Research Article

Fixed Points of Generalized F -Suzuki Type Contraction in Complete b -Metric Spaces

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We introduce the notion of generalized F -Suzuki type contraction in b -metric spaces and investigate the existence of fixed points of such mappings. The presented results generalize and improve several results of the topics in the literature.

1. Introduction and Preliminaries

The concept of b -metric was introduced by Czerwik [1] as a generalization of metric (see also Bakhtin [2, 3]) to extend celebrated Banach contraction mapping principle. Following this initial paper of Czerwik [1], a number of researchers in nonlinear analysis investigate the topology of the paper and proved several fixed point theorems in the context of complete b -metric spaces (see, e.g., [4–8] and the related references therein).

Definition 1 (see [1]). Let X be a nonempty set and let $s \geq 1$ be a given real number. A mapping $d : X \times X \rightarrow [0, \infty)$ is said to be a b -metric if for all $x, y, z \in X$ the following conditions are satisfied:

$$(bM_1) \quad d(x, y) = 0 \text{ if and only if } x = y;$$

$$(bM_2) \quad d(x, y) = d(y, x);$$

$$(bM_3) \quad d(x, z) \leq s[d(x, y) + d(y, z)].$$

In this case, the pair (X, d) is called a b -metric space (with constant s).

Remark 2. It is clear that the notion of b -metric is a real generalization of usual metric since a b -metric space is

a metric space when $s = 1$. For more details and examples on b -metric, see, for example, [1, 3, 9, 10].

Example 3. Let $X = \{0, 1, 2, 3\}$ and let a mapping $d : X \times X \rightarrow [0, \infty)$ be defined as follows:

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ |x - y|^{-1}, & \text{if } x \neq y. \end{cases} \quad (1)$$

Then, (X, d) is a b -metric space with coefficient $s = 7/12$. But it is not a metric space since the triangle inequality is not satisfied. Indeed,

$$d(0, 1) > d(0, 3) + d(3, 1), \quad d(0, 1) > d(0, 4) + d(4, 1),$$

$$d(3, 4) > d(3, 2) + d(2, 4). \quad (2)$$

Definition 4 (see [11]). Let (X, d) be a b -metric space. Then, a sequence $\{x_n\}_{n=1}^{\infty}$ in X is called

(A) convergent if and only if there exists $x \in X$ such that $d(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$ and in this case we write $\lim_{n \rightarrow \infty} x_n = x$;

(B) Cauchy if and only if $d(x_n, x_m) \rightarrow 0$ as $m, n \rightarrow \infty$.

Remark 5 (see [11]). Notice that in a b -metric space (X, d) the following assertions hold:

- (A) a convergent sequence has a unique limit;
- (B) each convergent sequence is Cauchy;
- (C) in general, a b -metric is not continuous;
- (D) in general, a b -metric does not induce a topology on X .

Definition 6 (see [11]). The b -metric space (X, d) is complete if every Cauchy sequence in X converges in X .

Definition 7. Let (X, d_X) and (Y, d_Y) be b -metric spaces; a mapping $f : X \rightarrow Y$ is called

- (A) continuous at a point $x \in X$, if for every sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $x_n \rightarrow x$, then $f(x_n) \rightarrow f(x)$;
- (B) continuous on X , if it is continuous at each point $x \in X$.

2. Main Result

In this section we state and prove our main results. Throughout the paper, we assume that b -metric is continuous.

Inspired by the notion of F -contraction, defined by Wardowski [12], we introduce the notion of generalized F -Suzuki type contraction as follows. $\mathbb{N}, \mathbb{R}, \mathbb{R}^+$ denote the set of natural numbers, set of real numbers, and the set of non-negative real numbers.

Definition 8. Let (X, d) be a b -metric space with constant $s \geq 1$. A mapping $T : X \rightarrow X$ is called a generalized F -Suzuki type contraction if there exists $\tau > 0$ such that for all $x, y \in X$ with $x \neq y$

$$\begin{aligned} \frac{1}{2s} d(x, Tx) &< d(x, y) \\ \implies \tau + F(d(Tx, Ty)) & \\ &\leq \alpha F(d(x, y)) + \beta F(d(x, Tx)) \\ &\quad + \gamma F(d(y, Ty)), \end{aligned} \quad (3)$$

where $\gamma \in [0, 1)$ and $\alpha, \beta \in [0, 1]$ are real numbers with $\alpha + \beta + \gamma = 1$ and $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a mapping satisfying the following conditions:

- (F₁) F is strictly increasing; that is, for all $\alpha, \beta \in \mathbb{R}^+$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$;
- (F₂) for every sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive real numbers $\lim_{n \rightarrow \infty} \alpha_n = 0$ if and only if $\lim_{n \rightarrow \infty} F(\alpha_n) = -\infty$.

