Research Article

Nonlinear Dynamics of a Nutrient-Phytoplankton Model with Time Delay

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Received 1 April 2015; Accepted 9 June 2015

Academic Editor: Luca Guerrini

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We consider a nutrient-phytoplankton model with a Holling type II functional response and a time delay. By selecting the time delay used as a bifurcation parameter, we prove that the system is stable if the delay value is lower than the critical value but unstable when it is above this value. First, we investigate the existence and stability of the equilibria, as well as the existence of Hopf bifurcations. Second, we consider the direction, stability, and period of the periodic solutions from the steady state based on the normal form and the center manifold theory, thereby deriving explicit formulas. Finally, some numerical simulations are given to illustrate the main theoretical results.

1. Introduction

Ecological systems are characterized by the relationships between species and their natural environment. One of the key factors that affect population dynamics is predator. Due to its universal existence and importance in nature, the dynamic interactions between predators and their prey have remained one of the dominant themes in ecological population dynamics since the origin of this discipline [1–4].

Phytoplankton has vital effects in aquatic ecosystems where it plays a significant role as the base of the food chain. Phytoplankton controls the global carbon cycle which has an important effect on climate regulation [5]. In some situations, marine waters, reservoir, and lakes may exhibit plankton or algal blooms [6, 7], where drinking water may be polluted and algal toxins can affect human health through the food chain, while the marine biodiversity may also be decreased.

Nutrient-phytoplankton systems have a very important effect on aquatic ecosystems as part of predator-prey systems. Recently, many researchers have investigated a predator-prey model that involves nutrients, phytoplankton, and zooplankton [8–11], and various studies have simulated the dynamics of plankton systems.

Huppert et al. [12] proposed a generic bottom-up nutrient-phytoplankton model:

$$\begin{aligned}\dot{N} &= a - b\beta(t)NP - eN, \\ \dot{P} &= c\beta(t)NP - dP,\end{aligned}\tag{1}$$

where the forced NP compartment model comprises two variables: the nutrient concentration N and the phytoplankton biomass P . The initial conditions for this model are $N(0) = N_0 > 0$ and $P(0) = P_0 > 0$. a is an external source of nutrients that flows into the system from the environment, b and c are uptake rates, and e is the loss rate of nutrients. Seasonal environment conditions such as the water temperature, salinity, light, and thermocline depth are often important to different degrees depending on the system under investigation. The modulation of phytoplankton growth β can be represented as a periodic function $\beta(t) = \beta(t + \tau)$, where τ is the period of forcing, which is assumed to be annual in the present study. Analyses and numerical simulations can obtain insights into the mechanism of bloom recurrence, where modifications to the equations via

the inclusion of appropriate functional forms can generate more realistic dynamics.

Mäler [13] considered a better formulation of the dynamics of the eutrophication process:

$$\frac{dx}{dt} = a - bx + \frac{x^2}{1 + x^2}, \quad (2)$$

where a is the runoff of nutrients into the lake and b is the constant rate of the natural removal of nutrients. After reducing the levels of green plants, the bottom sediments will be more vulnerable to winds, waves, and bottom-feeding fishes. Finally, nutrient sediments are released into the water and they contribute to the further growth of phytoplankton. The term $x^2/(1 + x^2)$ represents the feedback from the bottom sediments. Depending on the numerical values for the parameters and specific simulations of the system, different negative externalities can affect the system in various ways.

In this study, based on the idea of Huppert et al. [12] and Mäler [13], we consider a nutrient-phytoplankton model with delay:

$$\begin{aligned} \frac{dx}{dt} &= I - bx + \frac{x^2}{1 + x^2} - axy, \\ \frac{dy}{dt} &= akx(t - \tau)y - my - f_y \frac{y}{H_y + y}, \end{aligned} \quad (3)$$

where two variables x and y represent the concentrations of nutrients and phytoplankton, respectively. All of the parameters are positive constants, where τ is a positive delay, that is, the time required to convert nutrients into phytoplankton, I is a constant concentration, that is, the nutrient runoff from the environment, b is the loss rate of the nutrient concentration, a is the predation rate of nutrients by phytoplankton, k denotes the maximum nutrient intake rate of phytoplankton, m is the mortality of phytoplankton, f_y represents the maximum predation rate of zooplankton on phytoplankton, and H_y is the half-saturation constant for phytoplankton, and $f_y/(H_y + y)$ represents the predation of zooplankton on phytoplankton. The initial conditions are as follows: $x(\theta) = \phi(\theta) \geq 0$, $y(\theta) = \psi(\theta) \geq 0$, $\theta \in [-\tau, 0)$, $\phi(0) > 0$, and $\psi(0) > 0$, where $\phi(\theta)$, $\psi(\theta)$ are continuous bounded functions in the interval $[-\tau, 0)$.

Huppert et al. [12] described the dynamics of the system above without considering the effect of delay. Time delay plays an important role in reflecting the real dynamical behaviors of biological systems and many studies have addressed this issue in recent years [14–23].

In this study, in order to investigate the effects of a time delay on the system, we selected the delay τ as a bifurcation parameter. The remainder of this paper is organized as follows. In Section 2, we prove that positive equilibria exist under certain conditions and we then analyze the local stability of the boundary equilibrium. We also analyze the stability of the positive equilibria and the critical conditions when the Hopf bifurcation occurs. In Section 3, we derive an explicit algorithm and sufficient conditions for determining the direction of the Hopf bifurcation and the stability of the periodic solutions. In Section 4, we present some numerical

simulations to illustrate our theoretical results. In Section 5, we give some brief conclusions.

2. Stability of the Positive Equilibria and Existence of the Hopf Bifurcation

2.1. The Existence of Positive Equilibria. In this section, we consider the existence of the positive equilibria for system (3). We define $f(x, y) = I - bx + (x^2/(1 + x^2)) - axy$, $g(x, y) = akxy - my - f_y(y/(H_y + y))$. The equations for the equilibria can be depicted as follows:

$$\begin{aligned} f(x, y) &= 0, \\ g(x, y) &= 0. \end{aligned} \quad (4)$$

According to $f(x, y) = 0$, a vertical isocline can be obtained, $l_1: y = (1/ax)(I - bx + (x^2/(1 + x^2)))$, which can be written as $F(x) = (1/ax)(I - bx + (x^2/(1 + x^2)))$. For l_1 , when the value of x is close to positive infinite, the value of $F(x)$ is close to $-b/a$. By contrast, when the value of x approaches zero from the right-hand side, the value of $F(x)$ is approximately positive infinite. Due to the vertical isocline, l_1 is continuous and derivable when $x \in (0, +\infty)$ and $x \in (-\infty, 0)$; thus we can compute the derivative of $F(x)$; that is, $F'(x) = -((I + 1)x^4 + (2I - 1)x^2 + I)/(1 + x^2)^2 ax^2$, and four roots exist when $F'(x) = 0$, as follows:

$$\begin{aligned} x_1 &= \sqrt{-(2I + 2)(-1 + 2I - \sqrt{(1 - 8I)})}, \\ x_2 &= -\sqrt{-(2I + 2)(-1 + 2I - \sqrt{(1 - 8I)})}, \\ x_3 &= \sqrt{-(2I + 2)(-1 + 2I + \sqrt{(1 - 8I)})}, \\ x_4 &= -\sqrt{-(2I + 2)(-1 + 2I + \sqrt{(1 - 8I)})}. \end{aligned} \quad (5)$$

