Research Article

Hopf Bifurcation Analysis and Chaos Control of a Chaotic System without Šilnikov Orbits

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Received 14 September 2015; Accepted 23 November 2015

Academic Editor: Luca Guerrini

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This paper mainly investigates the dynamical behaviors of a chaotic system without Šilnikov orbits by the normal form theory. Both the stability of the equilibria and the existence of local Hopf bifurcation are proved in view of analyzing the associated characteristic equation. Meanwhile, the direction and the period of bifurcating periodic solutions are determined. Regarding the delay as a parameter, we discuss the effect of time delay on the dynamics of chaotic system with delayed feedback control. Finally, numerical simulations indicate that chaotic oscillation is converted into a steady state when the delay passes through a certain critical value.

1. Introduction

In the past decades, both the chaotic synchronization and chaos control have excited amusements in many areas of science and technology due to their potential application. As is known some chaos is desirable and needs not to be controlled, but others are undesirable and need to be controlled. Therefore, the investigation of chaotic control is of great significance. Moreover, many schemes have been presented to implement chaotic control [1–18], of which the time-delayed feedback controlling forces is proved to be a simple and viable method for a continuous dynamical system. Fortunately, the time-delayed feedback controller can also be used to realize the control of a bifurcation. As is well discussed the chaotic dynamical behaviors of the chaotic system will be varied if the steady state is stable or the bifurcating periodic solutions are orbitally asymptotically stable.

Some authors study the chaotic systems with Šilnikov orbit, such as Elhadj and Sprott [9]. Some others investigate chaotic systems without Šilnikov orbit, like Wang and Chen [19–21], and so on. Recently, a chaotic system of non-Šilnikov type has been introduced in [19, 22] as follows:

$$\begin{aligned}\dot{x} &= a(y - x), \\ \dot{y} &= -ax - byz, \\ \dot{z} &= -c + y^2,\end{aligned}\quad (1)$$

where $a > 0$, $b > 0$, and $c > 0$. The bifurcations and period orbits of (1) will appear taking b as the bifurcation parameter values and fixing $a = 40$, $c = 10$. When $b = 33$ especially, there appears a chaotic attractor (see Figure 1). The system has two equilibria: $E_1(\sqrt{c}, \sqrt{c}, -a/b)$, $E_2(-\sqrt{c}, -\sqrt{c}, -a/b)$.

Inspired by the above, in this paper, we focus on the control of chaotic attractor of chaotic system (1). In order to better reflect the dynamical behaviors of system (1) depending on the past system information, it is reasonable to incorporate time delay into this system. The signal error of the current state and the past state of the continuous time system will be fed back to the system itself. Following the idea of Pyragas [13], we add a time-delayed force $k[y(t) - y(t - \tau)]$ to the second equation of system (1) and then system (1) takes the form as follows:

$$\begin{aligned}\dot{x}(t) &= a(y(t) - x(t)), \\ \dot{y}(t) &= -ax(t) - by(t)z(t) + k[y(t) - y(t - \tau)], \\ \dot{z}(t) &= -c + y^2(t),\end{aligned}\quad (2)$$

where k is a real constant.

The structure of this paper is organized as follows. In Section 2, the stability and the existence of Hopf bifurcation are determined. In Section 3, based on the normal form method and the center manifold theorem presented in

FIGURE 1: The trajectories and phase graphs of system (2) with $\tau = 0$.

Hassard et al. [23], the formulae of determining the direction, the stability, and the period of the bifurcating period solutions are presented. To verify the theoretic analysis, numerical simulations are given in Section 4. Finally, a summary of the main conclusions is provided in Section 5.

2. Bifurcation Analysis of the Chaotic System without Šilnikov Orbits

In this section, we analyze the influence of time delay on the stability of the steady state by taking τ as bifurcation parameter and fixing the value of k . The stability of system (2) will be changed and a family of periodic orbits will bifurcate from the equilibria with the increasing of τ .

By the linear transformation

$$\begin{aligned} x_1 &= x - x_0, \\ y_1 &= y - y_0, \\ z_1 &= z - z_0, \end{aligned} \quad (3)$$

system (2) becomes

$$\begin{aligned} \dot{x}_1(t) &= a(y_1(t) - x_1(t)), \\ \dot{y}_1(t) &= -ax_1(t) - bz_0y_1(t) - by_0z_1(t) - by_1(t)z_1(t) \\ &\quad + k[y_1(t) - y_1(t - \tau)], \\ \dot{z}_1(t) &= 2y_0y_1(t) + y_1^2(t). \end{aligned} \quad (4)$$

Then, the Jacobian matrix of (4) at any point (x, y) can be written as

$$J = \begin{bmatrix} -a & a & 0 \\ -a & -bz_0 + k - ke^{-\lambda\tau} & -by_0 \\ 0 & 2y_0 & 0 \end{bmatrix}. \quad (5)$$

The characteristic equation of (4) is given by

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 + (b_2\lambda^2 + b_1\lambda + b_0)e^{-\lambda\tau} = 0, \quad (6)$$

where $a_2 = bz_0 + a - k$, $a_1 = 2by_0^2 + a^2 + abz_0 - ak$, $a_0 = 2aby_0^2$, $b_2 = k$, $b_1 = ka$, and $b_0 = 0$. Thus, we will study the distribution of the roots of the third-degree exponential polynomial equation of (6), obviously, if $i\omega$ ($\omega > 0$) is a root of (6) and ω satisfies

$$\begin{aligned} -i\omega^3 - a_2\omega^2 + ia_1\omega + a_0 \\ + (ib_1\omega - b_2\omega^2)(\cos(\omega\tau) - i\sin(\omega\tau)) = 0. \end{aligned} \quad (7)$$

