Research Article

Two Bi-Accelerator Improved with Memory Schemes for Solving Nonlinear Equations

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The present paper is devoted to the improvement of the R -order convergence of with memory derivative free methods presented by Lotfi et al. (2014) without doing any new evaluation. To achieve this aim one more self-accelerating parameter is inserted, which is calculated with the help of Newton's interpolatory polynomial. First theoretically it is proved that the R -order of convergence of the proposed schemes is increased from 6 to 7 and 12 to 14, respectively, without adding any extra evaluation. Smooth as well as nonsmooth examples are discussed to confirm theoretical result and superiority of the proposed schemes.

1. Introduction

Finding the root of a nonlinear equation frequently occurs in scientific computation. Newton's method is the most well-known method for solving nonlinear equations and has quadratic convergence. However, the existence of the derivative in the neighborhood of the required root is compulsory for convergence of Newton's method, which restricts its applications in practice. To overcome this difficulty, Steffensen replaced the first derivative of the function in Newton's iterate by forward finite difference approximation. This method also possesses the quadratical convergence and the same efficiency just like Newton's method. Kung and Traub are pioneers in constructing optimal general multistep methods without memory. Moreover, they conjectured any multistep methods without memory using n function evaluations that may reach the convergence order at most 2^{n-1} [1]. Thus both Newton's and Steffensen's methods are optimal in the sense of Kung and Traub. But the superiority of Steffensen's method over Newton's method is that it is derivative free. So it can be applied to nondifferentiable equations also. To compare iterative methods theoretically, Owtrowski [2] introduced the idea of efficiency index given by $p^{1/n}$, where p is the order of convergence and n number of function evaluations per iteration. In other words we can say that an iterative method with higher efficiency index is more efficient. To improve convergence order as well as efficiency index without adding

any new function evaluations, Traub in his book introduced the method with memory. In fact he changed Steffensen's method slightly as follows (see [3, pp. 185–187]).

x_0, γ_0 are given suitably:

$$\gamma_k = -\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} \quad \text{for } k \geq 1,$$

$$x_{k+1} = x_k - \frac{\gamma_k f(x_k)^2}{f(x_k + \gamma_k f(x_k)) - f(x_k)}, \quad (k = 0, 1, 2, \dots). \quad (1)$$

The parameter γ_n is called self-accelerator and method (1) has R -order of convergence 2.414. The possibility to increase the convergence order cannot be denied by using more suitable parameters. Many researchers from the last few years, are trying to construct iterative methods without memory which support Kung and Traub conjecture such as [4–13] which are few of them. Although construction of optimal methods without memory is still an active field, from the last one year, many authors are shifting their attention for developing more efficient methods with memory.

In the convergence analysis of the new method, we employ the notation used in Traub's book [3]: if m_k and n_k are null sequences and $m_k/n_k \rightarrow C$, where C is a nonzero constant, we will write $m_k = O(n_k)$ or $m_k \sim Cn_k$. We also use the concept of R -order of convergence introduced

by Ortega and Rheinboldt [14]. Let x_k be a sequence of approximations generated by an iterative method (IM). If this sequence converges to a zero ξ of function f with the R -order $O_R((IM), \xi) \geq \alpha$, we will write

$$e_{k+1} \sim A_{k,\alpha} e_k^\alpha, \quad (2)$$

where $A_{k,\alpha}$ tends to the asymptotic error constant A_α of the iterative method (IM) when $k \rightarrow \infty$.

The rest of the paper is organized as follows: in Section 2 we describe the existing two- and three-point with memory derivative-free schemes and then their convergence orders are accelerated from six to seven and twelve to fourteen, respectively, without doing any extra evaluation. Proposed methods are obtained by imposing one more suitable iterative parameter. The parameter is calculated using Newton's interpolatory polynomial.

The numerical study is also presented in the next section to confirm the theoretical results. Finally, we give the concluding remarks.

