y_1(n) - z_1(n)| \\
&\quad + (a_1(n) e^{\eta_1(n-\tau_1)} + 2c_1(n) e^{2\varsigma_1(n-\tau_1)}) \\
&\quad \times |y_1(n-\tau_1) - z_1(n-\tau_1)| \\
&\quad + |1 - e^{\theta_2(n)}| \cdot |y_2(n) - z_2(n)| \\
&\quad + (a_2(n) e^{\eta_2(n-\tau_2)} + 2c_2(n) e^{2\varsigma_2(n-\tau_2)}) \\
&\quad \times |y_2(n-\tau_2) - z_2(n-\tau_2)|
\end{aligned}$$

$$\begin{aligned}
&\leq |1 - e^{\theta_1(n)}| \cdot |y_1(n) - z_1(n)| \\
&\quad + (a_1^U H_2 + 2c_1^U H_2^2) |y_2(n-\tau_2) - z_2(n-\tau_2)| \\
&\quad + |1 - e^{\theta_2(n)}| \cdot |y_2(n) - z_2(n)| \\
&\quad + (a_2^U H_1 + 2c_2^U H_1^2) |y_1(n-\tau_1) - z_1(n-\tau_1)|.
\end{aligned} \tag{48}$$

Combining with (48) and calculating the ΔV of V along the solution of (38), one has

$$\begin{aligned}
\Delta V(n) &= V(n+1) - V(n) \\
&= |y_1(n+1) - z_1(n+1)| + |y_2(n+1) - z_2(n+1)| \\
&\quad + \sum_{\zeta=n+1-\tau_1}^n (a_2^U H_1 + 2c_2^U H_1^2) |y_1(\zeta) - z_1(\zeta)| \\
&\quad + \sum_{\zeta=n+1-\tau_2}^n (a_1^U H_2 + 2c_1^U H_2^2) |y_2(\zeta) - z_2(\zeta)| \\
&\quad - |y_1(n) - z_1(n)| - |y_2(n) - z_2(n)| \\
&\quad - \sum_{\zeta=n-\tau_1}^{n-1} (a_2^U H_1 + 2c_2^U H_1^2) |y_1(\zeta) - z_1(\zeta)| \\
&\quad - \sum_{\zeta=n-\tau_2}^{n-1} (a_1^U H_2 + 2c_1^U H_2^2) |y_2(\zeta) - z_2(\zeta)| \\
&\leq |1 - e^{\theta_1(n)}| \cdot |y_1(n) - z_1(n)| \\
&\quad + (a_1^U H_2 + 2c_1^U H_2^2) |y_2(n-\tau_2) - z_2(n-\tau_2)| \\
&\quad + |1 - e^{\theta_2(n)}| \cdot |y_2(n) - z_2(n)| \\
&\quad + (a_2^U H_1 + 2c_2^U H_1^2) |y_1(n-\tau_1) - z_1(n-\tau_1)| \\
&\quad + \sum_{\zeta=n+1-\tau_1}^n (a_2^U H_1 + 2c_2^U H_1^2) |y_1(\zeta) - z_1(\zeta)| \\
&\quad + \sum_{\zeta=n+1-\tau_2}^n (a_1^U H_2 + 2c_1^U H_2^2) |y_2(\zeta) - z_2(\zeta)| \\
&\quad - |y_1(n) - z_1(n)| - |y_2(n) - z_2(n)| \\
&\quad - \sum_{\zeta=n-\tau_1}^{n-1} (a_2^U H_1 + 2c_2^U H_1^2) |y_1(\zeta) - z_1(\zeta)| \\
&\quad - \sum_{\zeta=n-\tau_2}^{n-1} (a_1^U H_2 + 2c_1^U H_2^2) |y_2(\zeta) - z_2(\zeta)| \\
&= \{ |1 - e^{\theta_1(n)}| - 1 + (a_2^U H_1 + 2c_2^U H_1^2) \} \cdot |y_1(n) - z_1(n)| \\
&\quad + \{ |1 - e^{\theta_2(n)}| - 1 + (a_1^U H_2 + 2c_1^U H_2^2) \} \cdot |y_2(n) - z_2(n)|
\end{aligned}$$

FIGURE 1: Positive almost periodic solution of system (50). (a) Time-series of $x_1^*(n)$ with initial values $x_1^*(-1) = 0.79$ and $x_1^*(0) = 0.80$ for $n \in [0, 100]$. (b) Time-series of $x_2^*(n)$ with initial values $x_2^*(-2) = 0.85$, $x_2^*(-1) = 0.80$, and $x_2^*(0) = 0.79$ for $n \in [0, 100]$.

