Research Article

On Factorizations of Upper Triangular Nonnegative Matrices of Order Three

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Let $T_3(\mathbb{N}_0)$ denote the semigroup of 3×3 upper triangular matrices with nonnegative integral-valued entries. In this paper, we investigate factorizations of upper triangular nonnegative matrices of order three. Firstly, we characterize the atoms of the subsemigroup S of the matrices in $T_3(\mathbb{N}_0)$ with nonzero determinant and give some formulas. As a consequence, problems 4a and 4c presented by Baeth et al. (2011) are each half-answered for the case $n = 3$. And then, we consider some factorization cases of matrix A in S with $\rho(A) = 1$ and give formulas for the minimum factorization length of some special matrices in S .

1. Introduction and Preliminaries

Upper triangular matrices are an important class of matrices, which lead to a broader study of all integral-valued matrices. There are many papers in the literature considering these matrices and similar topics.

Factoring such matrices plays a vital role in the study of upper triangular matrices (see [1]). The problem of factoring matrices was studied by Cohn [2] as early as 1963. Later, Jacobson and Wisner [3, 4] and Ch'uan and Chuan [5, 6] investigated these factorization problems in the context of semigroups of matrices. Motivated by these results, Baeth et al. [7] applied the concepts of contemporary factorization theory to semigroups of integral-valued matrices and calculated certain important invariants to give a sense of how unique or nonunique factorization is in each of these semigroups. In [7], Baeth et al. presented six open problems.

In this paper, we will investigate factorizations of upper triangular nonnegative matrices of order three. Also, we will consider open problem 4 presented in [7].

Throughout this paper, $\mathbb{N}$ will denote the set of all positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Also, $T_3(\mathbb{N}_0)$ will denote the semigroup of 3×3 upper triangular matrices with nonnegative integral-valued entries.

In the following, analogous with [7] or [8], some concepts and preliminaries are recalled.

A *semigroup* is a pairing $(S, \cdot)$ where S is a set and $\cdot$ is an associative binary operation on S . When the binary operation is clear from context and $A, B \in S$, we will simply write AB instead of $A \cdot B$. If S contains an element I such that $AI = IA = A$ for all $A \in S$, then I is the *identity* of S .

Let S be a semigroup with identity I . An element $A \in S$ is a *unit* of S if there exists an element $B \in S$ such that $AB = BA = I$. A nonunit $A \in S$ is called an *atom* of S if whenever $A = BC$ for some elements $B, C \in S$, either B or C is a unit of S . The semigroup S is said to be *atomic* provided each nonunit element in S can be written as a product of atoms of S .

Let S denote an atomic semigroup and let A be a nonunit element of S . The set

$$\mathcal{L}(A) = \{t : A = A_1 A_2 \cdots A_t \text{ with each } A_i \text{ an atom of } S\} \quad (1)$$

is called the *set of lengths* of A .

We denote by $L(A) = \sup \mathcal{L}(A)$ the longest (if finite) factorization length of A and $l(A) = \min \mathcal{L}(A)$ the minimum factorization length of A . The *elasticity* of A , denoted by

$$\rho(A) = \frac{L(A)}{l(A)}, \quad (2)$$

gives a coarse measure of how far away A is from having unique factorization. It is not hard to see that if A has a unique factorization $A = A_1 A_2 \cdots A_t$, whence $\mathcal{L}(A) = \{t\}$ and so

$$l(A) = L(A) = t, \quad \rho(A) = \frac{t}{t} = 1. \quad (3)$$

The elasticity of the semigroup S , denoted by $\rho(S)$, is given by

$$\rho(S) = \sup \{\rho(A) : A \in S\}. \quad (4)$$

If $\mathcal{L}(A) = \{t_1, t_2, \dots\}$ with $t_i < t_{i+1}$ for each i , then the delta set of A is given by

$$\Delta(A) = \{t_{i+1} - t_i : t_i, t_{i+1} \in \mathcal{L}(A)\} \quad (5)$$

and $\Delta(S) = \cup_{A \in S} \Delta(A)$.