Separating the real and imaginary parts of (7), we get

$$a_2\omega^2 - a_0 = -b_2\omega^2 \cos(\omega\tau) + b_1\omega \sin(\omega\tau), \quad (8)$$

$$\omega^3 - a_1\omega = b_1\omega \cos(\omega\tau) + b_2\omega^2 \sin(\omega\tau), \quad (9)$$

which is equivalent to

$$\begin{aligned} \omega^6 + (a_2^2 - b_2^2 - 2a_1)\omega^4 + (a_1^2 - 2a_0a_2 - b_1^2)\omega^2 + a_0^2 \\ = 0. \end{aligned} \quad (10)$$

Let $z = \omega^2$ and denote $p = a_2^2 - b_2^2 - 2a_1$, $q = a_1^2 - 2a_0a_2 - b_1^2$, and $r = a_0^2$; (10) becomes

$$z^3 + pz^2 + qz + r = 0. \quad (11)$$

From (11), we have

$$h(z) = z^3 + pz^2 + qz + r, \quad (12)$$

where $r > 0$.

Then, $h'(z) = 3z^2 + 2pz + q$.

Denote $\Delta = p^2 - 3q \leq 0$ and $\Delta > 0$; the equation $3z^2 + 2pz + q = 0$ has two real roots as follows:

$$\begin{aligned} z_1^* &= \frac{-p + \sqrt{\Delta}}{3}, \\ z_2^* &= \frac{-p - \sqrt{\Delta}}{3}. \end{aligned} \quad (13)$$

Notice that $\lim_{z \rightarrow \infty} h(z) = +\infty$ and $r = a_0^2 > 0$. We introduce the following results proved by Lü and Chen [24].

Lemma 1. *For the polynomial equation (11), we have the following results:*

- (1) If $\Delta = p^2 - 3q \leq 0$, then (11) does not have positive real root.
- (2) If $\Delta = p^2 - 3q > 0$, then (11) has positive real roots if and only if $z_1^* = (-p + \sqrt{\Delta})/3 > 0$ and $h(z_1^*) \leq 0$.

Without loss of generality, we give the following assumption:

$$\begin{aligned} \Delta &= p^2 - 3q > 0, \\ z_1^* &= \frac{-p + \sqrt{\Delta}}{3} \\ h(z_1^*) &\leq 0. \end{aligned} \quad (14)$$

If we make an assumption that (14) is true, then (11) has two positive roots z_1 and z_2 . Supposing $z_1 < z_2$, we have $h'(z_1^*) \leq 0$. Substituting $\omega_k = \sqrt{z_k}$ ($k = 1, 2$) into (8), we find that

$$\begin{aligned} \tau_k^j &= \frac{1}{\omega_k} \left\{ \cos^{-1} \left(\frac{(b_1 - a_2 b_2) \omega_k^2 - (a_1 b_1 - a_0 b_2)}{b_2^2 \omega_k^2 + b_1^2} \right) \right. \\ &\quad \left. + 2j\pi \right\}, \end{aligned} \quad (15)$$

where ($k = 1, 2$ and $j = 0, 1, 2, \dots$); then $\pm i\omega_k$ is a pair of pure imaginary roots of (7) with τ_k^j . Define

$$\tau_0 = \tau_{k_0}^0 = \min_{k \in \{1, 2, 3\}} \{\tau_k^0\}. \quad (16)$$

Lemma 2. *If (14) holds, when $\tau = \tau_k^j$ ($k = 1, 2$; $j = 0, 1, 2, \dots$), then (11) has a pair of pure imaginary roots ω_k and all the other roots of (11) have nonzero real parts.*

Lemma 3. *If (14) holds, then we have the following transversality condition:*

- (i) $[d(\operatorname{Re} \lambda)/d\tau]_{\tau=\tau_k}^{-1} < 0$ and $[d(\operatorname{Re} \lambda)/d\tau]_{\tau=\tau_k}^{-1} > 0$.
- (ii) $[d(\operatorname{Re} \lambda)/d\tau]_{\tau=\tau_k^j}^{-1}$ ($j = 0, 1, 2, \dots$) and $h'(z_k)$ have the same sign.

Proof. Substituting $\lambda(\tau)$ into (6) and taking the derivative with respect to τ , we obtain

$$\left(\frac{d\lambda}{d\tau} \right)^{-1} = \frac{(3\lambda^2 + 2a_2\lambda + a_1)}{\lambda(b_2\lambda^2 + b_1\lambda)e^{-\lambda\tau}} + \frac{2b_2\lambda + b_1}{\lambda(b_2\lambda^2 + b_1\lambda)} - \frac{\tau}{\lambda}. \quad (17)$$

From (9), we find that

$$\begin{aligned} \left(\frac{d\lambda}{d\tau} \right)^{-1}_{\tau=\tau_k} &= \operatorname{Re} \left[\frac{(3\lambda^2 + 2a_2\lambda + a_1)}{\lambda(b_2\lambda^2 + b_1\lambda)e^{-\lambda\tau}} \right]_{\tau=\tau_k} \\ &\quad + \operatorname{Re} \left[\frac{2b_2\lambda + b_1}{\lambda(b_2\lambda^2 + b_1\lambda)} \right]_{\tau=\tau_k} \\ &= \operatorname{Re} \left[\frac{(3\lambda^2 + 2a_2\lambda + a_1)}{\lambda(b_2\lambda^2 + b_1\lambda)} (\cos(\omega_k \tau_k^j) \right. \\ &\quad \left. + i \sin(\omega_k \tau_k^j)) \right]_{\tau=\tau_k} + \operatorname{Re} \left[\frac{2b_2\lambda + b_1}{\lambda(b_2\lambda^2 + b_1\lambda)} \right]_{\tau=\tau_k} \\ &= \frac{1}{\Lambda} [3\omega_k^6 + 2(a_2^2 - b_2^2 - 2a_1)\omega_k^4 + (a_1^2 - 2a_0a_2 \\ &\quad - b_1^2)\omega_k^2] = \frac{1}{\Lambda} [3\omega_k^6 + 2p\omega_k^4 + q\omega_k^2] = \frac{z_k}{\Lambda} [3z_k^2 \\ &\quad + 2pz_k + q] = \frac{z_k}{\Lambda} [h'(z_k)], \end{aligned} \quad (18)$$

where $\Lambda = b_1^2\omega_k^4 + b_2^2\omega_k^6$. Since $z_k > 0$, we conclude that $(d\lambda/d\tau)_{\tau=\tau_k^j}$ and $h'(z_k)$ have the same sign. This completes the proof. $\square$