2. Brief Literature Review and Improving with Memory Schemes

Two-step (double) and three-step (triple) Newton's methods can be, respectively, written as

$$y_k = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, 2, \dots, \quad (3)$$

$$x_{k+1} = y_k - \frac{f(y_k)}{f'(y_k)},$$

$$y_k = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, 2, \dots,$$

$$z_k = y_k - \frac{f(y_k)}{f'(y_k)}, \quad (4)$$

$$x_{k+1} = z_k - \frac{f(z_k)}{f'(z_k)}.$$

The orders of convergence of the schemes (3) and (4) have been increased to four and eight, respectively, but both of them have no improvement in efficiency as compared to the original Newton's method. One major drawback of the same schemes is that they also involved derivatives. For the purpose of obtaining efficient as well as free from derivatives schemes, Lotfi et al. [15] approximated the derivatives $f'(x_k)$, $f'(y_k)$, and $f'(z_k)$ by Lagrange's interpolatory polynomials of degrees one, two, and three, respectively, which are given by

$$f'(x_k) \approx P'_1(x_k) = \frac{f(x_k)}{x_k - w_k} + \frac{f(w_k)}{w_k - x_k} = f[x_k, w_k], \quad (5)$$

where $w_k = x_k + \gamma f(x_k)$ and $\gamma \in \mathbb{R} - \{0\}$,

$$\begin{aligned} f'(y_k) &\approx P'_2(y_k) \\ &= \frac{2y_k - w_k - y_k}{(x_k - w_k)(x_k - y_k)} f(x_k) \\ &\quad + \frac{2y_k - x_k - y_k}{(w_k - x_k)(w_k - y_k)} f(w_k) \\ &\quad + \frac{2y_k - w_k - x_k}{(y_k - w_k)(y_k - x_k)} f(y_k), \\ f'(z_k) &\approx P'_3(z_k) \\ &= \frac{w_k y_k + w_k z_k + y_k z_k - 2(w_k + y_k + z_k)z_k + 3z_k^2}{(x_k - w_k)(x_k - y_k)(x_k - z_k)} \\ &\quad \cdot f(x_k) \\ &\quad + \frac{x_k y_k + x_k z_k + y_k z_k - 2(x_k + y_k + z_k)z_k + 3z_k^2}{(w_k - x_k)(w_k - y_k)(w_k - z_k)} \\ &\quad \cdot f(w_k) \\ &\quad + \frac{x_k w_k + x_k z_k + w_k z_k - 2(x_k + w_k + z_k)z_k + 3z_k^2}{(y_k - w_k)(y_k - x_k)(y_k - z_k)} \\ &\quad \cdot f(y_k) \\ &\quad + \frac{x_k w_k + x_k y_k + w_k y_k - 2(x_k + w_k + y_k)z_k + 3z_k^2}{(z_k - w_k)(z_k - x_k)(z_k - y_k)} \\ &\quad \cdot f(z_k). \end{aligned} \quad (6)$$

Thus modified versions of schemes (3) and (4), respectively, become

$$y_k = x_k - \frac{f(x_k)}{P'_1(x_k)}, \quad k = 0, 1, 2, \dots, \quad (7)$$

$$x_{k+1} = y_k - \frac{f(y_k)}{P'_2(y_k)},$$

$$y_k = x_k - \frac{f(x_k)}{P'_1(x_k)}, \quad k = 0, 1, 2, \dots,$$

$$z_k = y_k - \frac{f(y_k)}{P'_2(y_k)}, \quad (8)$$

$$x_{k+1} = z_k - \frac{f(z_k)}{P'_3(z_k)}.$$

The authors have shown that without memory methods (7) and (8) preserve the convergence order with reduced number

of function evaluation. Their corresponding error expressions are given by

$$e_{k+1} = \frac{1}{c_1^3} (1 + \gamma c_1)^2 c_2 (c_2^2 - c_1 c_3) e_k^4 + O(e_k^5), \quad (9)$$