This paper will be divided into two sections. In Section 2, we will consider the semigroup S of upper triangular 3×3 matrices with nonnegative entries and nonzero determinant. Firstly, we characterize the atoms of the subsemigroup S of the matrices in $T_3(\mathbb{N}_0)$ with nonzero determinant and give some formulas. As a result, problems 4a and 4c presented by Baeth et al. in [7] are each half-answered for the case $n = 3$. And then, we consider some factorization cases of matrix A in S with $\rho(A) = 1$ and give formulas for the minimum factorization length of some special matrices in S .

2. Upper Triangular Nonnegative Matrices of Order Three

In this section we consider the semigroup S of upper triangular 3×3 matrices with nonnegative entries and nonzero determinant. In this case, I_3 is the only unit of S .

For each pair $i, j \in \{1, 2, 3\}$, let $E_{i,j}$ denote the matrix whose only nonzero entry is $e_{ij} = 1$.

The following theorem gives some characterizations about the atoms of S .

Theorem 1. *Let S denote the subsemigroup of the matrices in $T_3(\mathbb{N}_0)$ with nonzero determinant. The set of atoms of S consists of the matrices*

$$X_{12} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad X_{13} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (6)$$

$$\text{or } X_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

and, for each prime p , the matrices

$$Y_{11} = \begin{pmatrix} p & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Y_{22} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (7)$$

$$\text{or } Y_{33} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & p \end{pmatrix}.$$

Proof. Suppose that $X_{ij} = X_1 X_2$ for some $X_1, X_2 \in S$. Since $\det(X_{ij}) = 1$, $\det(X_1) = \det(X_2) = 1$ and we can write

$$X_{ij} = X_1 X_2 = \begin{pmatrix} 1 & b_{12} & b_{13} \\ 0 & 1 & b_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & c_{12} & c_{13} \\ 0 & 1 & c_{23} \\ 0 & 0 & 1 \end{pmatrix}, \quad (8)$$

where $\sum_{k=1}^3 b_{ik} c_{kj} = 1$ and $\sum_{k=1}^3 b_{ik} c_{kl} = 0$ if $h \neq i$ and $l \neq j$. As a result, either X_1 or X_2 is the identity and hence X_{ij} is an atom.

Note that the proofs that Y_{ii} ($1 \leq i \leq 3$) are atoms are similar; we only prove that Y_{22} is an atom in the following. Suppose now that p is prime and $Y_{22} = Y_1 Y_2$, $Y_1, Y_2 \in S$. Since p is prime, either

$$Y_1 Y_2 = \begin{pmatrix} 1 & b_{12} & b_{13} \\ 0 & p & b_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & c_{12} & c_{13} \\ 0 & 1 & c_{23} \\ 0 & 0 & 1 \end{pmatrix} \quad (9)$$

or

$$Y_1 Y_2 = \begin{pmatrix} 1 & b_{12} & b_{13} \\ 0 & 1 & b_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & c_{12} & c_{13} \\ 0 & p & c_{23} \\ 0 & 0 & 1 \end{pmatrix}. \quad (10)$$

In either case, $b_{ij} = c_{ij} = 0$ for $1 \leq i < j \leq 3$. Consequently, either Y_1 or Y_2 is the identity and hence Y_{ii} is an atom of S .

Finally, we will show that these are the only atoms of S .