Theorem 4. *Suppose (14) and Routh-Hurwitz criteria for (7) are satisfied. Moreover, if $\tau_1^0 > \tau_2^0$, there exists $m \in N$ such that $\tau_2^0 < \tau_1^0 < \tau_2^1 < \tau_1^1 < \tau_2^2 < \tau_1^2 < \dots < \tau_2^m < \tau_1^m < \tau_2^{m+1} < \tau_1^{m+1}$.*

- (i) For $\tau \in [0, \tau_2^0] \cup [\tau_1^0, \tau_2^1] \cup \dots \cup [\tau_1^{m-1}, \tau_2^m] \cup [\tau_1^m, \tau_2^{m+1}]$, the equilibriums $E_{1,2}$ of system (2) are unstable.
- (ii) For $\tau \in [\tau_2^0, \tau_1^0] \cup [\tau_2^1, \tau_1^1] \cup \dots \cup [\tau_2^m, \tau_1^m] \cup [\tau_2^{m+1}, +\infty]$, the equilibriums $E_{1,2}$ are asymptotically stable.
- (iii) When $\tau = \tau_k^j$ ($k = 1, 2$ and $j = 0, 1, 2, \dots$), system (2) undergoes a Hopf bifurcation at the equilibriums $E_{1,2}$.

3. Direction and Stability of Hopf Bifurcation Period Solution

In the previous section, we obtained the conditions of Hopf bifurcation occurring. In this section, the direction and the stability of the bifurcations are analyzed using the central manifold theorem. Assuming that system (2) always undergoes Hopf bifurcation at the equilibrium E_1 for $\tau = \tau_k^j$ ($k = 1, 2$ and $j = 0, 1, 2, \dots$), then, letting $u_1 = x - x_0$, $u_2 = y - y_0$, $u_3 = z - z_0$, $\bar{u}_i = u_i(\tau\tau)$, and $\tau = \tau_k^j$ and dropping the bars for simplifications of notations, the nonlinear system (2) can be transformed into an FDE in $C \in C([-1, 0], R^3)$ as

$$\dot{u}(t) = L_\mu(u_t) + f(\mu, u_t), \quad (19)$$

where $u(t) = (u_1(t), u_2(t), u_3(t))^T \in R^3$ and $L_\mu : C \rightarrow R^3$ and $f : R \times C \rightarrow R$ are given, respectively, by

$$\begin{aligned} L_\mu(\varphi) &= (\mu + \tau_k) \begin{pmatrix} -a & a & 0 \\ -a & k - bz_0 & -by_0 \\ 0 & 2y_0 & 0 \end{pmatrix} \begin{pmatrix} \varphi_1(0) \\ \varphi_2(0) \\ \varphi_3(0) \end{pmatrix} \\ &\quad + (\mu + \tau_k) \begin{pmatrix} 0 & 0 & 0 \\ 0 & -k & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi_1(-1) \\ \varphi_2(-1) \\ \varphi_3(-1) \end{pmatrix}, \quad (20) \\ f(\tau, \varphi) &= (\mu + \tau_k) \begin{pmatrix} 0 \\ -b\varphi_2(0)\varphi_3(0) \\ \varphi_2^2(0) \end{pmatrix}. \end{aligned}$$

If we choose

$$\begin{aligned} \eta(\theta, \mu) &= (\mu + \tau_k) \begin{pmatrix} -a & a & 0 \\ -a & k - bz_0 & -by_0 \\ 0 & 2y_0 & 0 \end{pmatrix} \delta(\theta) \\ &\quad - (\mu + \tau_k) \begin{pmatrix} 0 & 0 & 0 \\ 0 & -k & 0 \\ 0 & 0 & 0 \end{pmatrix} \delta(\theta + 1), \quad (21) \end{aligned}$$

where $\delta(\cdot)$ is a Dirac delta function and $\theta \in [-1, 0]$, then there is

$$L_\mu(\varphi) = \int_{-1}^0 d\eta(\theta, \mu) \varphi(\theta), \quad \text{for } \theta \in C. \quad (22)$$

For $\varphi \in C([-1, 0], R^3)$, define

$$\begin{aligned} A(\mu)\varphi &= \begin{cases} d\frac{\varphi(\theta)}{d\theta}, & \theta \in (0, 1], \\ \int_{-1}^0 d\eta(\theta, s) \varphi(s), & \theta = 0, \end{cases} \quad (23) \\ R(\mu)\varphi &= \begin{cases} 0, & \theta \in (0, 1], \\ F(\mu, \varphi), & \theta = 0. \end{cases} \end{aligned}$$

For convenience, we can write system (9) into an operator equation

$$\dot{u}(t) = A(\mu)u_t + R(\mu)u_t, \quad (24)$$

where $u_t(\theta) = u(t + \theta)$, $\theta \in [-1, 0]$.