From the analysis above, (i) if $I \geq 1/8$, we can obtain $F'(x) \leq 0$ for an arbitrary real number x ; thus $F(x)$ decreases monotonically when $x \in (0, +\infty)$; (ii) if $I < 1/8$, four real number roots exist when $F'(x) = 0$. Obviously, $x_1 > x_3 > 0 > x_4 > x_2$. According to the second derivative of $F(x)$, we can find that x_1, x_4 are the points with the extreme maximum value and x_2, x_3 are the points with the extreme minimum value. We can find that $F(x)$ decreases monotonically when $x \in (0, x_3) \cup (x_1, +\infty)$ and increases monotonically when $x \in (x_3, x_1)$.

According to $g(x, y) = 0$, the horizontal isocline $l_2: y = (f_y/(akx - m)) - H_y$ and $y = 0$, which can be written as $G(x) = (f_y/(akx - m)) - H_y$ and $G(x) = 0$. When the horizontal isocline $l_2: G(x) \neq 0$, that is, $G(x) = (f_y/(akx - m)) - H_y$, it is easy to find that the line, $x = m/ak$, is the asymptote of $G(x)$. Obviously, $G(x)$ is continuous when $x \in (0, m/ak) \cup (m/ak, +\infty)$, and we can find that $G(x)$ decreases monotonically when $x \in (0, m/ak) \cup (m/ak, +\infty)$. For $G(x)$, $G(x) < 0$ holds when $x \in (0, m/ak) \cup ((f_y + mH_y)/akH_y, +\infty)$, whereas $G(x) > 0$ holds when $x \in (m/ak, (f_y + mH_y)/akH_y)$.

First, when the system comprises only nutrients without phytoplankton in the system, according to the vertical isocline, $l_1: F(x) = (1/ax)(I - bx + (x^2/(1+x^2)))$, and according to the horizontal isocline, $l_2: G(x) = 0$. From the analysis above, if $I \geq 1/8$, we find that there is always a boundary equilibrium. If $I < 1/8$, one boundary equilibrium exists at least. The boundary equilibrium can be written as $E_0(x_0, 0)$, $x_0 > 0$.

In addition, according to the vertical isocline, $l_1: F(x) = (1/ax)(I - bx + (x^2/(1+x^2)))$, and according to the horizontal isocline, $l_2: G(x) = (f_y/(akx - m)) - H_y$. From the discussion above, if $I \geq 1/8$, we find that a positive equilibrium exists in the system when the condition that $(f_y + mH_y)/akH_y < x_0$ holds. Thus, we can find that there is no positive equilibrium in the system when the condition that $m/ak \geq x_0$ holds. If $I < 1/8$, one positive equilibrium exists at least when $F(x_1) \geq G(x_1) > 0$ holds. The positive equilibrium can be written as $E_*(x^*, y^*)$.

In the analysis above, we only prove that positive equilibria exist in the system when certain conditions are satisfied but we do not indicate that positive equilibria do not exist when the conditions are not satisfied.

The Jacobian matrix of the system without time delay at the equilibrium $E_0(x_0, 0)$ is

$$T_1 = \begin{pmatrix} -b + \frac{2x_0}{(1+x_0^2)^2} & -ax_0 \\ 0 & aex_0 - m - f_y \frac{1}{H_y} \end{pmatrix}. \quad (6)$$

The index of equilibrium E_0 is +1 when the condition $2x_0 < b(1+x_0^2)$, $aex_0H_y < mH_y + f_y$ is satisfied, which is stable. According to the eigenvalues of T_1 , it is locally asymptotically stable when the above conditions are satisfied. In particular, the equilibrium $E_0(x_0, 0)$ is unstable, when $2x_0 > b(1+x_0^2)$ or $aex_0H_y > mH_y + f_y$.

From the above discussion, we find a positive equilibrium in the system without time delay under some preconditions, defined by $E_*(x^*, y^*)$. The Jacobian matrix of the system at the equilibrium $E_*(x^*, y^*)$ is

$$T_2 = \begin{pmatrix} -b + \frac{2x^*}{(1+x^{*2})^2} - aPy^* & -ax^* \\ aey^* & f_y \frac{y^*}{(H_y + y^*)^2} \end{pmatrix}. \quad (7)$$

When $2y^*(H_y + y^*)^2 + f_y y^*(1+x^{*2})^2 < (b + ay^*)(1+x^{*2})^2(H_y + y^*)^2$, the index of equilibrium N_2 is +1 if $a^2ex^*y^*(H_y + y^*)^2(1+x^{*2})^2 > f_y y^*(2x^* - b(1+x^{*2})^2 - ax^*(1+x^{*2})^2)$. It is locally asymptotically stable using the Routh-Hurwitz criteria when the above conditions are satisfied. In particular, the equilibrium $E_*(x^*, y^*)$ is unstable, when $a^2ex^*y^*(H_y + y^*)^2(1+x^{*2})^2 < f_y y^*(2x^* - b(1+x^{*2})^2 - ax^*(1+x^{*2})^2)$, which is a saddle.

2.2. Local Stability of Equilibrium E_0 . The linearization of (3) at the equilibrium E_0 is

$$\begin{aligned} \frac{dx}{dt} &= \left(-b + \frac{2x_0}{1+(x_0)^2} \right) x - ax_0 y, \\ \frac{dy}{dt} &= \left(akx_0 - m - \frac{f_y}{H_y} \right) y. \end{aligned} \quad (8)$$

Therefore, the characteristic matrix at equilibrium $E_0(x_0, 0)$ is

$$T_0 = \begin{pmatrix} \lambda - \left(-b + \frac{2x_0}{1+(x_0)^2} \right) & ax_0 \\ 0 & \lambda - \left(akx_0 - m - \frac{f_y}{H_y} \right) \end{pmatrix}. \quad (9)$$

Its characteristic equation is

$$\lambda^2 - (b_{11} + b_{22})\lambda + b_{11}b_{22} = 0, \quad (10)$$

where $b_{11} = -b + (2x_0/(1+(x_0)^2))$, $b_{12} = -ax_0$, $b_{21} = 0$, and $b_{22} = akx_0 - m - f_y/H_y$.

Theorem 1. If $-b + (2x_0/(1+(x_0)^2)) < 0$ and $akx_0 - m - f_y/H_y < 0$ hold, the boundary equilibrium E_0 is locally asymptotically stable.