For any

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \in S, \quad (11)$$

we can write

$$\begin{aligned} A &= [I_3 + (a_{33} - 1)E_{3,3}] [I_3 + a_{23}E_{2,3}] [I_3 + a_{13}E_{1,3}] \\ &\quad \cdot [I_3 + (a_{22} - 1)E_{2,2}] [I_3 + a_{12}E_{1,2}] [I_3 + (a_{11} - 1)E_{1,1}] \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & a_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &\quad \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_{12} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned} \quad (12)$$

Thus, the set of atoms of S consists of the matrices $A = I_3 + E_{i,j}$ for each pair i and j with $1 < i < j < 3$ or $A = I_3 + (p - 1)E_{ii}$ for some prime p and $1 \leq i \leq 3$. $\square$

Recall that a unitriangular matrix is a matrix in $T_3(\mathbb{N}_0)$ whose diagonal elements are all 1s. Denote $\Sigma(A) = \sum_{1 \leq i < j \leq 3} a_{ij}$. From the proof of Theorem 1, we can immediately obtain the following corollary.

Corollary 2. Let S denote the unitriangular matrices in $T_3(\mathbb{N}_0)$ and $A \in S$. Then A is an atom if and only if $\Sigma(A) = 1$.

Hereafter, for any given $A \in T_3(\mathbb{N}_0)$ with nonzero determinant, we let $r(A)$ denote the number of (not necessarily distinct) prime factors of $\det(A)$.

Proposition 3. Let S denote the subsemigroup of the matrices in $T_3(\mathbb{N}_0)$ with nonzero determinant. If A can be factored as $A = A_1 A_2 \cdots A_t$ with each A_i being an atom of S , then $t = r(A) + k$, where

$$k = \left| \left\{ i : A_i \in \left\{ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right\} \right\} \right|. \quad (13)$$

Proof. For each i , A_i is an atom and thus $\det(A_i)$ is either 1 or prime. Since

$$\det(A) = \det(A_1) \det(A_2) \cdots \det(A_t), \quad (14)$$

we have

$$|\{i : \det(A_i) \text{ is prime}\}| = r(A). \quad (15)$$

If $k = |\{i : \det(A_i) = 1\}|$, then the length of this factorization of A is

$$t = |\{i : \det(A_i) \text{ is prime}\}| + |\{i : \det(A_i) = 1\}| = r(A) + k. \quad (16)$$

□

Lemma 4 (see [7, Theorem 4.4]). Let S denote the subsemigroup of $T_n(\mathbb{N}_0)$ of unitriangular matrices and let $A \in S$. Then $L(A) = \Sigma(A)$.

Theorem 5. Let S denote the subsemigroup of the matrices in $T_3(\mathbb{N}_0)$ with nonzero determinant and

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \in S. \quad (17)$$

Then the following statements hold:

- (1) $L(A) = r(A) + a_{12} + a_{13} + a_{23}$;
- (2) if $a_{12} = a_{13} = a_{23} = 0$, then $l(A) = r(A) = L(A)$ and $\rho(A) = 1$;
- (3) if $a_{12} \mid a_{11}a_{22}$ ($a_{12} > 0$), $a_{13} = a_{23} = 0$, and then

$$l(A) = r(A) + 1, \quad \rho(A) = \frac{r(A) + a_{12}}{r(A) + 1}; \quad (18)$$

- (4) if $a_{13} \mid a_{11}a_{33}$ ($a_{13} > 0$), $a_{12} = a_{23} = 0$, and then

$$l(A) = r(A) + 1, \quad \rho(A) = \frac{r(A) + a_{13}}{r(A) + 1}; \quad (19)$$

- (5) if $a_{23} \mid a_{22}a_{33}$ ($a_{23} > 0$), $a_{12} = a_{13} = 0$, and then

$$l(A) = r(A) + 1, \quad \rho(A) = \frac{r(A) + a_{23}}{r(A) + 1}. \quad (20)$$

Proof. (1) Suppose that $A = A_1 A_2 \cdots A_t$ with each A_i being an atom of S . By Proposition 3, $t = r(A) + k$ where

$$k = |\{i : A_i \in \{I_3 + E_{12}, I_3 + E_{13}, I_3 + E_{23}\}\}|. \quad (21)$$