$$e_{k+1} = \frac{1}{c_1^7} (1 + \gamma c_1)^4 c_2^2 (c_2^2 - c_1 c_3) \cdot (c_2^3 - c_1 c_2 c_3 + c_1^2 c_4) e_k^8 + O(e_k^9), \quad (10)$$

respectively, where $c_i = f^{(i)}(\xi)/i!$. The above two without memory schemes are optimal in the sense of Kung and Traub. Initially, Traub showed that the convergence of the without memory methods can be increased by use of information from current and previous iterations without adding any evaluation, which are known as with memory methods. To get increased order of convergence the authors in the same paper [15] first replaced γ by γ_k and then $\gamma_k = -1/\bar{f}'(\xi)$ for $k = 1, 2, \dots$, where ξ is the exact root and $\bar{f}'(\xi)$ is the approximation of $f'(\xi)$. In addition, they used the approximation

$$\gamma_k = -\frac{1}{\bar{f}'(\xi)} = -\frac{1}{N_3'(x_k)}, \quad (11)$$

for method (7) and

$$\gamma_k = -\frac{1}{\bar{f}'(\xi)} = -\frac{1}{N_4'(x_k)} \quad (12)$$

for method (8), where

$$\begin{aligned} N_3(t) &= f(x_k) + f[x_k, x_{k-1}](t - x_k) \\ &\quad + f[x_k, x_{k-1}, y_{k-1}](t - x_k)(t - x_{k-1}) \\ &\quad + f[x_k, x_{k-1}, y_{k-1}, w_{k-1}] \\ &\quad \cdot (t - x_k)(t - x_{k-1})(t - y_{k-1}), \\ N_4(t) &= f(x_k) + f[x_k, z_{k-1}](t - x_k) \\ &\quad + f[x_k, z_{k-1}, y_{k-1}](t - x_k)(t - z_{k-1}) \\ &\quad + f[x_k, z_{k-1}, y_{k-1}, x_{k-1}] \\ &\quad \cdot (t - x_k)(t - z_{k-1})(t - y_{k-1}) \\ &\quad + f[x_k, z_{k-1}, y_{k-1}, x_{k-1}, w_{k-1}] \\ &\quad \cdot (t - x_k)(t - z_{k-1})(t - y_{k-1})(t - x_{k-1}), \end{aligned} \quad (13)$$

are Newton's interpolatory polynomial of degrees three and four, respectively. Here the single prime ($'$) denotes the first derivative and double prime ($''$) will later denote the second

derivative. So that the one-parametric version of the methods (7) and (8) can be written as

$$\begin{aligned} w_k &= x_k + \gamma_k f(x_k), & \gamma_{k+1} &= -\frac{1}{N_3'(x_k)}, \\ k &= 0, 1, 2, \dots, \\ y_k &= x_k - \frac{f(x_k)}{P_1'(x_k)}, \\ x_{k+1} &= y_k - \frac{f(y_k)}{P_2'(y_k)}, \end{aligned} \quad (14)$$

$$\begin{aligned} w_k &= x_k + \gamma_k f(x_k), & \gamma_{k+1} &= -\frac{1}{N_4'(x_k)}, \\ k &= 0, 1, 2, \dots, \\ y_k &= x_k - \frac{f(x_k)}{P_1'(x_k)}, \\ z_k &= y_k - \frac{f(y_k)}{P_2'(y_k)}, \\ x_{k+1} &= z_k - \frac{f(z_k)}{P_3'(z_k)}. \end{aligned} \quad (15)$$

The authors showed that the convergence order of methods (14) and (15) is increased from 4 to 6 and from 8 to 12, respectively. The aim of this paper is to find more efficient methods using the same number of evaluations. For this purpose we introduce one more iterative parameter in the above methods; then the modified with memory methods are given by

$$\begin{aligned} w_k &= x_k + \gamma_k f(x_k), & \gamma_{k+1} &= -\frac{1}{N_3'(x_k)}, \\ k &= 0, 1, 2, \dots, \\ y_k &= x_k - \frac{f(x_k)}{P_1'(x_k) + \alpha_k f(w_k)}, \\ x_{k+1} &= y_k - \frac{f(y_k)}{P_2'(y_k) + \alpha_k f(y_k)} \end{aligned} \quad (16)$$