Proof. There are two eigenvalues where $-b + (2x_0/(1+(x_0)^2))$ and $akx_0 - m - f_y/H_y$. When all the roots of the characteristic equation at E_0 are negative, the boundary equilibrium E_0 is a stable node. Thus, the boundary equilibrium E_0 is locally asymptotically stable. The proof is complete. $\square$

2.3. Local Stability and the Hopf Bifurcation of Equilibrium E_* . In the following, we focus on the existence of a local Hopf bifurcation at positive equilibrium $E_*(x^*, y^*)$. If we let $P(x) = x^2/(1+x^2)$, the linearization of (3) at the equilibrium E_* is as follows:

$$\begin{aligned} \frac{dx}{dt} &= (-b + P'(x^*) - ay^*)x - ax^*y, \\ \frac{dy}{dt} &= ak y^* x(t - \tau) + f_y \frac{y^*}{(H_y + y^*)^2} y. \end{aligned} \quad (11)$$

Thus, the characteristic matrix at equilibrium $E_*(x^*, y^*)$ is

$$J_* = \begin{pmatrix} \lambda - (-b + P'(x^*) - ay^*) & ax^* \\ -aky^* e^{-\lambda\tau} & \lambda - \left(f_y \frac{y^*}{(H_y + y^*)^2} \right) \end{pmatrix}. \quad (12)$$

Set $a_{11} = -b + (2x^*/(1+(x^*)^2)) - ay^*$, $a_{12} = -ax^*$, $a_{21} = ak y^*$, and $a_{22} = f_y(y^*/(H_y + y^*)^2)$.

Its characteristic equation is

$$\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21}e^{-\lambda\tau} = 0 \quad (13)$$

or

$$\lambda^2 + A\lambda + Be^{-\lambda\tau} + B_0 = 0, \quad (14)$$

where $A = -(a_{11} + a_{22})$, $B = -a_{12}a_{21}$, and $B_0 = a_{11}a_{22}$.

Theorem 2. For system (3), the positive equilibrium $E_*(x^*, y^*)$ is locally asymptotically stable for all $0 \leq \tau < \tau_0$ and the system undergoes a Hopf bifurcation at the positive equilibrium $E_*(x^*, y^*)$ for $\tau = \tau_k$, $k = 0, 1, 2, \dots$, when the conditions that $-A < 0$ and $B + B_0 > 0$ hold.

Lemma 3. If $\tau = 0$, $-A < 0$, and $B + B_0 > 0$ hold, the positive equilibrium $E_*(x^*, y^*)$ is locally asymptotically stable.

Proof. When $\tau = 0$, the characteristic equation is

$$\lambda^2 + A\lambda + B + B_0 = 0. \quad (15)$$

If $-A < 0$ and $B + B_0 > 0$, then all the roots of characteristic equation (15) have negative real parts. Thus, the positive equilibrium $E_*(x^*, y^*)$ is locally asymptotically stable. The proof is complete. $\square$

Lemma 4. Let

$$w_0 = \sqrt{\frac{-(A^2 - 2B_0)^2 + \sqrt{\Delta}}{2}}, \quad -A < 0, \quad B + B_0 > 0. \quad (16)$$

If

$$\tau = \tau_k = \frac{1}{w_0} \left(\arccos \frac{w_0^2 - B_0}{B} + 2k\pi \right), \quad (17)$$

$$k = 0, 1, 2, \dots,$$

then (13) has a pair of purely imaginary roots $\pm iw_0$.

Proof. When $\tau \neq 0$, we assume that $\lambda = \mu(\tau) + iw(\tau)$ is a root of (13). By substituting this into (13), we obtain

$$\begin{aligned} \mu^2(\tau) - w^2(\tau) + 2\mu(\tau)w(\tau)i + \mu(\tau)A + Aw(\tau)i \\ + Be^{-\mu(\tau)\tau}(\cos \tau w(\tau) - i \sin \tau w(\tau)) + B_0 = 0 \end{aligned} \quad (18)$$

or

$$\begin{aligned} \mu^2(\tau) - w^2(\tau) + \mu(\tau)A + B_0 = -Be^{-\mu(\tau)\tau} \cos \tau w(\tau), \\ 2\mu(\tau)w(\tau) + Aw(\tau) = Be^{-\mu(\tau)\tau} \sin \tau w(\tau). \end{aligned} \quad (19)$$

If we let $\mu(\tau) = 0$ and $w(\tau) = w_0 > 0$, (19) is equal to

$$\begin{aligned} w_0^2 - B_0 &= B \cos \tau w_0, \\ Aw_0 &= B \sin \tau w_0 \end{aligned} \quad (20)$$

and thus

$$w_0^4 + (A^2 - 2B_0)w_0^2 + (B_0)^2 - B^2 = 0. \quad (21)$$

If we let $-A < 0$, $B + B_0 > 0$, we can obtain $4((B_0)^2 - B^2) < 0$; therefore, $\Delta = (A^2 - 2B_0)^2 - 4((B_0)^2 - B^2) > 0$, and then $w_0 = \sqrt{(-(A^2 - 2B_0) + \sqrt{\Delta})/2}$.

However, we can solve the first formula in (20) to obtain

$$\begin{aligned} \tau = \tau_k = \frac{1}{w_0} \left(\arccos \frac{w_0^2 - B_0}{B} + 2k\pi \right), \\ k = 0, 1, 2, \dots \end{aligned} \quad (22)$$

Thus, when $\tau = \tau_k$, (13) has a pair of purely imaginary roots $\pm iw_0$. The proof is complete. $\square$

Lemma 5. For arbitrary $\tau = \tau_k$, $k = 0, 1, 2, \dots$, that satisfy the above conditions, one can find that the derivative of the real number part of the eigenvalue is greater than zero; that is,

$$\left. \frac{d\operatorname{Re}(\lambda(\tau))}{d\tau} \right|_{\tau=\tau_k} > 0, \quad k = 0, 1, 2, \dots \quad (23)$$

We can find that the roots of (13) cross the imaginary axis in a transverse direction from left to right as τ increases.

Proof. Differentiating characteristic equation (13) with respect to τ , we obtain

$$\begin{aligned} 2\lambda(\tau) \frac{d\lambda(\tau)}{d\tau} + A \frac{d\lambda(\tau)}{d\tau} \\ + Be^{-\lambda(\tau)\tau} \left(-\frac{\tau d\lambda(\tau)}{d\tau} - \lambda(\tau) \right) = 0. \end{aligned} \quad (24)$$

Thus, we have

$$\frac{d\lambda(\tau)}{d\tau} = \frac{B\lambda(\tau)e^{-\lambda(\tau)\tau}}{2\lambda(\tau) + A - B\tau e^{-\lambda(\tau)\tau}}. \quad (25)$$