For $\psi \in ([-1, 0], (R^3)^*)$, define

$$A^*\varphi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in (0, 1], \\ \int_{-1}^0 d\eta^T(t, 0) \psi(-t), & s = 0, \end{cases} \quad (25)$$

and a bilinear inner product

$$\begin{aligned} \langle \psi, \varphi \rangle &= \overline{\psi(0)}\varphi(0) \\ &\quad - \int_{\theta=-1}^0 \int_{\xi=0}^{\theta} \overline{\psi^T}(\xi - \theta) d\eta(\theta) \varphi(\xi) d\xi, \quad (26) \end{aligned}$$

where $\eta(\theta) = \eta(\theta, 0)$. Obviously, A^* and $A(0)$ are adjoint operators. In line with the discussion in Section 2, we know that $\pm i\omega_k \tau_k$ are eigenvalues of $A(0)$. Thus they are eigenvalues of A^* . Next, we need to calculate the eigenvectors of A^* and $A(0)$ corresponding to $-i\omega_k \tau_k$ and $i\omega_k \tau_k$, respectively. Let $q(\theta) = (1, \alpha, \beta)^T e^{i\theta\omega\tau}$ be the eigenvalue of $A(0)$; that is, $A(0)q(0) = A(\theta) = i\omega_k q(\theta)$; we have

$$\begin{aligned} \tau \begin{pmatrix} i\omega_k + a & -a & 0 \\ a & i\omega_k - k + bz_0 + ke^{-i\omega_k\tau} & by_0 \\ 0 & -2y_0 & i\omega_k \end{pmatrix} q(0) \\ = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (27) \end{aligned}$$

Then it is easy to obtain

$$q(0) = (1, \alpha, \beta)^T = \left(1, \frac{i\omega_k + a}{a}, \frac{2y_0(i\omega_k + a)}{i\omega_k a}\right)^T. \quad (28)$$

Similarly, supposing $q^*(\theta) = (1, \alpha^*, \beta^*)^T e^{i\theta\omega\tau}$, from the definition of A^* , we have

$$\begin{aligned} q^*(0) &= D(1, \alpha^*, \beta^*)^T \\ &= D\left(1, \frac{a - i\omega_k}{a}, \frac{by_0(a - i\omega_k)}{i\omega_k a}\right)^T. \quad (29) \end{aligned}$$

So we have

$$\begin{aligned} q(\theta) &= (1, \alpha, \beta)^T \\ &= \left(1, \frac{i\omega_k + a}{a}, \frac{2y_0(i\omega_k + a)}{i\omega_k a}\right)^T e^{i\omega_0 \tau_k \theta}, \\ &\quad \theta \in [-1, 0], \quad (30) \end{aligned}$$

$$\begin{aligned} q^*(s) &= D(1, \alpha^*, \beta^*)^T \\ &= D\left(1, \frac{a - i\omega_k}{a}, \frac{by_0(a - i\omega_k)}{i\omega_k a}\right)^T e^{i\omega_0 \tau_k s}, \end{aligned}$$

$$s \in [0, 1],$$

where D is a constant such that $\langle q^*(s), q(\theta) \rangle = 1$. By (26), we get

$$\begin{aligned} \langle q^*(s), q(\theta) \rangle &= \overline{D(1, \alpha^*, \beta^*)} (1, \alpha, \beta)^T \\ &\quad - \int_{-1}^0 \int_{\xi=0}^{\theta} \overline{D(1, \alpha^*, \beta^*)} e^{-i(\xi-\theta)\omega\tau} d\eta(\theta) (1, \alpha, \beta)^T \\ &\quad \cdot e^{i\xi\omega\tau} d\xi = \overline{D(1 + \overline{\alpha^*}\alpha + \overline{\beta^*}\beta)} \\ &\quad - \int_{-1}^0 \int_{\xi=0}^{\theta} \overline{D(1, \alpha^*, \beta^*)} \theta e^{i\theta\omega\tau} d\eta(\theta) (1, \alpha, \beta)^T \\ &= \overline{D(1 + \overline{\alpha^*}\alpha + \overline{\beta^*}\beta + k\tau_k \overline{\alpha^*}\alpha e^{-i\omega_k \tau_k})} = 1. \quad (31) \end{aligned}$$

Therefore, we can choose D as

$$D = \frac{1}{(1 + \bar{\alpha}^* \alpha + \bar{\beta}^* \beta + k \tau_k \bar{\alpha}^* \alpha e^{-i\omega_k \tau_k})}. \quad (32)$$

Below we apply the idea in Hassard et al. [23] to compute the coordinates describing the center manifold $\mu = 0$. Let x_t be the solution of (2) when $x = 0$. Define

$$\begin{aligned} z(t) &= \langle q^*, x_t \rangle, \\ W(t, \theta) &= x_t(\theta) - 2 \operatorname{Re} z(t) q(\theta). \end{aligned} \quad (33)$$

On the center manifold C_0 , we get

$$W(t, \theta) = W(z(t), \bar{z}(t), \theta), \quad (34)$$

where

$$\begin{aligned} W(t, \theta) &= W(z(t), \bar{z}(t), \theta) = W(z, \bar{z}) \\ &= W_{20} \frac{z^2}{2} + W_{11} z \bar{z} + W_{02} \frac{\bar{z}^2}{2} + \dots \end{aligned} \quad (35)$$

and z and $\bar{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\bar{z}^*$. Noting that W is also real if $x_t \in C_0$ of (2), so we deal with real solutions only. For solution x_t , since $x = 0$, we have

$$\begin{aligned} \dot{z} &= \langle q^*(s), \dot{x}_t \rangle = \langle q^*(s), A(0)x_t + R(0)x_t \rangle \\ &= \langle q^*(s), A(0)x_t \rangle + \langle q^*(s), R(0)x_t \rangle \\ &\quad - \int_{-1}^0 \int_{\xi=0}^{\theta} \bar{q}^*(\xi - \theta) \eta(\theta) A(0) R(0) x_t(\xi) d(\xi) \quad (36) \\ &= \langle i\omega_0 \tau_k q^*(s), x_t \rangle + \bar{q}^*(0) f(0, x_t(\theta)) \\ &= i\omega_0 \tau_k z(t) + \bar{q}^*(0) f_0(z(t), \bar{z}(t)). \end{aligned}$$