with their error expression

$$\begin{aligned} e_{k+1} &= \frac{1}{c_1^3} (1 + \gamma_k c_1)^2 (\alpha_k c_1 + c_2) \\ &\quad \cdot (\alpha_k^2 c_1 + c_2^2 + c_1 (2\alpha_k c_2 - c_3)) e_k^4 + O(e_k^5), \\ w_k &= x_k + \gamma_k f(x_k), & \gamma_{k+1} &= -\frac{1}{N_4'(x_k)}, \\ k &= 0, 1, 2, \dots, \end{aligned} \quad (17)$$

$$y_k = x_k - \frac{f(x_k)}{P_1'(x_k) + \alpha_k f(w_k)},$$

$$\begin{aligned} z_k &= y_k - \frac{f(y_k)}{P'_2(y_k) + \alpha_k f(y_k)}, \\ x_{k+1} &= z_k - \frac{f(z_k)}{P'_3(z_k) + \alpha_k f(z_k)}. \end{aligned} \quad (18)$$

With its error expression,

$$\begin{aligned} e_{k+1} &= \frac{1}{c_1^7} (1 + \gamma_k c_1)^4 (\alpha_k c_1 + c_2)^2 \\ &\cdot (\alpha_k^2 c_1 + c_2^2 + c_1 (2\alpha_k c_2 - c_3)) \\ &\cdot (\alpha_k c_1^3 + c_2^3 + c_1 c_2 (3\alpha_k c_2 - c_3) \\ &\quad + c_1^3 (3\alpha_k^2 c_2 - \alpha_k c_3 + c_4)) e_k^8 + O(e_k^9). \end{aligned} \quad (19)$$

Since the above error equations contain the iterative parameters both γ_k and α_k , we should approximate these parameters in such a way that they will increase the convergence order. To this end, we approximate γ_k and α_k as follows.

For method (16)

$$\begin{aligned} \gamma_k &= -\frac{1}{c_1} \approx -\frac{1}{\widetilde{c_1}} = -\frac{1}{N'_3(x_k)}, \\ \alpha_k &= -\frac{c_2}{c_1} \approx -\frac{\widetilde{c_2}}{\widetilde{c_1}} = -\frac{N''_4(w_k)}{2N'_4(w_k)}, \end{aligned} \quad (20)$$

and for method 2

$$\begin{aligned} \gamma_k &= -\frac{1}{c_1} \approx -\frac{1}{\widetilde{c_1}} = -\frac{1}{\widetilde{N'_4(x_k)}}, \\ \alpha_k &= -\frac{c_2}{c_1} \approx -\frac{\widetilde{c_2}}{\widetilde{c_1}} = -\frac{\widetilde{N''_5(w_k)}}{2\widetilde{N'_5(w_k)}}, \end{aligned} \quad (21)$$

where

$$\begin{aligned} N_3(t) &= N_3(t; x_k, y_{k-1}, x_{k-1}, w_{k-1}) \\ &= f(x_k) + f[x_k, y_{k-1}](t - x_k) \\ &\quad + f[x_k, y_{k-1}, x_{k-1}](t - x_k)(t - y_{k-1}) \\ &\quad + f[x_k, y_{k-1}, x_{k-1}, w_{k-1}] \\ &\quad \cdot (t - x_k)(t - y_{k-1})(t - x_{k-1}), \\ N_4(t) &= N_4(t; x_k, w_k, y_{k-1}, x_{k-1}, w_{k-1}) \\ &= f(x_k) + f[x_k, w_k](t - x_k) \\ &\quad + f[x_k, w_k, y_{k-1}](t - x_k)(t - w_k) \\ &\quad + f[x_k, w_k, y_{k-1}, x_{k-1}] \\ &\quad \cdot (t - x_k)(t - w_k)(t - y_{k-1}) \\ &\quad + f[x_k, w_k, y_{k-1}, x_{k-1}, w_{k-1}] \\ &\quad \cdot (t - x_k)(t - w_k)(t - y_{k-1})(t - x_{k-1}), \end{aligned}$$