The following is the main result of this paper.

Theorem 9. Let (X, d) be a complete b -metric space with constant $s \geq 1$. If T is a generalized F -Suzuki type contraction, then T has a fixed point $v \in X$.

Proof. Fix $x \in X$. We establish an iterative sequence $\{x_n\}$ in the following way:

$$x_1 = Tx, x_2 = Tx_1 = T^2x, \dots, x_n = Tx_{n-1} = T^n x, \quad \forall n \in \mathbb{N}. \quad (4)$$

Throughout the proof, we assume that

$$d(x_n, Tx_n) > 0, \quad \text{for every } n \in \mathbb{N}. \quad (5)$$

Indeed, if there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0}, Tx_{n_0}) = 0$, then the proof is completed trivially.

Due to assumption (5), we have

$$\frac{1}{2s} d(x_n, Tx_n) < d(x_n, Tx_n), \quad \forall n \in \mathbb{N}. \quad (6)$$

Thus, by hypotheses of theorem, we have

$$\begin{aligned} \tau + F(d(Tx_n, T^2x_n)) &\leq \alpha F(d(x_n, Tx_n)) + \beta F(d(x_n, Tx_n)) \\ &\quad + \gamma F(d(Tx_n, T^2x_n)), \end{aligned} \quad (7)$$

which is equivalent to

$$\tau + (1 - \gamma) F(d(Tx_n, T^2x_n)) \leq (\alpha + \beta) F(d(x_n, Tx_n)). \quad (8)$$

Since $\alpha + \beta + \gamma = 1$, inequality (8) turns into

$$\begin{aligned} F(d(Tx_n, T^2x_n)) &\leq F(d(x_n, Tx_n)) - \frac{\tau}{\alpha + \beta} \\ &< F(d(x_n, Tx_n)). \end{aligned} \quad (9)$$

From (F₁), we conclude that

$$d(x_{n+1}, Tx_{n+1}) = d(Tx_n, T^2x_n) < d(x_n, Tx_n), \quad \forall n \in \mathbb{N}. \quad (10)$$

Therefore, $\{d(x_n, Tx_n)\}_{n=1}^{\infty}$ is a decreasing sequence of real numbers which is bounded from below. Therefore, $\{d(x_n, Tx_n)\}_{n=1}^{\infty}$ converges and

$$\lim_{n \rightarrow \infty} d(x_n, Tx_n) = \beta = \inf \{d(x_n, Tx_n) : n \in \mathbb{N}\}. \quad (11)$$

We will show that $\beta = 0$. Suppose, on the contrary, that $\beta > 0$. In other words, for every $\epsilon > 0$ there exists $m \in \mathbb{N}$, such that

$$d(x_m, Tx_m) < \beta + \epsilon. \quad (12)$$

From (F₁), we find that

$$F(d(x_m, Tx_m)) < F(\beta + \epsilon). \quad (13)$$

On the other hand, we have

$$\frac{1}{2s} d(x_m, Tx_m) < d(x_m, Tx_m), \quad (14)$$

due to (6). Since T is generalized F -Suzuki type contraction, we derive

$$\begin{aligned} \tau + F(d(Tx_m, T^2x_m)) &\leq \alpha F(d(x_m, Tx_m)) \\ &+ \beta F(d(x_m, Tx_m)) + \gamma F(d(Tx_m, T^2x_m)), \end{aligned} \quad (15)$$

which yields that

$$\tau + (1 - \gamma) F(d(Tx_m, T^2x_m)) \leq (\alpha + \beta) F(d(x_m, Tx_m)). \quad (16)$$

Taking the fact that $\alpha + \beta + \gamma = 1$ into account, we find

$$F(d(Tx_m, T^2x_m)) \leq F(d(x_m, Tx_m)) - \frac{\tau}{\alpha + \beta}. \quad (17)$$

Analogously, again by (6), we have $(1/2s)d(Tx_m, T^2x_m) < d(Tx_m, T^2x_m)$. Owing to the fact that T is generalized F -Suzuki type contraction, we conclude that

$$\begin{aligned} \tau + F(d(T^2x_m, T^3x_m)) &\leq \alpha F(d(Tx_m, T^2x_m)) \\ &+ \beta F(d(Tx_m, T^2x_m)) + \gamma F(d(T^2x_m, T^3x_m)). \end{aligned} \quad (18)$$