That is,

$$\dot{z}(t) = i\omega_0 \tau_k z + g(z, \bar{z}), \quad (37)$$

where

$$g(z, \bar{z}) = g_{20} \frac{z^2}{2} + g_{11} z \bar{z} + g_{02} \frac{\bar{z}^2}{2} + \dots \quad (38)$$

Then we have

$$\begin{aligned} g(z, \bar{z}) &= \bar{q}^*(0) f_0(z, \bar{z}) = f(0, x_t) \\ &= \bar{D} \tau_k (1, \alpha^*, \beta^*) (f_1(0, x_t), f_2(0, x_t), f_3(0, x_t))^T, \end{aligned} \quad (39)$$

where

$$\begin{aligned} f_1(0, x_t) &= 0, \\ f_2(0, x_t) &= -\tau_k b y_{1t}(0) z_{1t}(0), \\ f_3(0, x_t) &= \tau_k y_{1t}^2(0). \end{aligned} \quad (40)$$

Noticing $x_t(\theta) = (x_{1t}(\theta), y_{1t}(\theta), z_{1t}(\theta))^T = W(t, 0) + zq(\theta) + \bar{z}\bar{q}(\theta)$ and $q(\theta) = (1, \alpha, \beta)^T e^{i\omega_0 \theta}$, we have

$$\begin{aligned} x_{1t}(0) &= z + \bar{z} + W_{20}^{(1)}(0) \frac{z^2}{2} + W_{11}^{(1)}(0) z \bar{z} \\ &\quad + W_{02}^{(1)}(0) \frac{\bar{z}^2}{2} + \dots, \\ y_{1t}(0) &= \alpha z + \bar{\alpha} \bar{z} + W_{20}^{(2)}(0) \frac{z^2}{2} + W_{11}^{(2)}(0) z \bar{z} \\ &\quad + W_{02}^{(2)}(0) \frac{\bar{z}^2}{2} + \dots, \\ z_{1t}(0) &= \beta z + \bar{\beta} \bar{z} + W_{20}^{(3)}(0) \frac{z^2}{2} + W_{11}^{(3)}(0) z \bar{z} \\ &\quad + W_{02}^{(3)}(0) \frac{\bar{z}^2}{2} + \dots. \end{aligned} \quad (41)$$

In view of (38), (39), and (41), we get

$$\begin{aligned} g(z, \bar{z}) &= \bar{q}^*(0) f_0(z, \bar{z}) = f(0, x_t) \\ &= \bar{D} \tau_k (1, \alpha^*, \beta^*) (f_1(0, x_t), f_2(0, x_t), f_3(0, x_t))^T \\ &= \bar{D} \tau_k [M z^2 + N z \bar{z} + P \bar{z}^2 + Q z^2 \bar{z} + \dots], \end{aligned} \quad (42)$$

where

$$\begin{aligned} M &= b \bar{\alpha}^* \alpha \beta + \alpha^2 \bar{\beta}^*, \\ N &= b \bar{\alpha}^* \alpha \beta + 2 \alpha \bar{\alpha} \bar{\beta}^*, \\ P &= b \bar{\alpha} \bar{\beta} \alpha^* + \bar{\beta}^* \bar{\alpha}^2, \\ Q &= b \bar{\alpha}^* \left[\alpha W_{11}^{(3)}(0) + \frac{1}{2} \bar{\alpha} W_{20}^{(3)}(0) + \frac{1}{2} \bar{\beta} W_{20}^{(2)}(0) \right. \\ &\quad \left. + \beta W_{11}^{(2)}(0) \right] + \bar{\beta}^* \left[\bar{\alpha} W_{20}^{(2)}(0) + 2 \alpha W_{11}^{(2)}(0) \right], \end{aligned} \quad (43)$$

where

$$\begin{aligned} g_{20} &= 2 \bar{D} \tau_k M, \\ g_{11} &= \bar{D} \tau_k N, \\ g_{02} &= 2 \bar{D} \tau_k P, \\ g_{21} &= 2 \bar{D} \tau_k Q. \end{aligned} \quad (44)$$

Since $W_{11}^{(3)}(0)$, $W_{20}^{(3)}(0)$, $W_{20}^{(2)}(0)$, and $W_{11}^{(2)}(0)$ in g_{21} are unknown, we still need to compute them. By (24) and (36), we drive

$$\begin{aligned} \dot{W} &= \begin{cases} AW - 2 \operatorname{Re} \{ \bar{q}^*(0) f_0 q(\theta) \}, & \theta \in [-1, 0), \\ AW - 2 \operatorname{Re} \{ \bar{q}^*(0) f_0 q(\theta) \} + f_0, & \theta = 0. \end{cases} \end{aligned} \quad (45)$$

We can rewrite

$$\dot{W} = AW + H(z, \bar{z}, \theta), \quad (46)$$

where

$$H(z, \bar{z}, \theta) = H_{20} \frac{z^2}{2} + H_{11} z \bar{z} + H_{02} \frac{\bar{z}^2}{2} + H_{21} \frac{z^2 \bar{z}}{2} + \dots \quad (47)$$