Note that the numbers of factors $I_3 + E_{12}$ ($I_3 + E_{13}$, $I_3 + E_{23}$, resp.) of A are not more than a_{12} (a_{13} , a_{23} , resp.). Then we have

$$k \leq a_{12} + a_{13} + a_{23} = \Sigma(A) \quad (22)$$

and thus

$$L(A) \leq r(A) + \Sigma(A). \quad (23)$$

On the other hand, from the proof of Theorem 1, we know that

$$\begin{aligned} A &= A_1 A_2 A_3 A_4 A_5 A_6 \\ &= [I_3 + (a_{33} - 1)E_{3,3}] [I_3 + a_{23}E_{2,3}] \\ &\quad \cdot [I_3 + a_{13}E_{1,3}] [I_3 + (a_{22} - 1)E_{2,2}] [I_3 + a_{12}E_{1,2}] \\ &\quad \cdot [I_3 + (a_{11} - 1)E_{1,1}], \end{aligned} \quad (24)$$

and then,

$$\begin{aligned} L(A) &\geq L(A_1) + L(A_2) + L(A_3) + L(A_4) \\ &\quad + L(A_5) + L(A_6) \\ &= r(A_1) + r(A_4) + r(A_6) + L(A_2) + L(A_3) + L(A_5) \\ &= r(A) + \Sigma_{1 \leq i < j \leq n} a_{ij} = r(A) + \Sigma(A). \end{aligned} \quad (25)$$

Thus, combining (23) and (25), we have $L(A) = r(A) + \Sigma(A)$.

(2) Suppose that A is a diagonal matrix; that is, $a_{ij} = 0$ for all $i, j \in \{1, 2, 3\}$ with $i < j$, and write $t = r(A) + k$ as in Proposition 3. If $k \geq 1$, then A contains at least one factor of $X_{12} = I_3 + E_{12}$, $X_{13} = I_3 + E_{13}$, or $X_{23} = I_3 + E_{23}$, and then there is at least one superdiagonal entry of A that is not 0. This contradicts with the fact that A is a diagonal matrix. Thus, $t = r(A) = l(A) = L(A)$ and, in this case,

$$\rho(A) = \frac{L(A)}{l(A)} = 1. \quad (26)$$

(3) Suppose that $a_{12} \mid a_{11}a_{22}$ ($a_{12} > 0$), $a_{13} = a_{23} = 0$. Denote $l(A) = r(A) + k$ as in Proposition 3. Notice that $r(A)$ is the number of (not necessarily distinct) prime factors of $\det(A)$ and $a_{12} > 0$. Then $k \geq 1$, and so $l(A) \geq r(A) + 1$. Since $a_{12} > 0$, $a_{12} \mid a_{11}a_{22}$, a_{12} can be factored as a product of one factor of a_{11} and one factor of a_{22} ; say $a_{12} = a'_{11}a'_{22}$ where

$a_{11} = m_{11}a'_{11}$ and $a_{22} = n_{22}a'_{22}$ for some positive integers m_{11} and n_{22} . Factor A as

$$\begin{aligned} A &= A_1 A_2 \cdots A_8 \\ &= \begin{pmatrix} a'_{33} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & n_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \\ &\quad \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} m_{11} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &\quad \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & a'_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned} \quad (27)$$

By the above factorization of A and (2), it is not hard to see that

$$l(A) \leq l(A_1) + l(A_2) + \cdots + l(A_8) = r(A) + 1. \quad (28)$$

Thus, $l(A) = r(A) + 1$.

In this case we immediately get

$$\rho(A) = \frac{r(A) + \Sigma(A)}{r(A) + 1} = \frac{r(A) + a_{12}}{r(A) + 1}. \quad (29)$$

Hence, (3) holds.

Similarly, we can show that (4) and (5) hold. $\square$

Remark 6. Given $A \in S$, Theorem 5 provides a formula for $l(A)$. Consequently, open problems 4a and 4c in [7] are each half-answered for the case $n = 3$.