It implies that

$$F(d(T^2x_m, T^3x_m)) \leq F(d(Tx_m, T^2x_m)) - \frac{\tau}{\alpha + \beta}, \quad (19)$$

since $\alpha + \beta + \gamma = 1$. Furthermore, by combining (17) and (20), we get

$$\begin{aligned} F(d(T^2x_m, T^3x_m)) &\leq F(d(Tx_m, T^2x_m)) - \frac{\tau}{\alpha + \beta} \\ &\leq F(d(x_m, Tx_m)) - \frac{2\tau}{\alpha + \beta}. \end{aligned} \quad (20)$$

Iteratively, we obtain that

$$\begin{aligned} F(d(T^n x_m, T^{n+1} x_m)) &\leq F(d(T^n x_m, T^{n-1} x_m)) - \frac{\tau}{\alpha + \beta} \\ &\leq F(d(T^{n-1} x_m, T^{n-2} x_m)) - \frac{2\tau}{\alpha + \beta} \\ &\vdots \\ &\leq F(d(Tx_m, x_m)) - \frac{n\tau}{\alpha + \beta} \\ &< F(\beta + \epsilon) - \frac{n\tau}{\alpha + \beta}. \end{aligned} \quad (21)$$

By letting $n \rightarrow \infty$, we find that

$$\lim_{n \rightarrow \infty} F(d(T^n x_m, T^{n+1} x_m)) = -\infty. \quad (22)$$

Consequently, from (F_2) , we derive that $\lim_{n \rightarrow \infty} d(T^n x_m, T^{n+1} x_m) = 0$. Thus, there exists $N_1 \in \mathbb{N}$ such that

$$d(T^n x_m, T^{n+1} x_m) < \beta, \quad \forall n \geq N_1, \quad (23)$$

and from (4) we get

$$d(x_{m+n}, Tx_{m+n}) < \beta, \quad \forall n \geq N_1. \quad (24)$$

This is a contradiction with the definition of β . Hence, we have

$$\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0. \quad (25)$$

In what follows, we will prove that

$$\lim_{n, m \rightarrow \infty} d(x_n, x_m) = 0. \quad (26)$$

Suppose, on the contrary, that there exist $\epsilon > 0$ and sequences $\{p(n)\}_{n=1}^{\infty}$ and $\{q(n)\}_{n=1}^{\infty}$ of natural numbers such that

$$\begin{aligned} p(n) &> q(n) > n, \\ d(x_{p(n)}, x_{q(n)}) &\geq \epsilon, \\ d(x_{p(n)-1}, x_{q(n)}) &< \epsilon, \\ \forall n &\in \mathbb{N}. \end{aligned} \quad (27)$$

From the triangle inequality, we have

$$\begin{aligned} d(x_{p(n)}, x_{q(n)}) &\leq s[d(x_{p(n)}, x_{p(n)-1}) + d(x_{p(n)-1}, x_{q(n)})] \\ &\leq sd(x_{p(n)}, x_{p(n)-1}) + s\epsilon \\ &= sd(x_{p(n)-1}, Tx_{p(n)-1}) + s\epsilon, \quad \forall n \in \mathbb{N}. \end{aligned} \quad (28)$$

Owing to (25), there exists $N_2 \in \mathbb{N}$ such that

$$d(x_{p(n)}, Tx_{p(n)}) < \epsilon, \quad \forall n > N_2. \quad (29)$$

Taking (29) into account, (28) yields that

$$d(x_{p(n)}, x_{q(n)}) < 2s\epsilon, \quad \forall n > N_2. \quad (30)$$

So from (F_2) , we obtain

$$F(d(x_{p(n)}, x_{q(n)})) < F(2s\epsilon), \quad \forall n > N_2. \quad (31)$$

On the other hand, we can easily get that

$$\frac{1}{2s} d(x_{p(n)}, Tx_{p(n)}) < \frac{\epsilon}{2s} < \epsilon \leq d(x_{p(n)}, x_{q(n)}), \quad \forall n > N_2, \quad (32)$$

from (27) and (29). Since T is generalized F -Suzuki type contraction, for all $n > N_2$ we find that

$$\begin{aligned} \tau + F(d(Tx_{p(n)}, Tx_{q(n)})) &\leq \alpha F(d(x_{p(n)}, x_{q(n)})) \\ &+ \beta F(d(x_{p(n)}, Tx_{p(n)})) + \gamma F(d(x_{q(n)}, Tx_{q(n)})). \end{aligned} \quad (33)$$