Comparing the coefficients, we get

$$(A - 2i\omega_0)W_{20}(\theta) = -H_{20}, \quad (48)$$

$$AW_{11}(\theta) = -H_{11}(\theta). \quad (49)$$

For $\theta \in [-1, 0)$,

$$\begin{aligned} H(z, \bar{z}, \theta) &= -\bar{q}^*(0) f_0 q(\theta) - q^*(0) \bar{f}_0 \bar{q}(\theta) \\ &= -g(z, \bar{z}) q(\theta) - \bar{g}(z, \bar{z}) \bar{q}(\theta). \end{aligned} \quad (50)$$

Comparing the coefficients of (49) with (50), we find that

$$H_{20}(\theta) = -g_{20} q(\theta) - \bar{g}_{02} \bar{q}(\theta), \quad (51)$$

$$H_{11}(\theta) = -g_{11} q(\theta) - \bar{g}_{11} \bar{q}(\theta). \quad (52)$$

By (48) and (51) and the definition of A , we obtain

$$\dot{W}_{20} = 2i\omega_0 \tau_k W_{20}(\theta) + g_{20} q(\theta) + \bar{g}_{02} \bar{q}(\theta). \quad (53)$$

Since $q(\theta) = q(0)e^{i\omega_0 \tau_k \theta}$, we get

$$\begin{aligned} W_{20}(\theta) &= \frac{ig_{20}}{\omega_0 \tau_k} q(0) e^{i\omega_0 \tau_k \theta} + \frac{i\bar{g}_{02}}{3\omega_0 \tau_k} \bar{q}(0) e^{-i\omega_0 \tau_k \theta} \\ &\quad + E_1 e^{2i\omega_0 \tau_k \theta}, \end{aligned} \quad (54)$$

where $E_1 = (E_1^{(1)}, E_1^{(2)}, E_1^{(3)})^T$ is a constant vector.

Similarly, in view of (49) and (52) and the definition of A , we get

$$\dot{W}_{11}(\theta) = g_{11} q(\theta) + \bar{g}_{11} \bar{q}(\theta), \quad (55)$$

$$\begin{aligned} W_{11}(\theta) &= -\frac{ig_{11}}{\omega_0 \tau_k} q(0) e^{i\omega_0 \tau_k \theta} + \frac{i\bar{g}_{11}}{\omega_0 \tau_k} \bar{q}(0) e^{-i\omega_0 \tau_k \theta} \\ &\quad + E_2, \end{aligned} \quad (56)$$

where $E_1 = (E_2^{(1)}, E_2^{(2)}, E_2^{(3)})^T$ is a constant vector.

Next, we will seek appropriate E_1, E_2 in (54) and (55), respectively. By means of the definition of A , (51) and (52), we have

$$\dot{W}_{20}(\theta) = \int_{-1}^0 d\eta(\theta) W_{20}(\theta) = 2i\omega_0 \tau_k W_{20} - H_{20}(0), \quad (57)$$

$$\dot{W}_{11}(\theta) = \int_{-1}^0 d\eta(\theta) W_{11}(\theta) = -H_{11}(0), \quad (58)$$

where $\eta(\theta) = \eta(0, \theta)$. In view of (48), we get

$$H_{20}(\theta) = -g_{20} q(\theta) - \bar{g}_{02} \bar{q}(\theta) + 2\tau_k (0, b\alpha\beta, \alpha^2)^T, \quad (59)$$

$$H_{11}(\theta) = -g_{11} q(\theta) - \bar{g}_{11} \bar{q}(\theta) + 2\tau_k (0, b\alpha\beta, 2\alpha\bar{\alpha})^T. \quad (60)$$

For $i\omega_k \tau_k$ is the eigenvalues of $A(0)$ and $q(0)$ is the corresponding eigenvector, we obtain

$$\left(i\omega_k \tau_k - \int_{-1}^0 e^{i\theta\omega\tau} d\eta(\theta) \right) q(0) = 0, \quad (61)$$

$$\left(-i\omega_k \tau_k - \int_{-1}^0 e^{-i\theta\omega\tau} d\eta(\theta) \right) E_1 = 2\tau_k (0, b\alpha\beta, \alpha^2)^T.$$

Substituting (54) and (57) into (59), we obtain

$$\left(2i\omega_k \tau_k - \int_{-1}^0 e^{2i\theta\omega\tau} d\eta(\theta) \right) E_1 = 2\tau_k (0, b\alpha\beta, \alpha^2)^T. \quad (62)$$

That is,

$$\begin{aligned} &\begin{pmatrix} 2i\omega_k + a & -a & 0 \\ a & 2i\omega_k - k + bz_0 + ke^{-i\omega_k \tau} & by_0 \\ 0 & -2y_0 & 2i\omega_k \end{pmatrix} E_1 \\ &= \begin{pmatrix} 0 \\ b\alpha\beta \\ \alpha^2 \end{pmatrix}. \end{aligned} \quad (63)$$

It follows that

$$\begin{aligned} E_1^{(1)} &= \frac{\Delta_{11}}{\Delta_1}, \\ E_1^{(2)} &= \frac{\Delta_{12}}{\Delta_1}, \\ E_1^{(3)} &= \frac{\Delta_{13}}{\Delta_1}, \end{aligned} \quad (64)$$

where

$$\begin{aligned} \Delta_{11} &= 2 \begin{vmatrix} 0 & -a & 0 \\ b\alpha\beta & 2i\omega_k - k + bz_0 + ke^{-i\omega_k \tau} & by_0 \\ \alpha^2 & -2y_0 & 2i\omega_k \end{vmatrix}, \\ \Delta_{12} &= 2 \begin{vmatrix} 2i\omega_k + a & 0 & 0 \\ a & b\alpha\beta & by_0 \\ 0 & \alpha^2 & 2i\omega_k \end{vmatrix}, \\ \Delta_{13} &= 2 \begin{vmatrix} 2i\omega_k + a & -a & 0 \\ a & 2i\omega_k - k + bz_0 + ke^{-i\omega_k \tau} & b\alpha\beta \\ 0 & -2y_0 & \alpha^2 \end{vmatrix}, \\ \Delta_1 &= \begin{vmatrix} 2i\omega_k + a & -a & 0 \\ a & 2i\omega_k - k + bz_0 + ke^{-i\omega_k \tau} & by_0 \\ 0 & -2y_0 & 2i\omega_k \end{vmatrix}. \end{aligned} \quad (65)$$