Theorem 7. Let S denote the subsemigroup of the matrices in $T_3(\mathbb{N}_0)$ with nonzero determinant and

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \in S. \quad (30)$$

If A satisfies one of the following conditions,

- (1) $a_{12} = a_{13} = a_{23} = 0$;
- (2) $a_{11} = a_{22} = a_{33} = 1, a_{12} = 0$;
- (3) $a_{11} = a_{22} = a_{33} = 1, a_{13} = 0$;
- (4) $a_{11} = a_{22} = a_{33} = 1, a_{23} = 0$;
- (5) $a_{12} = a_{23} = 1, a_{13} = 0$;
- (6) $a_{12} = a_{23} = 1, a_{23} = 0$;
- (7) $a_{13} = a_{23} = 1, a_{12} = 0$,

then $\rho(A) = 1$.

Proof. (1) If $a_{12} = a_{13} = a_{23} = 0$, by Theorem 5, $l(A) = r(A) = L(A)$. In this case, $\rho(A) = 1$.

(2) Suppose that $a_{11} = a_{22} = a_{33} = 1$ and $a_{12} = 0$. If $a_{13} = a_{23} = 0$, by (1), the conclusion holds. Now, assume that $a_{13} = a_{23} \neq 0$; the cases of that $a_{13} = 0, a_{23} \neq 0$ or $a_{13} \neq 0, a_{23} = 0$ can be proved similarly. In this case, X_{12} is not an atomic factor of A and X_{13} and X_{23} are atomic factors of A . Note that X_{13} can commute with X_{23} ; then A only has the following factorization:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{a_{13}} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{a_{23}}, \quad (31)$$

with $\rho(A) = 1$ and the minimum factorization length $l(A) = a_{13} + a_{23}$.

Analogous with the proof of (2), we can show that (4) holds.

(3) Suppose that $a_{11} = a_{22} = a_{33} = 1$ and $a_{13} = 0$. If $a_{12} = a_{23} = 0$, by (1), the conclusion holds. Now, assume that $a_{12} = a_{23} \neq 0$; the cases of that $a_{12} = 0, a_{23} \neq 0$ or $a_{12} \neq 0, a_{23} = 0$ can be proved similarly. In this case, X_{13} is not an atomic factor of A and X_{12} and X_{23} are atomic factors of A . Note that $X_{12}X_{23} = X_{23}X_{12}X_{13}$; the factorizations of A must satisfy that all the X_{23} matrices are before X_{12} matrices, and then A only has the following factorization:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{a_{23}} \cdot \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{a_{12}}, \quad (32)$$

with $\rho(A) = 1$ and the minimum factorization length $l(A) = a_{12} + a_{23}$.

(5) Suppose that $a_{12} = a_{23} = 1$ and $a_{13} = 0$. In this case, X_{13} is not an atomic factor of A , and X_{12} and X_{23} are atomic factors of A . Assume that A can be factored as $A = A_1 A_2 \cdots A_t$ with each A_i being an atom; then $t = r(A) + k$, where

$$k = |\{i : A_i \in \{X_{12}, X_{23}\}\}|. \quad (33)$$

By Theorem 5 (1), $k \leq a_{12} + a_{23} + a_{13} = 2$. On the other hand, since $a_{12} = a_{23} = 1$, X_{12}, X_{23} are atomic factors of A , and we have $k \geq 2$. Thus, $k = 2$. In this case, A can factorize as follows:

$$\begin{aligned} A &= [I_3 + (a_{33} - 1)E_{3,3}] [I_3 + E_{2,3}] [I_3 + (a_{22} - 1)E_{2,2}] \\ &\quad \cdot [I_3 + E_{1,2}] [I_3 + (a_{11} - 1)E_{1,1}] \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &\quad \cdot \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned} \quad (34)$$

So if we give the atomic factorizations of $[I_3 + (a_{33} - 1)E_{3,3}]$, $[I_3 + (a_{22} - 1)E_{2,2}]$, and $[I_3 + (a_{11} - 1)E_{1,1}]$, then we will obtain the atomic factorization of A with $\rho(A) = 1$ and the minimum factorization length $l(A) = r(A) + 2$.