Taking (31) into account, (33) turns into

$$\begin{aligned} \tau + F(d(Tx_{p(n)}, Tx_{q(n)})) &< \alpha F(2s\epsilon) + \beta F(d(x_{p(n)}, Tx_{p(n)})) \\ &+ \gamma F(d(x_{q(n)}, Tx_{q(n)})), \quad \forall n > N_2. \end{aligned} \quad (34)$$

Regarding (25) and (F_2) , we obtain that

$$\lim_{n \rightarrow \infty} F(d(Tx_{p(n)}, Tx_{q(n)})) = -\infty. \quad (35)$$

From (F_2) , we get that

$$\lim_{n \rightarrow \infty} d(Tx_{p(n)}, Tx_{q(n)}) = 0 \iff \lim_{n \rightarrow \infty} d(x_{p(n)+1}, x_{q(n)+1}) = 0. \quad (36)$$

This is a contradiction with relations in (27). Hence, $\lim_{m,n \rightarrow \infty} d(x_n, x_m) = 0$; that is, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X . On account of the completeness of (X, d) , there exists $v \in X$ such that

$$\lim_{n \rightarrow \infty} d(x_n, v) = 0. \quad (37)$$

We claim that, for every $n \in \mathbb{N}$,

$$\begin{aligned} \frac{1}{2s} d(x_n, Tx_n) &< d(x_n, v) \quad \text{or} \\ \frac{1}{2s} d(Tx_n, T^2x_n) &< d(Tx_n, v), \end{aligned} \quad (38)$$

$$\forall n \in \mathbb{N}.$$

We will prove the claim above by the method of *reductio ad absurdum*. Suppose, on the contrary, that there exists $m \in \mathbb{N}$ such that

$$\begin{aligned} \frac{1}{2s} d(x_m, Tx_m) &\geq d(x_m, v), \\ \frac{1}{2s} d(Tx_m, T^2x_m) &\geq d(Tx_m, v). \end{aligned} \quad (39)$$

From (10) and (F_1) , we have

$$d(Tx_m, T^2x_m) < d(x_m, Tx_m). \quad (40)$$

It follows from (39) and (40) that

$$\begin{aligned} d(x_m, Tx_m) &\leq sd(x_m, v) + sd(v, Tx_m) \\ &\leq \frac{1}{2} d(x_m, Tx_m) + \frac{1}{2} d(Tx_m, T^2x_m) \\ &< \frac{1}{2} d(x_m, Tx_m) + \frac{1}{2} d(x_m, Tx_m) \\ &= d(x_m, Tx_m). \end{aligned} \quad (41)$$

This is a contradiction. Hence, (38) holds. Since T is generalized F -Suzuki type contraction, (38) yields that, for every $n \in \mathbb{N}$, either

$$\begin{aligned} \tau + F(d(Tx_n, Tv)) &\leq \alpha F(d(x_n, v)) + \beta F(d(x_n, Tx_n)) \\ &+ \gamma F(d(v, Tv)) \end{aligned} \quad (42)$$

or

$$\begin{aligned} \tau + F(d(T^2x_n, Tv)) &\leq \alpha F(d(Tx_n, v)) \\ &+ \beta F(d(Tx_n, T^2x_n)) + \gamma F(d(v, Tv)) \end{aligned} \quad (43)$$

holds. On account of (F_2) , the limits in (25) and (37) imply that

$$\lim_{n \rightarrow \infty} F(\sigma(x_n, v)) = -\infty, \quad \lim_{n \rightarrow \infty} F(\sigma(x_n, Tx_n)) = -\infty. \quad (44)$$

Thus, letting $n \rightarrow \infty$ in (42), we conclude that

$$\lim_{n \rightarrow \infty} F(d(Tx_n, Tv)) = -\infty. \quad (45)$$

Again by using (F_2) , we observe that

$$\lim_{n \rightarrow \infty} d(Tx_n, Tv) = 0. \quad (46)$$

By regarding the triangle inequality with (4), we derive that

$$\begin{aligned} d(v, Tv) &\leq s[d(v, Tx_n) + d(Tx_n, Tv)] \\ &= sd(v, x_{n+1}) + sd(Tx_n, Tv). \end{aligned} \quad (47)$$

By letting $n \rightarrow \infty$ in the inequality above together with the limits in (37) and (46), we conclude that $d(v, Tv) = 0$. Thus, v is a fixed point of T ; that is, $v = Tv$.