From (20) and (21), we obtain

$$\begin{aligned} \left. \frac{d\lambda(\tau)}{d\tau} \right|_{\tau=\tau_k} &= \frac{iBw_0e^{-iw_0\tau_k}}{2iw_0 + A - B\tau_k e^{-iw_0\tau_k}} \\ &= \frac{iBw_0(\cos w_0\tau_k - i \sin w_0\tau_k)}{2iw_0 + A - B\tau_k(\cos w_0\tau_k - i \sin w_0\tau_k)} \\ &= \frac{Bw_0 \sin w_0\tau_k + iBw_0 \cos w_0\tau_k}{(A - B\tau_k \cos w_0\tau_k) + i(2w_0 + B\tau_k \sin w_0\tau_k)} \\ &= \frac{Aw_0^2 + iw_0(w_0^2 - B_0)}{(A - \tau_k(w_0^2 - B_0)) + i(2w_0 + \tau_k Aw_0)}, \end{aligned}$$

$$\begin{aligned}
& \operatorname{Re} \left(\frac{d\lambda(\tau)}{d\tau} \right) \Big|_{\tau=\tau_k} \\
&= \frac{Aw_0^2 (A - \tau_k (w_0^2 - B_0)) + w_0 (w_0^2 - B_0) (2w_0 + \tau_k Aw_0)}{(A - \tau_k (w_0^2 - B_0))^2 + (2w_0 + \tau_k Aw_0)^2} \\
&= \frac{(A^2 - 2B_0)w_0^2 + 2w_0^4}{(A - \tau_k (w_0^2 - B_0))^2 + (2w_0 + \tau_k Aw_0)^2} \\
&= \frac{(a_{11}^2 + a_{22}^2)w_0^2 + 2w_0^4}{(A - \tau_k (w_0^2 - B_0))^2 + (2w_0 + \tau_k Aw_0)^2} > 0
\end{aligned} \tag{26}$$

because of

$$\frac{d \operatorname{Re}(\lambda(\tau))}{d\tau} \Big|_{\tau=\tau_k} = \operatorname{Re} \left(\frac{d\lambda(\tau)}{d\tau} \right) \Big|_{\tau=\tau_k} \tag{27}$$

and thus

$$\begin{aligned}
& \frac{d \operatorname{Re}(\lambda(\tau))}{d\tau} \Big|_{\tau=\tau_k} \\
&= \frac{(a_{11}^2 + a_{22}^2)w_0^2 + 2w_0^4}{(A - \tau_k (w_0^2 - B_0))^2 + (2w_0 + \tau_k Aw_0)^2} > 0.
\end{aligned} \tag{28}$$

This shows that all of the roots cross the imaginary axis at $i\omega$ from left to right as τ increases. The proof is complete. $\square$

From Lemmas 3–5 and the Hopf bifurcation theorem, the proof is complete.

3. Direction and Stability of the Hopf Bifurcations

In the previous section, we obtained some conditions for the occurrence of Hopf bifurcations. In this section, we consider the direction, stability, and period of the periodic solutions from the steady state, as well as deriving the explicit formulae that determine these factors at the critical value $\tau = \tau_k$, $k = 0, 1, 2, \dots$, using the method introduced by Hassard et al. [24].

If we let $u_1(t) = x(\tau t) - x^*$, $u_2(t) = y(\tau t) - y^*$, $\tau = \mu + \tau_k$, and $Q(x) = f_y(y/(H_y + y))$, system (3) can be written as

$$u'(t) = L(\mu)u_t + f(\mu)u_t, \tag{29}$$

where $u = (u_1, u_2)^T$, $u_t(\theta) = u(t + \theta)$, $\theta \in [-1, 0]$, for $\phi = (\phi_1, \phi_2) \in C([-1, 0], R^2)$, and

$$\begin{aligned}
L(\mu)\phi &= (\mu + \tau_k) \begin{pmatrix} -b + P'(x^*) - ay^* & -ax^* \\ 0 & Q'(y^*) \end{pmatrix} \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 \\ ak y^* & 0 \end{pmatrix} \begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \end{pmatrix},
\end{aligned} \tag{30}$$

$$f(\mu)\phi = (\mu + \tau_k)$$

$$\begin{pmatrix} -a\phi_1(0)\phi_2(0) + \frac{P''(x^*)}{2!}\phi_1^2(0) + \frac{P'''(x^*)}{3!}\phi_1^3(0) \\ ak\phi_1(-1)\phi_2(0) - \frac{Q''(x^*)}{2!}\phi_2^2(0) - \frac{Q'''(x^*)}{3!}\phi_1^3(0) \end{pmatrix} \tag{31}$$

+ H.O.T.,

where H.O.T. denotes “higher order terms.” By the Riesz representation theorem, a function $\eta(\theta, \mu)$ of bounded variation exists for $\theta \in [-1, 0]$ such that

$$L(\mu)\phi = \int_{-1}^0 d\eta(\theta, \mu)\phi(\theta) \tag{32}$$

for $\phi \in C([-1, 0], R^2)$.

In fact, we can choose

$$\begin{aligned}
& \eta(\theta, \mu) \\
&= (\mu + \tau_k) \begin{pmatrix} -b + P'(x^*) - ay^* & -ax^* \\ 0 & Q'(y^*) \end{pmatrix} \delta(\theta) \\
&- (\mu + \tau_k) \begin{pmatrix} 0 & 0 \\ ak y^* & 0 \end{pmatrix} \delta(\theta + 1),
\end{aligned} \tag{33}$$

where

$$\delta(\theta) = \begin{cases} 1, & \theta = 0 \\ 0, & \theta \neq 0. \end{cases} \tag{34}$$

In the following discussion, we omit the H.O.T. term in (31) because determining the direction and stability of the Hopf bifurcation only requires up to the third term. For $\phi \in C^1([-1, 0], R^2)$, we define

$$A(\mu)\phi = \begin{cases} \frac{d\phi}{d\theta}, & \theta \in [-1, 0) \\ L(\mu)\phi, & \theta = 0, \end{cases} \tag{35}$$

$$F(\mu)\phi = \begin{cases} 0, & \theta \in [-1, 0) \\ f(\mu)\phi, & \theta = 0. \end{cases} \tag{36}$$

Then, (29) can be rewritten as

$$\dot{u}_t = A(\mu)u_t + F(\mu)u_t. \tag{37}$$

For $\psi \in C^1([0, 1], (R^2)^*)$, we define the adjoint operator A^* as follows:

$$A^*\psi(s) = \begin{cases} -\frac{d\psi}{ds}, & s \in (0, 1] \\ \int_{-1}^0 \psi(-\theta) d\eta(\theta, 0), & s = 0. \end{cases} \tag{38}$$

For $\psi \in C^1([0, 1], (R^2)^*)$ and $\phi \in C^1([0, 1], R^2)$, we define the bilinear form:

$$\begin{aligned}
\langle \psi, \phi \rangle &= \bar{\psi}(0)\phi(0) \\
&- \int_{-1}^0 \int_{\xi=0}^{\theta} \psi(\xi - \theta) d\eta(\theta)\phi(\xi) d\xi,
\end{aligned} \tag{39}$$

where $\eta(\theta) = \eta(\theta, 0)$. In Section 2, we found that $i\omega_0\tau_k$ and $-i\omega_0\tau_k$ are eigenvalues of $A(0)$ and A^* . Thus, we need to

calculate the eigenvectors of $A(0)$ and A^* that correspond to $i\omega_0\tau_k$ and $-i\omega_0\tau_k$, respectively. Suppose that $q(\theta)$ and $q^*(s)$ are the eigenvectors that correspond to $i\omega_0\tau_k$ and $-i\omega_0\tau_k$, respectively. Let

$$q(\theta) = \begin{pmatrix} 1 \\ C \end{pmatrix} e^{i\omega_0\tau_k\theta}. \quad (40)$$