$$\begin{aligned} \widetilde{N}_4(t) &= \widetilde{N}_4(t; x_k, z_{k-1}, y_{k-1}, x_{k-1}, w_{k-1}) \\ &= f(x_k) + f[x_k, z_{k-1}](t - x_k) \\ &\quad + f[x_k, z_{k-1}, y_{k-1}](t - x_k)(t - z_{k-1}) \\ &\quad + f[x_k, z_{k-1}, y_{k-1}, x_{k-1}] \\ &\quad \cdot (t - x_k)(t - z_{k-1})(t - y_{k-1}), \\ &\quad + f[x_k, z_{k-1}, y_{k-1}, x_{k-1}, w_{k-1}] \\ &\quad \cdot (t - x_k)(t - z_{k-1})(t - y_{k-1})(t - x_{k-1}), \\ \widetilde{N}_5(t) &= \widetilde{N}_5(t; x_k, w_k, z_{k-1}, y_{k-1}, x_{k-1}, w_{k-1}) \\ &= f(x_k) + f[x_k, w_k](t - x_k) \\ &\quad + f[x_k, w_k, z_{k-1}](t - x_k)(t - w_k) \\ &\quad + f[x_k, w_k, z_{k-1}, y_{k-1}] \\ &\quad \cdot (t - x_k)(t - w_k)(t - z_{k-1}) \\ &\quad + f[x_k, w_k, z_{k-1}, y_{k-1}, x_{k-1}] \\ &\quad \cdot (t - x_k)(t - w_k)(t - z_{k-1})(t - y_{k-1}), \\ &\quad + f[x_k, w_k, z_{k-1}, y_{k-1}, x_{k-1}, w_{k-1}] \\ &\quad \cdot (t - x_k)(t - w_k)(t - z_{k-1})(t - y_{k-1})(t - x_{k-1}) \end{aligned} \quad (22)$$

are Newton's interpolatory polynomial of degrees three, four, four and five, respectively. Now we denote

$$\begin{aligned} e_k &= x_k - \xi, & e_{k,z} &= z_k - \xi, \\ e_{k,y} &= y_k - \xi, & e_{k,w} &= w_k - \xi, \end{aligned} \quad (23)$$

where ξ is the exact root. Before going to prove the main result, we state the following two lemmas which can be obtained by using the error expression of Newton's interpolation, in the same manner as in [16].

Lemma 1. If $\gamma_k = -1/N'_3(x_k)$ and $\alpha_k = -N''_4(w_k)/2N'_4(w_k)$, then the estimates

- (i) $1 + \gamma_k c_1 \sim (c_4/c_1)e_{k-1,y}e_{k-1,w}e_{k-1}$,
- (ii) $\alpha_k c_1 + c_2 \sim -c_5 e_{k-1,y}e_{k-1,w}e_{k-1}$.

Lemma 2. If $\gamma_k = -1/\widetilde{N'_4(x_k)}$ and $\alpha_k = -\widetilde{N''_5(w_k)}/2\widetilde{N'_5(w_k)}$, then the estimates

- (i) $1 + \alpha_k c_1 \sim -(c_5/c_1)e_{k-1,z}e_{k-1,y}e_{k-1,w}e_{k-1}$,
- (ii) $\beta_k c_1 + c_2 \sim c_6 e_{k-1,z}e_{k-1,y}e_{k-1,w}e_{k-1}$.

The theoretical proof of the order of convergence of the proposed methods is given by the following theorem.

TABLE 1: Comparison of the absolute error in first, second, and third iteration.