Let us analyze the second case (42). Regarding (4), we have

$$\begin{aligned} F(d(T^2x_n, Tv)) &< \tau + F(d(T^2x_n, Tv)) \\ &\leq \alpha F(d(Tx_n, v)) + \beta F(d(Tx_n, T^2x_n)) + \gamma F(d(v, Tv)) \\ &= \alpha F(d(x_{n+1}, v)) + \beta F(d(x_{n+1}, Tx_{n+1})) + \gamma F(d(v, Tv)). \end{aligned} \quad (48)$$

As it was discussed above, from (25), (37), and (F_2) , we conclude that

$$\lim_{n \rightarrow \infty} F(d(T^2x_n, Tv)) = -\infty. \quad (49)$$

From (F_2) equivalently, we get

$$\lim_{n \rightarrow \infty} d(T^2x_n, Tv) = 0. \quad (50)$$

Again by the triangle inequality together with (4), we find that

$$\begin{aligned} d(v, Tv) &\leq s[d(v, T^2x_n) + d(T^2x_n, Tv)] \\ &= sd(v, x_{n+2}) + d(T^2x_n, Tv). \end{aligned} \quad (51)$$

By letting $n \rightarrow \infty$ in the inequality above together with the limits in (37) and (50), we obtain $d(v, Tv) = 0$. Thus, v is a fixed point of T and that completes the proof. $\square$

Definition 10. Let (X, d) be a b -metric space with constant $s \geq 1$. A mapping $T : X \rightarrow X$ is said to be a F -Suzuki type contraction if there exists $\tau > 0$ such that for all $x, y \in X$ with $x \neq y$

$$\frac{1}{2s}d(x, Tx) < d(x, y) \implies \tau + F(d(Tx, Ty)) \leq F(d(x, y)), \quad (52)$$

where $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a mapping satisfying the following conditions.

(F_1) F is strictly increasing; that is, for all $\alpha, \beta \in \mathbb{R}^+$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$.

(F_2) For every sequence $\{\alpha_n\}_{n=1}^\infty$ of positive real numbers $\lim_{n \rightarrow \infty} \alpha_n = 0$ if and only if $\lim_{n \rightarrow \infty} F(\alpha_n) = -\infty$.

Theorem 11. Let (X, d) be complete b -metric space with constant $s \geq 1$ and let $T : X \rightarrow X$ be a F -Suzuki type contraction mapping. Then, T has a fixed point.

Proof. By taking $\alpha = 1$ and $\beta = \gamma = 0$ in Theorem 9, the proof is complete. $\square$

Corollary 12. Let T be a self-mapping on a complete metric space (X, d) . Let $\gamma \in [0, 1)$ and $\alpha, \beta \in [0, 1]$ be real numbers such that $\alpha + \beta + \gamma = 1$. Assume that there exists $\tau > 0$ such that, for all $x, y \in X$ with $x \neq y$,

$$\begin{aligned} \frac{1}{2}d(x, Tx) < d(x, y) \\ \implies \tau + F(d(Tx, Ty)) \\ \leq \alpha F(d(x, y)) + \beta F(d(x, Tx)) \\ + \gamma F(d(y, Ty)), \end{aligned} \quad (53)$$

where $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ is satisfied in conditions (F_1) and (F_2). Then, T has a fixed point $v \in X$; that is, $Tv = v$.

Proof. Since any metric space is a b -metric space with constant $s = 1$, so from Theorem 9 the proof is complete. $\square$

Corollary 13. Let T be a self-mapping on a complete metric space (X, d) . Assume that there exists $\tau > 0$ such that, for all $x, y \in X$ with $x \neq y$,

$$\frac{1}{2}d(x, Tx) < d(x, y) \implies \tau + F(d(Tx, Ty)) \leq F(d(x, y)), \quad (54)$$

where $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ is satisfied in conditions (F_1) and (F_2). Then, T has a fixed point $v \in X$; that is, $Tv = v$.

Proof. Since any metric space is a b -metric space with constant $s = 1$, so by taking $\alpha = 1$ and $\beta = \gamma = 0$ in Theorem 9 the proof is complete. $\square$

Theorem 14. Let (X, d) be complete b -metric space with constant $s \geq 1$ and let T be a continuous self-mapping on X . Let $\gamma \in [0, 1)$ and $\alpha, \beta \in [0, 1]$ be real numbers such that $\alpha + \beta + \gamma = 1$. Assume that there exists $\tau > 0$ such that for all $x, y \in X$

$$\begin{aligned} d(Tx, Ty) > 0 \\ \implies \tau + F(d(Tx, Ty)) \\ < \alpha F(d(x, y)) + \beta F(d(x, Tx)) \\ + \gamma F(d(y, Ty)), \end{aligned} \quad (55)$$

where $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ is satisfied in conditions (F_1) and (F_2). Then, T has a fixed point $v \in X$; that is, $Tv = v$.