Using the condition that $A(0)q(\theta) = i\omega_0\tau_k q(\theta)$, that is,

$$\begin{aligned} \tau_k \begin{pmatrix} i\omega_0 + b - P'(x^*) + ay^* & ax^* \\ -aky^* e^{-i\omega_0\tau_k} & i\omega_0 - Q'(y^*) \end{pmatrix} \begin{pmatrix} 1 \\ C \end{pmatrix} \\ = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \end{aligned} \quad (41)$$

thus we can obtain

$$C = -\frac{i\omega_0 + b - P'(x^*) + ay^*}{ax^*}. \quad (42)$$

Similarly, suppose that $q^*(s) = E(1, D)e^{i\omega_0\tau_k s}$ is the eigenvector of A^* that corresponds to $-i\omega_0\tau_k$; then $A^*q^*(s) = -i\omega_0\tau_k q^*(s)$, and from the definition of A^* , we have

$$\begin{aligned} \tau_k \begin{pmatrix} -i\omega_0 + b - P'(x^*) + ay^* & -aky^* e^{i\omega_0\tau_k} \\ ax^* & -i\omega_0 - Q'(y^*) \end{pmatrix} \begin{pmatrix} 1 \\ D \end{pmatrix} \\ = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned} \quad (43)$$

Therefore, we obtain

$$D = \frac{ax^*}{i\omega_0 + Q'(y^*)}. \quad (44)$$

Let $\langle q^*, q \rangle = 1$, $\langle q^*, \bar{q} \rangle = 0$ and due to

$$\begin{aligned} \langle q^*, q \rangle &= \bar{q}^*(0) q(0) - \int_{-1}^0 \int_{\xi=0}^{\theta} \bar{q}^*(\xi - \theta) d\eta(\theta) \\ &\quad \cdot q(\xi) d\xi = \bar{E} \begin{pmatrix} 1 \\ \bar{D} \end{pmatrix} \begin{pmatrix} 1 \\ C \end{pmatrix} \\ &\quad - \int_{-1}^0 \int_{\xi=0}^{\theta} \bar{E} \begin{pmatrix} 1 \\ \bar{D} \end{pmatrix} e^{-i\omega_0\tau_k(\xi-\theta)} d\eta(\theta) \begin{pmatrix} 1 \\ C \end{pmatrix} e^{i\omega_0\tau_k\xi} d\xi \\ &= \bar{E} \left\{ 1 + \bar{D}C - \int_{-1}^0 \theta \begin{pmatrix} 1 \\ \bar{D} \end{pmatrix} d\eta(\theta) \begin{pmatrix} 1 \\ C \end{pmatrix} e^{i\omega_0\tau_k\theta} \right\} \\ &= \bar{E} \left\{ 1 + \bar{D}C - \begin{pmatrix} 1 \\ \bar{D} \end{pmatrix} \tau_k \begin{pmatrix} 0 & 0 \\ ak y^* & 0 \end{pmatrix} \right. \\ &\quad \cdot \left. \begin{pmatrix} 1 \\ C \end{pmatrix} e^{-i\omega_0\tau_k\theta} \right\} = \bar{E} \left\{ 1 + \bar{D}C + \tau_k \bar{D} a k y^* e^{-i\omega_0\tau_k} \right\} \end{aligned} \quad (45)$$

we can obtain

$$E = \frac{1}{1 + \bar{D}C + \tau_k \bar{D} a k y^* e^{i\omega_0\tau_k}}. \quad (46)$$

In the following, we first construct the coordinates to describe the center manifold C_0 at $\mu = 0$ (i.e., $\tau = \tau_k$). We define

$$\begin{aligned} z(t) &= \langle q^*, u_t \rangle, \\ W(t, \theta) &= u_t(\theta) - 2\text{Re}\{z(t) q(\theta)\}. \end{aligned} \quad (47)$$

On the center manifold C_0 , we have

$$W(t, \theta) = W(z(t), \bar{z}(t), \theta), \quad (48)$$

where

$$\begin{aligned} W(z, \bar{z}, \theta) &= W_{20}(\theta) \frac{z^2}{2} + W_{11}(\theta) z\bar{z} + W_{02}(\theta) \frac{\bar{z}^2}{2} \\ &\quad + W_{30}(\theta) \frac{z^3}{6} + \dots \end{aligned} \quad (49)$$

z and $\bar{z}$ are local coordinates for the center manifold C_0 at $\mu = 0$. From (36), (37), (38), (39), and (47), we can obtain

$$\begin{aligned} \dot{z}(t) &= \langle q^*, \dot{u}(t) \rangle \\ &= i\omega_0\tau_k z(t) \\ &\quad + \langle q^*(\theta), F(0)(W(z, \bar{z}, \theta) + 2\text{Re}\{z(t) q(\theta)\}) \rangle \\ &= i\omega_0\tau_k z(t) \\ &\quad + \bar{q}^*(0) f(0)(W(z, \bar{z}, 0) + 2\text{Re}\{z(t) q(0)\}) \\ &\triangleq i\omega_0\tau_k z(t) + g(z, \bar{z}), \end{aligned} \quad (50)$$

where

$$\begin{aligned} g(z, \bar{z}) &= \bar{q}^*(0) f(0)(W(z, \bar{z}, 0) + 2\text{Re}\{z(t) q(0)\}) \\ &= g_{20} \frac{z^2}{2} + g_{11} z\bar{z} + g_{02} \frac{\bar{z}^2}{2} + g_{21} \frac{z^2\bar{z}}{2} + \dots \end{aligned} \quad (51)$$

Based on (37) and (47), we have

$$\begin{aligned} \dot{W} &= \dot{u}_t - 2\text{Re}\{\dot{z}q\} = \dot{u}_t - \dot{z}q - \dot{\bar{z}}\bar{q} = Au_t + Fu_t - \dot{z}q - \dot{\bar{z}}\bar{q} \\ &= A(W + 2\text{Re}\{zq\}) + F(W + 2\text{Re}\{zq\}) \\ &\quad - 2\text{Re}\{g(z, \bar{z}) q(\theta)\} \\ &= \begin{cases} AW - 2\text{Re}\{\bar{q}^*(0) F(0) q(\theta)\}, & -1 \leq \theta < 0 \\ AW - 2\text{Re}\{\bar{q}^*(0) F(0) q(\theta)\} + F(0), & \theta = 0 \end{cases} \\ &\triangleq AW + H(z, \bar{z}, \theta), \end{aligned} \quad (52)$$

where

$$H(z, \bar{z}, \theta) = H_{20}(\theta) \frac{z^2}{2} + H_{11}(\theta) z\bar{z} + H_{02}(\theta) \frac{\bar{z}^2}{2} + \dots \quad (53)$$