Function	Method	$ x_1 - \xi $	$ x_2 - \xi $	$ x_3 - \xi $	COC
f_1 $\xi = 0$ $x_0 = 0.6$	(14) with $\gamma_0 = -1$	$0.67423e - 2$	$0.72063e - 12$	$0.23027e - 72$	6.0653
	(15) with $\gamma_0 = -1$	$0.23448e - 3$	$0.56614e - 45$	$0.86233e - 543$	11.962
	(16) with $\gamma_0 = -1$ and $\alpha_0 = -0.1$	$0.29657e - 2$	$0.28414e - 14$	$0.99885e - 101$	7.1926
	2 with $\gamma_0 = -1$ and $\alpha_0 = -0.1$	$0.98901e - 4$	$0.14013e - 44$	$0.41159e - 628$	14.285
f_2 $\xi = -1$ $x_0 = -1.65$	(14) with $\gamma_0 = 1$	$0.10743e - 0$	$0.40447e - 5$	$0.24898e - 34$	6.6381
	(15) with $\gamma_0 = 1$	$0.23223e - 1$	$0.62952e - 20$	$0.77469e - 241$	11.901
	(16) with $\gamma_0 = 1$ and $\alpha_0 = 0.65$	$0.21296e - 0$	$0.50014e - 4$	$0.73645e - 32$	7.7919
	2 with $\gamma_0 = 1$ and $\alpha_0 = 0.65$	$0.67736e - 1$	$0.22204e - 15$	$0.55387e - 214$	13.725
f_3 $\xi = -1$ $x_0 = -0.8$	(14) with $\gamma_0 = 1$	$0.13702e - 0$	$0.85922e - 4$	$0.99097e - 23$	5.7102
	(15) with $\gamma_0 = 1$	$0.35840e - 1$	$0.25132e - 14$	$0.61401e - 172$	11.954
	(16) with $\gamma_0 = 1$ and $\alpha_0 = -0.3$	$0.19721e - 0$	$0.19721e - 4$	$0.33883e - 29$	6.6720
	2 with $\gamma_0 = 1$ and $\alpha_0 = -0.3$	$0.68741e - 1$	$0.33307e - 15$	$0.26137e - 215$	13.922

Theorem 3. If an initial approximation x_0 is sufficiently close to a simple zero ξ of $f(x)$ and the parameters γ_k and α_k in the iterative scheme (16) and 2 are recursively calculated by the forms given in (20) and (21), respectively, then the R-orders of convergence of with memory schemes (16) and 2 are at least seven and fourteen, respectively.

Proof. First, we assume that the R-orders of convergence of the sequences x_k , w_k , y_k , and z_k are at least m , m_1 , m_2 , and m_3 , respectively. Hence

$$\begin{aligned} e_{k+1} &\sim A_{k,m} e_k^m \sim A_{k,m} (A_{k-1,m} e_{k-1}^m)^m \\ &\sim A_{k,m} A_{k-1,m}^m e_{k-1}^{m^2}, \end{aligned} \quad (24)$$

$$\begin{aligned} e_{k,w} &\sim A_{k,m_1} e_k^{m_1} \sim A_{k,m_1} (A_{k-1,m} e_{k-1}^m)^{m_1} \\ &\sim A_{k,m_1} A_{k-1,m}^{m_1} e_{k-1}^{mm_1}. \end{aligned} \quad (25)$$

Similarly

$$e_{k,y} \sim A_{k,m_2} A_{k-1,m}^{m_2} e_{k-1}^{mm_2}, \quad (26)$$

$$e_{k,z} \sim A_{k,m_3} A_{k-1,m}^{m_3} e_{k-1}^{mm_3}. \quad (27)$$

Now we will prove the results in two parts. First for method (16) and then for 2.