Proof. Choose $x \in X$. Set

$$x_1 = Tx, x_2 = Tx_1 = T^2x, \dots, x_n = Tx_{n-1} = T^n x_0, \quad \forall n \in \mathbb{N}. \quad (56)$$

If there exists $n \in \mathbb{N}$ such that $d(x_n, Tx_n) = 0$, the proof is complete. So we assume that

$$0 < d(x_n, Tx_n) = d(Tx_{n-1}, Tx_n), \quad \forall n \in \mathbb{N}. \quad (57)$$

So from the assumption of theorem, we have

$$\begin{aligned} \tau + F(d(Tx_{n-1}, Tx_n)) \\ \leq \alpha F(d(x_{n-1}, x_n)) + \beta F(d(x_{n-1}, Tx_{n-1})) \\ + \gamma F(d(x_n, Tx_n)) \\ = \alpha F(d(x_{n-1}, x_n)) + \beta F(d(x_{n-1}, x_n)) \\ + \gamma F(d(Tx_{n-1}, Tx_n)) \end{aligned} \quad (58)$$

and hence

$$\tau + (1 - \gamma) F(d(Tx_{n-1}, Tx_n)) \leq (\alpha + \beta) F(d(x_{n-1}, x_n)). \quad (59)$$

Since $\alpha + \beta + \gamma = 1$, we get

$$\begin{aligned} F(d(Tx_{n-1}, Tx_n)) &\leq F(d(x_{n-1}, x_n)) - \frac{\tau}{\alpha + \beta} \\ &< F(d(x_{n-1}, x_n)). \end{aligned} \quad (60)$$

So from (F_1), we conclude that

$$\begin{aligned} d(x_n, Tx_n) = d(Tx_{n-1}, Tx_n) < d(x_{n-1}, x_n) = d(x_{n-1}, Tx_{n-1}), \\ \forall n \in \mathbb{N}. \end{aligned} \quad (61)$$

Therefore, $\{d(x_n, Tx_n)\}_{n=1}^\infty$ is a decreasing sequence of real numbers which is bounded from below. Therefore, $\{d(x_n, Tx_n)\}_{n=1}^\infty$ converges and

$$\lim_{n \rightarrow \infty} d(x_n, Tx_n) = \beta = \inf \{d(x_n, Tx_n) : n \in \mathbb{N}\}. \quad (62)$$

We will show that $\beta = 0$. Arguing by contradiction, we assume $\beta > 0$. For every $\epsilon > 0$, there exists $m \in \mathbb{N}$, such that

$$d(x_m, Tx_m) < \beta + \epsilon. \quad (63)$$

Hence, from (F_1) , we get

$$F(d(x_m, Tx_m)) < F(\beta + \epsilon). \quad (64)$$

On the other hand (57), we have

$$0 < d(x_m, Tx_m) = d(Tx_{m-1}, Tx_m). \quad (65)$$

So from assumption of theorem, we obtain

$$\begin{aligned} & \tau + F(d(Tx_{m-1}, Tx_m)) \\ & \leq \alpha F(d(x_{m-1}, x_m)) + \beta F(d(x_{m-1}, Tx_{m-1})) \\ & \quad + \gamma F(d(x_m, Tx_m)) \\ & = \alpha F(d(x_{m-1}, x_m)) + \beta F(d(x_{m-1}, x_m)) \\ & \quad + \gamma F(d(Tx_{m-1}, Tx_m)), \end{aligned} \quad (66)$$

and hence

$$\tau + (1 - \gamma) F(d(Tx_{m-1}, Tx_m)) < (\alpha + \beta) F(d(x_{m-1}, x_m)). \quad (67)$$

Since $\alpha + \beta + \gamma = 1$, we get

$$F(d(Tx_{m-1}, Tx_m)) \leq F(d(x_{m-1}, x_m)) - \frac{\tau}{\alpha + \beta}. \quad (68)$$

Also from (57), we have $0 < d(Tx_m, Tx_{m+1})$, and thus, by assumption of theorem, we have

$$\begin{aligned} & \tau + F(d(Tx_m, Tx_{m+1})) \\ & \leq \alpha F(d(x_m, x_{m+1})) + \beta F(d(x_m, Tx_m)) \\ & \quad + \gamma F(d(x_{m+1}, Tx_{m+1})) \\ & = \alpha F(d(x_m, x_{m+1})) + \beta F(d(x_m, x_{m+1})) \\ & \quad + \gamma F(d(Tx_m, Tx_{m+1})), \end{aligned} \quad (69)$$

and therefore,

$$\tau + (1 - \gamma) F(d(Tx_m, Tx_{m+1})) \leq (\alpha + \beta) F(d(x_m, x_{m+1})). \quad (70)$$