Near the origin on the center manifold C_0 , by (49)–(51), we have

$$\begin{aligned} \dot{W}(z, \bar{z}) &= W_z \dot{z} + W_{\bar{z}} \dot{\bar{z}} \\ &= z W_{20}(\theta) \dot{z} + W_{11}(\theta) \bar{z} \dot{z} + \bar{z} W_{02}(\theta) \dot{\bar{z}} \\ &\quad + W_{11}(\theta) z \dot{\bar{z}} + O(|(z, \bar{z})|^3) \\ &= z W_{20}(\theta) (i\omega_0 \tau_k z + g(z, \bar{z})) \\ &\quad + W_{11}(\theta) \bar{z} (i\omega_0 \tau_k z + g(z, \bar{z})) \\ &\quad + \bar{z} W_{02}(\theta) (-i\omega_0 \tau_k z + \bar{g}(z, \bar{z})) \\ &\quad + W_{11}(\theta) z (-i\omega_0 \tau_k z + \bar{g}(z, \bar{z})) \\ &\quad + O(|(z, \bar{z})|^3) \\ &= i\omega_0 \tau_k W_{20}(\theta) z^2 - i\omega_0 \tau_k W_{02}(\theta) \bar{z}^2 \\ &\quad + O(|(z, \bar{z})|^3). \end{aligned} \quad (54)$$

By (49), (52), (53), and (54) and by comparing the coefficients, we have

$$(2i\omega_0 \tau_k I - A(0)) W_{20}(\theta) = H_{20}(\theta), \quad (55)$$

$$-A(0) W_{11}(\theta) = H_{11}(\theta), \quad (56)$$

$$(A(0) + 2i\omega_0 \tau_k I) W_{02}(\theta) = -H_{02}(\theta). \quad (57)$$

We consider

$$\begin{aligned} q(0) &= \begin{pmatrix} 1 \\ C \end{pmatrix}, \\ q^*(0) &= E(1 \ D) \end{aligned} \quad (58)$$

and because of

$$\begin{aligned} u_t(\theta) &= (u_{1t}(\theta), u_{2t}(\theta))^T \\ &= W(t, \theta) + 2\text{Re}\{z(t)q(\theta)\} \\ &= W(t, \theta) + zq(\theta) + \bar{z}\bar{q}(\theta) \end{aligned} \quad (59)$$

we obtain

$$u_t(\theta) = W(z, \bar{z}, \theta) = z \begin{pmatrix} 1 \\ C \end{pmatrix} e^{i\omega_0 \tau_k \theta} + \bar{z} \begin{pmatrix} 1 \\ \bar{C} \end{pmatrix} e^{-i\omega_0 \tau_k \theta} + W_{20}(\theta) \frac{z^2}{2} + W_{11}(\theta) z\bar{z} + W_{02}(\theta) \frac{\bar{z}^2}{2} + W_{30}(\theta) \frac{z^3}{6} + \dots,$$

$$u_{1t}(0) = z + \bar{z} + W_{20}^{(1)}(0) \frac{z^2}{2} + W_{11}^{(1)}(0) z\bar{z} + W_{02}^{(1)}(0) \frac{\bar{z}^2}{2} + \dots,$$

$$u_{2t}(0) = Cz + \bar{C}\bar{z} + W_{20}^{(2)}(0) \frac{z^2}{2} + W_{11}^{(2)}(0) z\bar{z} + W_{02}^{(2)}(0) \frac{\bar{z}^2}{2} + \dots,$$

$$u_{1t}(-1) = e^{-i\omega_0 \tau_k} z + e^{i\omega_0 \tau_k} \bar{z} + W_{20}^{(1)}(-1) \frac{z^2}{2} + W_{11}^{(1)}(-1) z\bar{z} + W_{02}^{(1)}(-1) \frac{\bar{z}^2}{2} + \dots,$$

$$F(0) = \tau_k \begin{pmatrix} -au_{1t}(0)u_{2t}(0) + \frac{P''(x^*)}{2!}u_{1t}^2(0) + \frac{P'''(x^*)}{3!}u_{1t}^3(0) \\ aku_{1t}(-1)u_{2t}(0) - \frac{Q''(y^*)}{2!}u_{2t}^2(0) - \frac{Q'''(y^*)}{3!}u_{2t}^3(0) \end{pmatrix},$$

$$g(z, \bar{z}) = \bar{q}^*(0) f(0) (W(z, \bar{z}, 0) + 2\text{Re}\{z(t)q(0)\}) = \bar{E}\tau_k(1, \bar{D})$$

$$\begin{aligned} &\cdot \begin{pmatrix} -au_{1t}(0)u_{2t}(0) + \frac{P''(x^*)}{2!}u_{1t}^2(0) + \frac{P'''(x^*)}{3!}u_{1t}^3(0) \\ aku_{1t}(-1)u_{2t}(0) - \frac{Q''(y^*)}{2!}u_{2t}^2(0) - \frac{Q'''(y^*)}{3!}u_{2t}^3(0) \end{pmatrix} \\ &= \bar{E}\tau_k \left\{ -a \left[z + \bar{z} + W_{20}^{(1)}(0) \frac{z^2}{2} + W_{11}^{(1)}(0) z\bar{z} + W_{02}^{(1)}(0) \frac{\bar{z}^2}{2} + \dots \right] \right\} \end{aligned}$$

$$\begin{aligned}
& \cdot \left[Cz + \bar{C}\bar{z} + W_{20}^{(2)}(0) \frac{z^2}{2} + W_{11}^{(2)}(0) z\bar{z} + W_{02}^{(2)}(0) \frac{\bar{z}^2}{2} + \dots \right] \\
& + \frac{P''(x^*)}{2!} \left[z + \bar{z} + W_{20}^{(1)}(0) \frac{z^2}{2} + W_{11}^{(1)}(0) z\bar{z} + W_{02}^{(1)}(0) \frac{\bar{z}^2}{2} + \dots \right]^2 \\
& + \frac{P'''(x^*)}{3!} \left[z + \bar{z} + W_{20}^{(1)}(0) \frac{z^2}{2} + W_{11}^{(1)}(0) z\bar{z} + W_{02}^{(1)}(0) \frac{\bar{z}^2}{2} + \dots \right]^3 \\
& + \bar{D} \left[ak \left(e^{-i\omega_0 \tau_k} z + e^{i\omega_0 \tau_k} \bar{z} + W_{20}^{(1)}(-1) \frac{z^2}{2} + W_{11}^{(1)}(-1) z\bar{z} + W_{02}^{(1)}(-1) \frac{\bar{z}^2}{2} + \dots \right) \right] \\
& \cdot \left[Cz + \bar{C}\bar{z} + W_{20}^{(2)}(0) \frac{z^2}{2} + W_{11}^{(2)}(0) z\bar{z} + W_{02}^{(2)}(0) \frac{\bar{z}^2}{2} + \dots \right] - \bar{D} \\
& \cdot \frac{Q''(y^*)}{2!} \left[Cz + \bar{C}\bar{z} + W_{20}^{(2)}(0) \frac{z^2}{2} + W_{11}^{(2)}(0) z\bar{z} + W_{02}^{(2)}(0) \frac{\bar{z}^2}{2} + \dots \right]^2 - \bar{D} \\
& \cdot \frac{Q'''(y^*)}{3!} \left[Cz + \bar{C}\bar{z} + W_{20}^{(2)}(0) \frac{z^2}{2} + W_{11}^{(2)}(0) z\bar{z} + W_{02}^{(2)}(0) \frac{\bar{z}^2}{2} + \dots \right]^3 \Big\}.
\end{aligned} \tag{60}$$