$$\begin{aligned}
&\leq -\left\{1 - \max(|1 - h_1|, |1 - H_1|) - (a_2^U H_1 + 2c_2^U H_1^2)\right\} \\
&\quad \cdot |y_1(n) - z_1(n)| \\
&\quad - \left\{1 - \max(|1 - h_2|, |1 - H_2|) - (a_1^U H_2 + 2c_1^U H_2^2)\right\} \\
&\quad \cdot |y_2(n) - z_2(n)| \\
&= -\lambda_1 |y_1(n) - z_1(n)| - \lambda_2 |y_2(n) - z_2(n)| \\
&\leq -\beta (|y_1(n) - z_1(n)| + |y_2(n) - z_2(n)|) \\
&\leq -\beta ((y_1(n) - z_1(n))^2 + (y_2(n) - z_2(n))^2)^{1/2} \\
&= -\beta (|Y_n(0) - Z_n(0)|),
\end{aligned} \tag{49}$$

where $\beta = \min\{\lambda_1, \lambda_2\}$. Since $\lambda_i > 0$, $\beta > 0$. Denote $c \in C(R^+, R^+)$, $c(x) = \beta x$; therefore, the condition in Remark 5 is satisfied. From Lemma 4 and Remark 5, system (32) has a unique uniformly asymptotically stable almost periodic solution denoted by $(y_1^*(n), y_2^*(n))$, which is equivalent to saying that system (3) has a unique uniformly asymptotically stable positive almost periodic solution denoted by $(x_1^*(n), x_2^*(n))$. This proof of Theorem 10 is completed. $\square$

If the coefficients $\{r_i(n)\}$, $\{a_i(n)\}$, and $\{c_i(n)\}$ are bounded positive periodic sequences, then system (3) becomes a periodic version. Applying Lemma 4 and Theorem 10, Corollary 11 is obtained directly.

Corollary 11. *Periodic system (3) shows a unique positive periodic solution which is uniformly asymptotically stable under the same assumptions of Theorem 10.*

4. Numerical Simulations

In this section, we give two specific numerical examples to verify our analytical results, that is, Theorem 10 and Corollary 11.

Example 12. Consider the following delay discrete almost periodic competitive system:

$$\begin{aligned}
x_1(n+1) &= x_1(n) \exp \left[\left(0.87 + 0.02 \sin(\sqrt{2}n\pi) \right) - x_1(n) \right. \\
&\quad \left. - \left(0.025 - 0.001 \cos(\sqrt{2}n\pi) \right) x_2(n-2) \right. \\
&\quad \left. - \left(0.015 + 0.002 \sin(\sqrt{2}n\pi) \right) x_2^2(n-2) \right], \\
x_2(n+1) &= x_2(n) \exp \left[\left(0.95 + 0.03 \cos(\sqrt{2}n\pi) \right) - x_2(n) \right. \\
&\quad \left. - \left(0.020 + 0.001 \sin(\sqrt{3}n\pi) \right) x_1(n-1) \right. \\
&\quad \left. - \left(0.027 + 0.002 \sin(\sqrt{2}n\pi) \right) x_1^2(n-1) \right].
\end{aligned} \tag{50}$$

A computation shows that

$$\begin{aligned}
r_1^U &= 0.89, & r_1^L &= 0.85, & r_2^U &= 0.98, & r_2^L &= 0.92, \\
a_1^U &= 0.026, & a_1^L &= 0.024, & a_2^U &= 0.021, & a_2^L &= 0.019, \\
c_1^U &= 0.017, & c_1^L &= 0.013, & c_2^U &= 0.029, & c_2^L &= 0.025,
\end{aligned} \tag{51}$$

FIGURE 2: Phase portrait. (a) Two-dimensional phase portrait of $x_1^*(n)$ and $x_2^*(n)$ with initial values $x_1^*(-1) = 0.79$ and $x_1^*(0) = 0.80$ and $x_2^*(-2) = 0.85$, $x_2^*(-1) = 0.80$, and $x_2^*(0) = 0.79$ for $n \in [0, 100]$. (b) Three-dimensional phase portrait of n , $x_1^*(n)$ and $x_2^*(n)$ with the above initial values for $n \in [0, 100]$.