Since $\alpha + \beta + \gamma = 1$, we get

$$F(d(Tx_m, Tx_{m+1})) \leq F(d(x_m, x_{m+1})) - \frac{\tau}{\alpha + \beta}. \quad (71)$$

Now by using (64) and continuing similar method as used in (68) and (71), we obtain

$$\begin{aligned} F(d(x_{m+n}, x_{m+n+1})) &= F(d(Tx_{m+n-1}, Tx_{m+n})) \\ &\leq F(d(x_{m+n-1}, x_{m+n})) - \frac{\tau}{\alpha + \beta} \\ &= F(d(Tx_{m+n-2}, Tx_{m+n-1})) - \frac{\tau}{\alpha + \beta} \\ &\leq F(d(x_{m+n-2}, x_{m+n-1})) - \frac{2\tau}{\alpha + \beta} \\ &= F(d(Tx_{m+n-3}, Tx_{m+n-2})) - \frac{2\tau}{\alpha + \beta} \\ &\leq F(d(x_{m+n-3}, x_{m+n-2})) - \frac{3\tau}{\alpha + \beta} \\ &\vdots \\ &\leq F(d(x_{m+1}, x_{m+2})) - \frac{(n-1)\tau}{\alpha + \beta} \\ &= F(d(Tx_m, Tx_{m+1})) - \frac{(n-1)\tau}{\alpha + \beta} \\ &\leq F(\beta + \epsilon) - \frac{n\tau}{\alpha + \beta}. \end{aligned} \quad (72)$$

This implies that

$$\lim_{n \rightarrow \infty} F(d(x_{m+n}, x_{m+n+1})) = -\infty. \quad (73)$$

So from (F_2) , we have $\lim_{n \rightarrow \infty} d(x_{m+n}, x_{m+n+1}) = 0$, so that there exists $N_1 \in \mathbb{N}$ such that

$$d(x_{m+n}, x_{m+n+1}) < \beta, \quad \forall n \geq N_1, \quad (74)$$

and so from (56) we get

$$d(x_{m+n}, Tx_{m+n}) < \beta, \quad \forall n \geq N_1. \quad (75)$$

This is a contradiction with definition of β . So, $\beta = 0$ and from (62) we have

$$\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0. \quad (76)$$

Now, we claim that

$$\lim_{n, m \rightarrow \infty} d(x_n, x_m) = 0. \quad (77)$$

Arguing by contradiction, we assume that there exist $\epsilon > 0$ and sequences $\{p(n)\}_{n=1}^{\infty}$ and $\{q(n)\}_{n=1}^{\infty}$ of natural numbers such that

$$\begin{aligned} & p(n) > q(n) > n, \\ & d(x_{p(n)}, x_{q(n)}) \geq \epsilon, \\ & d(x_{p(n)-1}, x_{q(n)}) < \epsilon, \\ & \forall n \in \mathbb{N}. \end{aligned} \quad (78)$$

So

$$\begin{aligned} d(x_{p(n)}, x_{q(n)}) &\leq s [d(x_{p(n)}, x_{p(n)-1}) + d(x_{p(n)-1}, x_{q(n)})] \\ &\leq sd(x_{p(n)}, x_{p(n)-1}) + s\epsilon \\ &= sd(x_{p(n)-1}, Tx_{p(n)-1}) + s\epsilon, \quad \forall n \in \mathbb{N}. \end{aligned} \quad (79)$$

From (76), there exists $N_2 \in \mathbb{N}$ such that

$$d(x_{p(n)}, Tx_{p(n)}) < \epsilon, \quad \forall n > N_2. \quad (80)$$

It follows from (79) and (80) that

$$d(x_{p(n)}, x_{q(n)}) < 2s\epsilon, \quad \forall n > N_2. \quad (81)$$

So, from (F_2) , we obtain

$$F(d(x_{p(n)}, x_{q(n)})) < F(2s\epsilon), \quad \forall n > N_2. \quad (82)$$

On the other hand from (78), we have

$$\epsilon \leq d(x_{p(n)+1}, x_{q(n)+1}) = d(Tx_{p(n)}, Tx_{q(n)}), \quad \forall n > N_2. \quad (83)$$