Thus, by comparing the coefficients with (51), we obtain

$$\begin{aligned}
g_{20} &= 2\tau_k \bar{E} \left[-aC + \frac{P''(x^*)}{2!} + \bar{D} \left(akCe^{-i\omega_0 \tau_k} \right. \right. \\
& \quad \left. \left. - \frac{Q''(x^*)}{2!} C^2 \right) \right], \\
g_{11} &= \tau_k \bar{E} \left[-a(C + \bar{C}) + P''(x^*) \right. \\
& \quad \left. + \bar{D} (ak(Ce^{i\omega_0 \tau_k} + \bar{C}e^{-i\omega_0 \tau_k}) - Q''(y^*)C\bar{C}) \right], \\
g_{02} &= 2\tau_k \bar{E} \left[\left(-a\bar{C} + \frac{P''(x^*)}{2!} \right) + \bar{D} \left(ak\bar{C}e^{i\omega_0 \tau_k} \right. \right. \\
& \quad \left. \left. - \frac{P''(x^*)}{2!} \bar{C}^2 \right) \right], \\
g_{21} &= 2\tau_k \bar{E} \left[-a \left(W_{11}^{(2)}(0) + \frac{W_{20}^{(2)}(0)}{2} + \frac{W_{20}^{(1)}(0)}{2} \bar{C} \right. \right. \\
& \quad \left. \left. + W_{11}^{(1)}(0)C \right) + \frac{P''(x^*)}{2!} (W_{20}^{(1)}(0) + 2W_{11}^{(1)}(0)) \right. \\
& \quad \left. + \left(\frac{P'''(x^*)}{3!} \times 3 \right) + \bar{D} ak \left(e^{-i\omega_0 \tau_k} W_{11}^{(2)}(0) \right. \right. \\
& \quad \left. \left. + e^{i\omega_0 \tau_k} \frac{W_{20}^{(2)}(0)}{2} + CW_{11}^{(1)}(-1) + \bar{C} \frac{W_{20}^{(1)}(-1)}{2} \right) \right]
\end{aligned}$$

$$\begin{aligned}
& - \bar{D} \frac{Q''(y^*)}{2!} (W_{20}^{(2)}(0)\bar{C} + 2W_{11}^{(2)}(0)C) - \bar{D} \\
& \cdot \frac{Q'''(y^*)}{3!} \times 3C^2\bar{C} \Big].
\end{aligned} \tag{61}$$

$W_{20}(\theta)$ and $W_{11}(\theta)$ are still unknown; thus, we need to compute them. For $\theta \in [-1, 0)$, we have

$$\begin{aligned}
H(z, \bar{z}, \theta) &= -2\text{Re} \{ \bar{q}^*(0) F(0) q(\theta) \} \\
&= -gq(\theta) - \bar{g}\bar{q}(\theta) \\
&= - \left(g_{20} \frac{z^2}{2} + g_{11} z\bar{z} + g_{02} \frac{\bar{z}^2}{2} + \dots \right) q(\theta) \\
& \quad - \left(\bar{g}_{20} \frac{\bar{z}^2}{2} + \bar{g}_{11} z\bar{z} + \bar{g}_{02} \frac{z^2}{2} + \dots \right) \bar{q}(\theta).
\end{aligned} \tag{62}$$

By comparing the coefficients, we have

$$\begin{aligned}
H_{20}(\theta) &= -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), \\
H_{11}(\theta) &= -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta).
\end{aligned} \tag{63}$$

It follows from (55) that

$$\begin{aligned}
\dot{W}_{20}(\theta) &= 2i\omega_0 \tau_k W_{20}(\theta) + g_{20}q(0) + \bar{g}_{02}\bar{q}(0) \\
&= 2i\omega_0 \tau_k W_{20}(\theta) + g_{20}q(0) e^{i\omega_0 \tau_k \theta} \\
& \quad + \bar{g}_{02}\bar{q}(0) e^{-i\omega_0 \tau_k \theta}.
\end{aligned} \tag{64}$$

Solving for $W_{20}(\theta)$, we obtain

$$W_{20}(\theta) = \frac{ig_{20}q(0)}{w_0\tau_k} e^{i\omega_0\tau_k\theta} + \frac{ig_{02}\bar{q}(0)}{3w_0\tau_k} e^{-i\omega_0\tau_k\theta} + E_1 e^{2i\omega_0\tau_k\theta}. \quad (65)$$

Similarly, we obtain

$$W_{11}(\theta) = \frac{-ig_{11}q(0)}{w_0\tau_k} e^{i\omega_0\tau_k\theta} + \frac{i\bar{g}_{11}\bar{q}(0)}{w_0\tau_k} e^{-i\omega_0\tau_k\theta} + E_2, \quad (66)$$

where E_1 and E_2 are both two-dimensional vectors, which can be determined by setting $\theta = 0$ in H . For $\theta = 0$, we have $H(z, \bar{z}, 0) = -2\text{Re}\{\bar{q}^*(0)F(0)q(0)\} + F(0)$; thus

$$\begin{aligned} H_{20}(0) &= -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) \\ &+ \tau_k \begin{pmatrix} -aC + \frac{P''(x^*)}{2!} \\ akCe^{-i\omega_0\tau_k} - \frac{Q''(x^*)}{2!}C^2 \end{pmatrix}, \\ H_{11}(0) &= -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) \\ &+ \tau_k \begin{pmatrix} -a(C + \bar{C}) + \frac{P''(x^*)}{2!} \times 2 \\ ak(Ce^{i\omega_0\tau_k} + \bar{C}e^{-i\omega_0\tau_k}) - \frac{P''(x^*)}{2!} - 2C\bar{C} \end{pmatrix}. \end{aligned} \quad (67)$$