Similarly, substituting (51), (52), and (60) into (58), we have

$$\begin{pmatrix} a & -a & 0 \\ a & bz_0 & by_0 \\ 0 & -2y_0 & 0 \end{pmatrix} E_2 = 2 \begin{pmatrix} 0 \\ b\alpha\beta \\ 2\alpha\bar{\alpha} \end{pmatrix},$$

$$E_2^{(1)} = \frac{\Delta_{21}}{\Delta_2},$$

$$E_2^{(2)} = \frac{\Delta_{22}}{\Delta_2},$$

$$E_2^{(3)} = \frac{\Delta_{23}}{\Delta_2},$$

$$\Delta_{21} = 2 \begin{vmatrix} 0 & -a & 0 \\ b\alpha\beta & bz_0 & by_0 \\ 2\alpha\bar{\alpha} & -2y_0 & 0 \end{vmatrix}, \quad (66)$$

$$\Delta_{22} = 2 \begin{vmatrix} a & 0 & 0 \\ a & b\alpha\beta & by_0 \\ 0 & 2\alpha\bar{\alpha} & 0 \end{vmatrix},$$

$$\Delta_{23} = 2 \begin{vmatrix} a & -a & 0 \\ a & bz_0 & b\alpha\beta \\ 0 & -2y_0 & 2\alpha\bar{\alpha} \end{vmatrix},$$

$$\Delta_2 = \begin{vmatrix} a & -a & 0 \\ a & bz_0 & by_0 \\ 0 & -2y_0 & 0 \end{vmatrix}.$$

Consequently, we can determine $W_{20}(0)$ and $W_{11}(0)$. Thus, all g_{ij} can be determined by [13]. Following the basic idea of [25] and the method in [26], one can draw the conclusion about the bifurcation direction and the stability of the Hopf bifurcation, which are determined by the following parameters:

$$C_1(0) = \frac{i}{2\omega_k\tau_k} \left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2 \right) + \frac{g_{21}}{2},$$

$$\mu_2 = -\frac{\operatorname{Re}\{C_1(0)\}}{\operatorname{Re}\{d\lambda/d\tau\}}, \quad (67)$$

$$T_2 = \frac{\operatorname{Im}\{C_1(0)\} + \mu_2 \operatorname{Im}\{d\lambda/d\tau\}}{\omega_k\tau_k},$$

$$\beta_2 = 2 \operatorname{Re}\{C_1(0)\}.$$

Thus, we have the main results of this section.

Theorem 5. μ_2 determines the direction of the Hopf bifurcation; that is, if $\mu_2 > 0$ ($\mu_2 < 0$), the Hopf bifurcation is supercritical (subcritical) and the bifurcating period solutions exist for $\tau > \tau_k$ ($\tau < \tau_k$); β_2 determines the stability of the bifurcating period solutions: the bifurcating period solutions are orbitally stable (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$), and T_2

determines the period of the bifurcating period solutions: the period increases (decreases) if $T_2 > 0$ ($T_2 < 0$).

4. Application to Control of Chaos

In this section, we apply the results in the previous sections to system (2) for the purpose of control of chaos. From Section 2, we know that, under certain conditions, a family of periodic solutions bifurcate from the steady states of system (2) at some critical values of τ and the stability of the steady state maybe changes along with increasing of τ . If the bifurcating periodic solution is orbitally asymptotically stable or some steady state becomes locally stable, then chaos may vanish. Let $a = 10$, $b = 33$, and $c = 40$, and system (2) will become the following delayed feedback control form:

$$\begin{aligned} \dot{x}(t) &= 40(y(t) - x(t)), \\ \dot{y}(t) &= -40x(t) - 33y(t)z(t) + k[y(t) - y(t - \tau)], \\ \dot{z}(t) &= -10 + y^2(t), \end{aligned} \quad (68)$$

which has the equilibria $E_1(\sqrt{10}, \sqrt{10}, -40/33)$, $E_2(-\sqrt{10}, -\sqrt{10}, -40/33)$, and the linearized system of system (68) at $E_{1,2}$ is

$$\begin{aligned} \dot{u}(t) &= u(t) \begin{pmatrix} -40 & a & 0 \\ -40 & k+40 & -33\sqrt{10} \\ 0 & 2\sqrt{10} & 0 \end{pmatrix} \\ &+ u(t - \tau) \begin{pmatrix} 0 & 0 & 0 \\ 0 & -k & 0 \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned} \quad (69)$$

where $u(t) = (x(t), y(t), z(t))^T$.

The characteristic equation of (69) is given by

$$\begin{aligned} \lambda^3 - k\lambda^2 + (660 - 40k)\lambda + 26400 \\ + (k\lambda^2 + 40k\lambda)e^{-\lambda\tau} = 0. \end{aligned} \quad (70)$$

When $k = 0$, $\tau = 0$ (70) becomes

$$\lambda^3 + 25\lambda^2 + 1660\lambda + 26400 - 25\lambda^2 - 1000\lambda = 0. \quad (71)$$

From Routh-Hurwitz criterion, it is easy to know that the equilibrium $E_{1,2}$ is unstable, and, using Matlab 7.8.0, we can get chaotic phenomena (see Figure 1).