Modified Method I. For method (16), it can be derived that

$$\begin{aligned} e_{k,w} &\sim (1 + \gamma_k c_1) e_k, \\ e_{k,y} &\sim L_1 (1 + \gamma_k c_1) (\alpha_k c_1 + c_2) e_k^2 \quad \text{where } L_1 = \frac{1}{c_1}, \\ e_{k+1} &\sim L_2 (1 + \gamma_k c_1)^2 (\alpha_k c_1 + c_2) e_k^4, \\ \text{where } L_2 &= \frac{1}{c_1^3} (\alpha_k^2 c_1 + c_2^2 + c_1 (2\alpha_k c_2 - c_3)). \end{aligned} \quad (28)$$

Using the results of Lemma 1 in (28), we have

$$\begin{aligned} e_{k,w} &\sim \frac{c_4}{c_1} (A_{k-1,m_2}) (A_{k-1,m_1}) \\ &\cdot (A_{k-1,m}) e_{k-1}^{m_2+m_1+m+1}, \end{aligned} \quad (29)$$

$$\begin{aligned} e_{k,y} &\sim -\frac{c_4 c_5}{c_1} L_1 (A_{k-1,m_2}^2) (A_{k-1,m_1}^2) \\ &\cdot (A_{k-1,m}^2) e_{k-1}^{2m_2+2m_1+2m+2}, \end{aligned} \quad (30)$$

$$\begin{aligned} e_{k+1} &\sim -\frac{c_4^2 c_5}{c_1^2} L_2 (A_{k-1,m_2}^3) (A_{k-1,m_1}^3) \\ &\cdot (A_{k-1,m}^4) e_{k-1}^{3m_2+3m_1+4m+3}. \end{aligned} \quad (31)$$

Now comparing the equal powers of e_{k-1} in three pairs of (29)-(25), (30)-(26), and (31)-(24), we get the following nonlinear system:

$$\begin{aligned} mm_1 - m_2 - m_1 - m - 1 &= 0, \\ mm_2 - 2m_2 - 2m_1 - 2m - 2 &= 0, \\ m^2 - 3m_2 - 3m_1 - 4m - 3 &= 0. \end{aligned} \quad (32)$$

After solving these equations, we get $m = 7$, $m_2 = 3$, and $m_1 = 2$. It confirms the convergence of method (16). This shows the first part.

Modified Method II. For method 2, it can be derived that

$$\begin{aligned} e_{k,w} &\sim (1 + \alpha_k c_1) e_k, \\ e_{k,y} &\sim L_1 (1 + \gamma_k c_1) (\alpha_k c_1 + c_2) e_k^2 \quad \text{where } L_1 = \frac{1}{c_1}, \end{aligned}$$

$$\begin{aligned}
e_{k,z} &\sim L_2 (1 + \gamma_k c_1)^2 (\alpha_k c_1 + c_2) e_k^4, \\
\text{where } L_2 &= \frac{1}{c_1^3} (\alpha_k^2 c_1 + c_2^2 + c_1 (2\alpha_k c_2 - c_3)), \\
e_{k+1} &\sim O_3 (1 + \gamma_k c_1)^4 (\alpha_k c_1 + c_2)^2 e_k^8, \\
\text{where } O_3 &= (\alpha_k^2 c_1 + c_2^2 + c_1 (2\alpha_k c_2 - c_3)) \\
&\quad \times (\alpha_k c_1^3 + c_2^3 + c_1 c_2 (3\alpha_k c_2 - c_3) \\
&\quad + c_1^3 (3\alpha_k^2 c_2 - \alpha_k c_3 + c_4)).
\end{aligned} \tag{33}$$

Now using the results of Lemma 2 in 2, we have

$$\begin{aligned}
e_{k,w} &\sim -\frac{c_5}{c_1} (A_{k-1,m_3}) (A_{k-1,m_2}) (A_{k-1,m_1}) \\
&\quad \cdot (A_{k-1,m}) e_{k-1}^{m_3+m_2+m_1+m+1},
\end{aligned} \tag{34}$$

$$\begin{aligned}
e_{k,y} &\sim -\frac{c_5 c_6}{c_1} L_1 (A_{k-1,m_3}^2) (A_{k-1,m_2}^2) (A_{k-1,m_1}^2) \\
&\quad \cdot (A_{k-1,m}^2) e_{k-1}^{2m_3+2m_2+2m_1+2m+2},
\end{aligned} \tag{35}$$