According to (55) and (56), we have

$$\begin{aligned} \tau_k L_1 W_{20}(0) + \tau_k L_2 W_{20}(-1) \\ = 2i\omega_0\tau_k W_{20}(0) - H_{20}(0), \end{aligned} \quad (68)$$

$$\tau_k L_1 W_{11}(0) + \tau_k L_2 W_{11}(-1) = -H_{11}(0). \quad (69)$$

By substituting (68) and (69) into (65) and (66), respectively, we obtain

$$\begin{aligned} E_1 &= \frac{1}{\tau_k} (2i\omega_0 I - L_1 - L_2 e^{-2i\omega_0\tau_k})^{-1} \left[H_{20}(0) \right. \\ &+ 2g_{20}q(0) + \frac{2\bar{g}_{02}\bar{q}(0)}{3} + \frac{ig_{20}L_1 q(0)}{w_0} \\ &+ \frac{i\bar{g}_{02}L_1 \bar{q}(0)}{3w_0} + \frac{ig_{20}e^{-i\omega_0\tau_k} L_2 q(0)}{w_0} \\ &\left. + \frac{i\bar{g}_{02}e^{i\omega_0\tau_k} L_2 \bar{q}(0)}{3w_0} \right], \\ E_2 &= \frac{1}{\tau_k} (-L_1 - L_2)^{-1} \left[H_{11}(0) - \frac{ig_{11}L_1 q(0)}{w_0} \right. \\ &+ \frac{i\bar{g}_{11}L_1 \bar{q}(0)}{w_0} - \frac{ig_{11}e^{-i\omega_0\tau_k} L_2 q(0)}{w_0} \\ &\left. + \frac{i\bar{g}_{11}e^{i\omega_0\tau_k} L_2 \bar{q}(0)}{w_0} \right], \end{aligned} \quad (70)$$

where I is the 2×2 identity matrix. From the above, we already know $W_{20}(\theta)$ and $W_{11}(\theta)$, and g_{21} can be expressed in terms of the parameters and delay τ ; thus, we can compute the values as follows:

$$\begin{aligned} c_1(0) &= \frac{i}{2w_0\tau_k} \left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3} \right) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{\text{Re}(c_1(0))}{\text{Re}\lambda'(\tau_k)}, \\ \beta_2 &= 2\text{Re}(c_1(0)), \\ T_2 &= -\frac{\text{Im}\{c_1(0)\} + \mu_2 \text{Im}\{\lambda'(\tau_k)\}}{w_0\tau_k}. \end{aligned} \quad (71)$$

Theorem 6. μ_2 , β_2 , and T_2 are defined above; thus the following are true:

- (i) if $\mu_2 > 0$ ($\mu_2 < 0$), then the Hopf bifurcation is supercritical (subcritical);
- (ii) if $\beta_2 < 0$ ($\beta_2 > 0$), then the periodic solutions are stable (unstable);
- (iii) if $T_2 > 0$ ($T_2 < 0$), then the period of the bifurcating periodic solution of system (2) increases (decreases).

4. Numerical Simulation

According to the analysis above, the positive equilibrium does not always exist. Thus, based on the numerical technology, we can obtain the positive equilibria that exist under certain conditions. When some parameters are set, we select $b = 0.3$, $a = 0.5$, $k = 0.75$, $m = 0.2$, $f_y = 0.6$, and $H_y = 4$, and we can obtain the different results shown in Figure 1. Figure 1(a) shows that different nutrient concentrations correspond to various vertical isoclines and we can determine the different intersection points (i.e., equilibria). Figure 1(b) shows the stability with the change in the nutrient concentration when the other parameters are fixed, where the blue solid line shows that the positive equilibrium is unstable, whereas the red solid line shows that the positive equilibrium is stable.

Based on the two previous sections, we know that the stability of the boundary equilibria E_0 is relatively simple. However, the stability of the positive equilibria E_* is complex. We studied the effect of a time delay τ on the positive equilibria of system (3) and our results showed that the time delay has a vital role in system. The critical value τ_0 can be obtained by $\tau_0 = (1/w_0) \arccos((w_0^2 - B_0)/B)$. We set the parameter $I = 1.3$ and the others are the same as the above. By combining the analysis above, we can obtain the critical value: $\tau_0 = 3.91$. The numerical solutions for nutrients are shown in Figures 2(a) and 2(b). In order to study the effect of a time delay, we set τ as 0 and 1 initially, and Figure 2(a) shows that the solution converged to a positive equilibrium in the end. The positive equilibrium is locally asymptotically stable in system (3) without delay. The oscillation does not occur when the system had a delay and the parameter $\tau < \tau_0 = 3.91$.

FIGURE 1: (a) The intersections between the vertical isocline and horizontal isocline with different nutrient concentrations I , where the red solid line represents the horizontal isocline, the grey dashed line represents the asymptote of the horizontal isocline, and the other lines with different colors represent the vertical isocline that corresponds to different nutrient concentrations I . (b) The stability of the positive equilibrium with the change in the nutrient concentration I .

FIGURE 2: The solution for nutrients where $I = 1.3$: (a) $\tau = 0$ (red dashed line), $\tau = 1$ (blue solid line); (b) $\tau = 0$ (red dashed line), $\tau = 4$ (blue solid line).

The Hopf bifurcation occurred at the critical value of τ_0 . Thus, the oscillation will occur if $\tau > \tau_0 = 3.91$. The numerical solutions for nutrients are shown in Figure 2(b), and we set τ as 0 and 4. Figure 2(b) shows that the oscillation occurs when $\tau = 4$.

Furthermore, in order to study the relationship between nutrients and phytoplankton, we set the parameter τ as 5 and the numerical solutions for nutrients and phytoplankton are shown in Figure 3. Figure 3(a) shows the relationship

between the spatial distributions of nutrients and phytoplankton, which indicates that the density of phytoplankton reaches its maximal value when the concentration of nutrients reaches its minimal value. The density of phytoplankton affects the concentration of nutrients. As the density of phytoplankton increases, the density of nutrients decreases. Therefore, the density of nutrients and phytoplankton are mutually constrained. The phase of system (3) is shown in Figure 3(b).

FIGURE 3: (a) Solutions for nutrients and phytoplankton where $I = 1.3$, $\tau = 5$. (b) Phase diagram.

5. Conclusion

In this study, we considered a biological system that comprises nutrients and phytoplankton, where we focused on the effects of a delay on the system dynamics.

In Section 2, we showed that the positive equilibrium E_* is locally asymptotically stable if there is no time delay. However, as the time delay increases above the critical value, the equilibrium state becomes unstable and a Hopf bifurcation occurs. Using the method introduced by Hassard et al. [24], we derived the direction and stability of the Hopf bifurcation.

Based on the computer simulation, we showed that the relationship between nutrients and phytoplankton is mutually constrained. Thus, an abundance of nutrients leads to a major increase of phytoplankton, which depleted the nutrients via consumption. Due to the effect of time delay, the system shows oscillation. It reveals that time delay has a vital effect on the system.

The main limitations of this study are that the main theoretical results were confirmed by numerical simulations with hypothetical parameter values. Thus, we aim to obtain some real data to confirm the validity of our system. Furthermore, we did not consider the diffusion and advection of nutrients and phytoplankton; therefore, these aspects should be considered comprehensively in our future research.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Grant no. 31170338), by the Key Program of Zhejiang Provincial Natural Science Foundation of China (Grant no. LZ12C03001), and by the National Key Basic Research Program of China (973 Program, Grant no. 2012CB426510).

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