In the following, we discuss the condition under which Hopf bifurcation occurs at the equilibria $E_{1,2}$; supposing $i\omega$ is a root of (71), we get

$$\begin{aligned} \omega^6 - 2(660 - 40k)\omega^4 \\ + [(660 - 40k)^2 - 226400(-k) - (40k)^2]\omega^2 \\ + 696960000 = 0. \end{aligned} \quad (72)$$

FIGURE 2: Bifurcation diagram of system (2) with $\tau \in (0.0001, 0.15)$.

Let $z = \omega^2$; (72) becomes

$$\begin{aligned} & z^3 - 2(660 - 40k)z^2 \\ & + [(660 - 40k)^2 - 226400(-k) - (40k)^2]z \quad (73) \\ & + 696960000 = 0. \end{aligned}$$

Let $p = -2(660 - 40k)$, $q = [(660 - 40k)^2 - 226400(-k) - (40k)^2]$, and $r = a_0^2$, $K = -25$; then $\Delta = p^2 - 3q = [-2(660 - 40k)]^2 - 3[(660 - 40k)^2 - 226400(-k) - (40k)^2] = 22735600 > 0$.

Equation (73) has two real roots:

$$\begin{aligned} z_1 &\approx 3107.7, \\ \omega_1 &\approx 55.7467, \\ \tau_1^j &\approx 0.0292 + \frac{2j\pi}{\omega_1}, \\ h'(z_1) &\approx 8.7739e^{+006} > 0, \\ z_2 &\approx 591.5, \\ \omega_2 &\approx 24.3208, \\ \tau_2^j &\approx 0.0397 + \frac{2j\pi}{\omega_2}, \\ h'(z_2) &\approx -2.4423e^{+006} < 0. \end{aligned} \quad (74)$$

From Lemma 3 we have

$$\begin{aligned} \frac{\operatorname{Re} \lambda(\tau_1^j)}{d\tau} &> 0, \\ \frac{\operatorname{Re} \lambda(\tau_2^j)}{d\tau} &< 0. \end{aligned} \quad (75)$$

In addition, notice that $\tau_1^0 = 0.0292 < \tau_2^0 = 0.0397 < \tau_1^1 = 0.083 < \tau_2^1 = 0.166 \dots$

By Theorem 4, the equilibria points $E_{1,2}$ are unstable; when $\tau \in [0, \tau_2^0)$ (see Figures 2 and 4), the system varies from the chaos to the inverse double period bifurcation, the unstable bifurcating period solution, and double period

FIGURE 3: Local amplification of Figure 2 with $\tau \in (0.036, 0.04)$.

(a) The trajectories

(b) Phase graph

FIGURE 4: The trajectories and phase graphs of system (2) with $\tau = 0.02 < 0.0397$, $k = -25$ at initial values of phase graphs: $E_1 (3.1, 3.1, -1.1)$ (red), $E_2 (-3.1, -3.1, -1.1)$ (blue).

bifurcation and the chaos occurs from the equilibria $E_{1,2}$ (see Figures 2 and 3). When τ pass through $\tau_1^0 = 0.0397$, Hopf bifurcation occurs. When $\tau \in [\tau_2^0, \tau_1^1)$, the equilibria $E_{1,2}$ are asymptotically stable (see Figure 5). When $\tau > 0.083$, the equilibria $E_{1,2}$ lose their stability and the stable bifurcating period solutions occur (see Figure 6). By the

FIGURE 5: The trajectories and phase graphs of system with $\tau = 0.0405 > 0.0397$, $k = -25$ at initial values of phase graphs: $E_1 (3.1, 3.1, -1.1)$ (red), $E_2 (-3.1, -3.1, -1.1)$ (blue).

FIGURE 6: The trajectories and phase graphs of system (2) with $\tau = 0.09$, $k = -25$ at initial values of phase graphs: $E_1 (3.1, 3.1, -1.1)$ (red), $E_2 (-3.1, -3.1, -1.1)$ (blue).

theory of Hale [11], from formula (62), we also determine the direction of Hopf bifurcation and the other properties of bifurcating periodic solutions. From the formulae in Section 4 we evaluate $C_1(0)$, μ_2 , β_2 , and T_2 ; it follows that

$$\begin{aligned}
 C_1(0) &\approx -0.0102 - 0.0073i, \\
 \mu_2 &\approx -3.0600e^{-005} < 0, \\
 \beta_2 &\approx -0.0204 < 0, \\
 T_2 &\approx -0.0076 < 0.
 \end{aligned} \tag{76}$$

Since $\mu_2 < 0$, $\beta_2 < 0$, and $T_2 < 0$, the Hopf bifurcation is subcritical; these bifurcating period solutions from $E_{1,2}$ are

stable and the period of the bifurcating periodic solutions decreases.

5. Conclusions

In this paper, our goal is to control a chaotic system to be of stable state by applying the time-delayed feedback control method. It is shown that time delays play an important role in controlling the chaotic behaviors of system (1). The sufficient conditions are obtained to ensure local stability of the equilibria E_1 , E_2 and the existence of local Hopf bifurcation. In virtue of functional differential equations and Hassard method, the nature of the Hopf bifurcation of the time delay chaotic system is determined by the designed

controller. Numerical simulation results demonstrate that the new feedback controller using the time delay is a specially effective method to control chaos behavior. Through further investigation, we expect to decide on which bifurcation arises in the chaos system by proper setting on the feedback parameters for building programs to suppress chaos.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors would like to thank the anonymous referee for the very helpful suggestions and comments which led to the improvement of the original paper. And this work is supported by the Education Department of Henan Province (Grant no. 15A110046) and the Research Innovation Project of Zhoukou Normal University (Grant nos. ZKNUA201410 and ZKNU2014126).

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