Hence, from (82) and assumption of theorem, we have

$$\begin{aligned} &\tau + F(d(Tx_{p(n)}, Tx_{q(n)})) \\ &\leq \alpha F(d(x_{p(n)}, x_{q(n)})) + \beta F(d(x_{p(n)}, Tx_{p(n)})) \\ &\quad + \gamma F(d(x_{q(n)}, Tx_{q(n)})) \\ &< \alpha F(2s\epsilon) + \beta F(d(x_{p(n)}, Tx_{p(n)})) \\ &\quad + \gamma F(d(x_{q(n)}, Tx_{q(n)})), \quad \forall n > N_2. \end{aligned} \quad (84)$$

Using (79) and (F_2) , we obtain

$$\lim_{n \rightarrow \infty} F(d(Tx_{p(n)}, Tx_{q(n)})) = -\infty. \quad (85)$$

Thus, from this and (F_2) , we get

$$\lim_{n \rightarrow \infty} d(Tx_{p(n)}, Tx_{q(n)}) = 0 \iff \lim_{n \rightarrow \infty} d(x_{p(n)+1}, x_{q(n)+1}) = 0. \quad (86)$$

This is contradiction with relation (78). Hence, $\lim_{m,n \rightarrow \infty} d(x_n, x_m) = 0$. By completeness of (X, d) , there exists $v \in X$ such that

$$\lim_{n \rightarrow \infty} d(x_n, v) = 0. \quad (87)$$

Since T is continuous, we get

$$\lim_{n \rightarrow \infty} d(Tx_n, Tv) = 0. \quad (88)$$

Since $d(v, Tv) \leq s[d(v, x_n) + d(x_n, Tv)]$, therefore $d(v, Tv) = 0$ and hence $v = Tv$. $\square$

Theorem 15. Let (X, d) be complete b -metric space with constant $s \geq 1$ and let T be a continuous self-mapping on X . Assume that there exists $\tau > 0$ such that for all $x, y \in X$

$$d(Tx, Ty) > 0 \implies \tau + F(d(Tx, Ty)) < F(d(x, y)), \quad (89)$$

where $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ is satisfied in conditions (F_1) and (F_2) . Then, T has a fixed point $v \in X$; that is, $Tv = v$.

Proof. By taking $\alpha = 1$ and $\beta = \gamma = 0$ in Theorem 14, the proof is complete. $\square$

Theorem 16. Let (X, d) be complete metric space and let T be a continuous self-mapping on X . Let $\gamma \in [0, 1)$ and $\alpha, \beta \in [0, 1]$ be real numbers such that $\alpha + \beta + \gamma = 1$. Assume that there exists $\tau > 0$ such that for all $x, y \in X$

$$\begin{aligned} &d(Tx, Ty) > 0 \\ &\implies \tau + F(d(Tx, Ty)) \\ &\quad < \alpha F(d(x, y)) + \beta F(d(x, Tx)) \\ &\quad \quad + \gamma F(d(y, Ty)), \end{aligned} \quad (90)$$

where $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ is satisfied in conditions (F_1) and (F_2) . Then, T has a fixed point $v \in X$; that is, $Tv = v$.

Proof. It is sufficient to take $s = 1$ in Theorem 14. $\square$

Theorem 17. Let (X, d) be complete metric space and let T be a continuous self-mapping on X . Assume that there exists $\tau > 0$ such that for all $x, y \in X$

$$d(Tx, Ty) > 0 \implies \tau + F(d(Tx, Ty)) < F(d(x, y)), \quad (91)$$

where $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ is satisfied in conditions (F_1) and (F_2) . Then, T has a unique fixed point $v \in X$; that is, $Tv = v$.

Proof. Since every metric space is a b -metric space with constant $s = 1$, from Theorem 15 T has a fixed point $v \in X$. Indeed, if there is another fixed point $w \in X$ of T , such that $v \neq w$, therefore $d(Tv, Tw) > 0$. Since $\tau > 0$, from assumption of theorem we obtain

$$\begin{aligned} &F(d(v, w)) = F(d(Tv, Tw)) < \tau + F(d(Tv, Tw)) \\ &\leq \alpha F(d(v, w)). \end{aligned} \quad (92)$$

This is contradiction. So T has a unique fixed point. $\square$

Remark 18. Theorem 17 gives all consequence of Theorem 2.1 of [12] without assumption (F_3) used by [12]. Notice also that the results in [12] can be also concluded from the main theorem in [13] that is also a proper extension of the results in [12] in different aspect, more precisely, in complete metric-like spaces.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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