Analogous with the proof of (5), we can show that (6) and (7) hold. $\square$

Example 8. Consider

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \in T_3(\mathbb{N}_0). \quad (35)$$

Then by Theorem 7 (2), we can obtain the atomic factorization of A with $\rho(A) = 1$ and the minimum factorization length $l(A) = 2 + 3 = 5$ as follows:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^3. \quad (36)$$

Example 9. Consider

$$A = \begin{pmatrix} 6 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 8 \end{pmatrix} \in T_3(\mathbb{N}_0). \quad (37)$$

Then by Theorem 7 (5), we can obtain the atomic factorization of A with $\rho(A) = 1$ and the minimum factorization length $l(A) = r(A) + 2 = 9$ as follows:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}^3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (38)$$

Theorem 10. Let S denote the unitriangular matrices in $T_3(\mathbb{N}_0)$ and

$$A = \begin{pmatrix} 1 & a_{12} & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 1 \end{pmatrix} \in S, \quad a_{12}, a_{13}, a_{23} > 0. \quad (39)$$

Then we have

$$l(A) = \begin{cases} a_{12} + a_{23}, & \text{if } a_{13} \leq a_{12}a_{23}, \\ a_{12} + a_{23} + a_{13} - a_{12}a_{23}, & \text{if } a_{13} \geq a_{12}a_{23}. \end{cases} \quad (40)$$

Proof. It is a routine way to check that $X_{12}X_{13} = X_{13}X_{12}$, $X_{23}X_{13} = X_{13}X_{23}$, and $X_{12}X_{23} = X_{23}X_{12}$. Hence the maximum length is achieved by putting all the X_{23} matrices before the X_{12} matrices. In this case, we have the form

$X_{23}^{a_{23}} X_{12}^{a_{12}} X_{13}^{a_{13}}$. The minimum length is achieved by having as few as possible X_{13} matrices, that is, by putting X_{23} matrices after X_{12} matrices. If $a_{13} \leq a_{12}a_{23}$, we can run out of X_{13} matrices, leaving a form $X_{12}^{a'_{12}} X_{23}^{a'_{23}} X_{12} X_{23}^{a''_{23}} X_{12}^{a''_{12}}$, where these powers satisfy $a'_{12} + 1 + a''_{12} = a_{12}$, $a'_{23} + a''_{23} = a_{23}$, and $a'_{12}a_{23} + a''_{12} = a_{13}$. And, in this case, we have $l(A) = a_{12} + a_{23}$. If $a_{13} \geq a_{12}a_{23}$, then we can put all the X_{23} matrices after all of the X_{12} matrices, getting the form $X_{12}^{a_{12}} X_{23}^{a_{23}} X_{13}^{a_{13} - a_{12}a_{23}}$. Thus, in this case, we have $l(A) = a_{12} + a_{23} + a_{13} - a_{12}a_{23}$. $\square$

Example 11. Consider

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \in T_3(\mathbb{N}_0). \quad (41)$$

Then by Theorem 10, we can obtain a factorization of A with the minimum factorization length $l(A) = a_{12} + a_{23} = 5$ as follows:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^2 \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (42)$$

Example 12. Consider

$$A = \begin{pmatrix} 1 & 2 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \in T_3(\mathbb{N}_0). \quad (43)$$

Then by Theorem 10, we can obtain a factorization of A with the minimum factorization length $l(A) = a_{12} + a_{23} = 5$ as follows:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^3. \quad (44)$$

Example 13. Consider

$$A = \begin{pmatrix} 1 & 2 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \in T_3(\mathbb{N}_0). \quad (45)$$

Then by Theorem 10, we can obtain a factorization of A with the minimum factorization length $l(A) = a_{12} + a_{23} + a_{13} - a_{12}a_{23} = 6$ as follows:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^3 \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (46)$$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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