FIGURE 3: Uniformly asymptotic stability. (a) Time-series of $x_1^*(n)$ and $x_1(n)$ with initial values $x_1^*(-1) = 0.79$ and $x_1^*(0) = 0.80$ and $x_1(-1) = 0.83$ and $x_1(0) = 0.92$ for $n \in [0, 100]$, respectively. (b) Time-series of $x_2^*(n)$ and $x_2(n)$ with initial values $x_2^*(-2) = 0.85$, $x_2^*(-1) = 0.80$, and $x_2^*(0) = 0.79$ and $x_2(-2) = 0.81$, $x_2(-1) = 0.79$, and $x_2(0) = 0.97$ for $n \in [0, 100]$, respectively.

and then we have

$$\begin{aligned} H_1 &\approx 0.8958, & H_2 &\approx 0.9802, & \Delta_1 &\approx 0.8082, \\ \Delta_2 &\approx 0.8779, & h_1 &\approx 0.7404, & h_2 &\approx 0.7925, \\ \Delta_1 &> 0, & \Delta_2 &> 0. \end{aligned} \quad (52)$$

Moreover,

$$\begin{aligned} \lambda_1 &= 1 - \max(|1 - h_1|, |1 - H_1|) \\ &- (a_2^U H_1 + 2c_2^U H_1^2) \approx 0.6750 > 0, \end{aligned}$$

$$\begin{aligned} \lambda_2 &= 1 - \max(|1 - h_2|, |1 - H_2|) \\ &- (a_1^U H_2 + 2c_1^U H_2^2) \approx 0.7343 > 0. \end{aligned} \quad (53)$$

It is easy to see that the assumptions of Theorem 10 are satisfied; that is to say, system (50) has a unique positive almost periodic solution denoted by $(x_1^*(t), x_2^*(t))$ which is uniformly asymptotically stable (see Figure 1), and the two-dimensional and three-dimensional phase portraits are displayed in Figure 2, respectively. In Figure 3, any positive

FIGURE 4: Positive periodic solution of system (54). (a) Time-series of $x_1^+(n)$ with initial values $x_1^+(-1) = 0.79$ and $x_1^+(0) = 0.80$ for $n \in [0, 100]$. (b) Time-series of $x_2^+(n)$ with initial values $x_2^+(-2) = 0.85$, $x_2^+(-1) = 0.80$, and $x_2^+(0) = 0.79$ for $n \in [0, 100]$.

FIGURE 5: Phase portrait. (a) Two-dimensional phase portrait of $x_1^+(n)$ and $x_2^+(n)$ with initial values $x_1^+(-1) = 0.79$ and $x_1^+(0) = 0.80$ and $x_2^+(-2) = 0.85$, $x_2^+(-1) = 0.80$, and $x_2^+(0) = 0.79$ for $n \in [0, 100]$. (b) Three-dimensional phase portrait of n , $x_1^+(n)$ and $x_2^+(n)$ with the above initial values for $n \in [0, 100]$.

solution denoted by $(x_1(n), x_2(n))$ tends to the above almost periodic solution $(x_1^*(n), x_2^*(n))$.

Example 13. Consider the following delay discrete periodic competitive system:

$$\begin{aligned} x_1(n+1) &= x_1(n) \exp \left[(0.87 + 0.02 \sin(n\pi)) - x_1(n) \right. \\ &\quad \left. - (0.025 - 0.001 \cos(n\pi)) x_2(n-2) \right. \\ &\quad \left. - (0.015 + 0.002 \sin(n\pi)) x_2^2(n-2) \right], \end{aligned}$$

$$\begin{aligned} x_2(n+1) &= x_2(n) \exp \left[(0.95 + 0.03 \cos(n\pi)) - x_2(n) \right. \\ &\quad \left. - (0.020 + 0.001 \sin(n\pi)) x_1(n-1) \right. \\ &\quad \left. - (0.027 + 0.002 \sin(n\pi)) x_1^2(n-1) \right]. \end{aligned} \quad (54)$$

Analogously, we can see that system (54) satisfies the assumptions of Theorem 10. From Corollary 11, there exists a unique positive periodic solution of system (54) which is uniformly asymptotically stable. From Figure 4, system (54) shows a positive periodic solution denoted by $(x_1^+(t), x_2^+(t))$,

FIGURE 6: Uniformly asymptotic stability. (a) Time-series of $x_1^+(n)$ and $x_1(n)$ with initial values $x_1^+(-1) = 0.79$ and $x_1^+(0) = 0.80$ and $x_1(-1) = 0.83$ and $x_1(0) = 0.92$ for $n \in [0, 100]$, respectively. (b) Time-series of $x_2^+(n)$ and $x_2(n)$ with initial values $x_2^+(-2) = 0.85$, $x_2^+(-1) = 0.80$, and $x_2^+(0) = 0.79$ and $x_2(-2) = 0.81$, $x_2(-1) = 0.79$, and $x_2(0) = 0.97$ for $n \in [0, 100]$, respectively.

and the two-dimensional and three-dimensional phase portraits are displayed in Figure 5, respectively. Figure 6 shows that any positive solution denoted by $(x_1(n), x_2(n))$ tends to the above periodic solution $(x_1^+(n), x_2^+(n))$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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