$$\begin{aligned}
e_{k,z} &\sim \left(\frac{c_5^2 c_6}{c_1^2} \right) L_2 (A_{k-1,m_3}^3) (A_{k-1,m_2}^3) (A_{k-1,m_1}^3) \\
&\quad \cdot (A_{k-1,m}^4) e_{k-1}^{3m_3+3m_2+3m_1+4m+3},
\end{aligned} \tag{36}$$

$$\begin{aligned}
e_{k+1} &\sim \left(\frac{c_5^4 c_6^2}{c_1^4} \right) O_3 (A_{k-1,m_3}^6) (A_{k-1,m_2}^6) (A_{k-1,m_1}^6) \\
&\quad \cdot (A_{k-1,m}^8) e_{k-1}^{6m_3+6m_2+6m_1+8m+6}.
\end{aligned} \tag{37}$$

Comparing the equal powers of e_{k-1} in four pairs of (34)-(25), (35)-(26), (36)-(27), and (37)-(24), we get the following nonlinear system:

$$\begin{aligned}
mm_1 - m_3 - m_2 - m_1 - m - 1 &= 0, \\
mm_2 - 2m_3 - 2m_2 - 2m_1 - 2m - 2 &= 0, \\
mm_3 - 3m_3 - 3m_2 - 3m_1 - 4m - 3 &= 0, \\
m_2 - 6m_3 - 6m_2 - 6m_1 - 8m - 6 &= 0.
\end{aligned} \tag{38}$$

After solving these equations we get $m = 14$, $m_3 = 7$, $m_2 = 3$, and $m_1 = 2$. And thus the proof is completed. $\square$

Note 1. The efficiency index of the proposed method (16) along with (20) is $7^{1/3} = 1.9129$ which is more than $6^{1/3} = 1.8171$ of method (14).

Note 2. The efficiency index of the proposed method 2 along with (21) is $14^{1/4} = 1.9343$ which is more than $8^{1/4} = 1.8612$ of method (15).

3. Numerical Examples and Conclusion

In this section the proposed derivative free methods are applied to solve smooth as well as nonsmooth nonlinear equations and compared with the existing with memory methods. Nowadays, high-order methods are important because numerical applications use high precision in their computations; for this reason numerical tests have been carried out using variable precision arithmetic in *MATHEMATICA 8* with 700 significant digits. The computational order of convergence (COC) is defined by [17, 18]

$$\text{COC} = \frac{\ln(|f(x_k)/f(x_{k-1})|)}{\ln(|f(x_{k-1})/f(x_{k-2})|)}. \tag{39}$$

To test the performance of new method consider the following three nonlinear functions (which are taken from [5, 15]):

- (1) $f_1(x) = \sin(\pi x) e^{x^2 + x \cos(x) - 1} + x \log(x \sin(x) + 1)$,
- (2) $f_2(x) = e^{x^3 - x} - \cos(x^2 - 1) + x^3 + 1$,
- (3) $f_3(x) = \begin{cases} 10(x^4 + x), & x < 0 \\ -10(x^3 + x), & x \geq 0. \end{cases}$

The absolute errors for the first three iterations are given in Table 1. In the table $ae \pm b$ stands for $a \times 10^{\pm b}$. Note that a large number of three-step derivative-free (with and without memory) methods are available in the literature. But the methods which have been tested for nonsmooth functions are rare and this clearly proves the significance of this paper.

The effectiveness of the new proposed derivative free with memory methods is confirmed by comparing this with the existing memory family. The numerical results shown in Table 1 are in concordance with the theory developed here. From the theoretical result, we can conclude that the order of convergence of the without memory family can be made more higher than the existing with memory family by imposing more self-accelerating parameter without any additional calculations and the computational efficiency of the presented with memory method is high. The R -orders of convergence are increased from 6 to 7 and 12 to 14 in accordance with the quality of the applied accelerating method proposed in this paper. We can see that the self-accelerating parameters play a key role in increasing the order of convergence of the iterative method.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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