Research Article

On Antiperiodic Nonlocal Three-Point Boundary Value Problems for Nonlinear Fractional Differential Equations

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We introduce boundary value conditions involving antiperiodic and nonlocal three-point boundary conditions. We solve a nonlinear fractional differential equation supplemented with those conditions. We obtain some existence results for the given problem by applying some standard tools of fixed point theory. These results are well illustrated with the aid of examples.

1. Introduction

In recent years, several kinds of boundary value problems of nonlinear fractional differential equations, supplemented with a variety of boundary conditions (including Dirichlet, Neumann, mixed, periodic, antiperiodic, multipoint, integral type, and nonlocal), have been investigated by several researchers. This investigation includes a wide collection of results ranging from theoretical to analytic and numerical methods. For details and examples, see [1–9] and the references therein. This surge has been mainly due to the extensive applications of fractional operators in basic and technical sciences and engineering. One can easily witness from literature (special issues and books) on the topic that the tools of fractional calculus have helped in improving the mathematical modeling of several phenomena of practical nature; for instance, see [10–17].

In this paper, we study a new class of problems of fractional differential equations supplemented with antiperiodic and three-point nonlocal boundary conditions. Precisely, we consider the following fractional problem:

\[ c \mathcal{D}^q x(t) + f(\alpha, t, x(t)) = 0, \quad t \in [0, 1], \quad 1 < q \leq 2, \]
\[ x(0) = -x(1), \]
\[ \alpha x'(0) + \beta x'(1) = -\gamma x(\theta), \quad 0 < \theta < 1, \]  

where \( c \mathcal{D}^q \) is the usual Caputo fractional derivative, \( f : [0, 1] \times \mathbb{R} \to \mathbb{R} \) is a continuous function, \( \alpha, \beta \) are positive real constants, and \( \gamma \) is a nonnegative constant. Further, it is assumed that \( \gamma(1-2\theta) - 2(\alpha + \beta) \neq 0 \).

We emphasize that the second boundary condition in (1) can be interpreted as the sum of scalar multiples of the values of the derivative of the unknown function at \( x = 0 \) and \( x = 1 \) is proportional to the value of the unknown function at an arbitrary value \( \theta \in (0, 1) \). In case of \( \alpha = \beta = 1, \gamma = 0 \), problem (1) reduces to the one with antiperiodic boundary conditions. Thus the proposed problem generalizes antiperiodic problems to semiantiperiodic three-point nonlocal problems.

2. Auxiliary Lemma and Notations

In order to define the solutions for the given problem, we consider the following lemma.

Lemma 1. Let \( g \in C[0, 1] \). Then the following linear semiantiperiodic three-point fractional boundary value problem

\[ c \mathcal{D}^q x(t) + g(\alpha, t) = 0, \quad t \in [0, 1], \quad 1 < q \leq 2, \]
\[ x(0) = -x(1), \]
\[ \alpha x'(0) + \beta x'(1) = -\gamma x(\theta), \quad 0 < \theta < 1, \]
has a unique solution given by

\[
  x(t) = - \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} g(s) \, ds + b_1 + b_2 t,
\]

where \( b_1, b_2 \in \mathbb{R} \) are unknown arbitrary constants. Since \( x(0) = -x(1) \), (4) gives

\[
  2b_1 + b_2 = \int_0^1 \frac{(1-s)^{\gamma-1}}{\Gamma(\gamma)} g(s) \, ds. \quad (5)
\]

Using the boundary conditions \( \alpha x'(0) + \beta x'(1) = -\gamma x(\theta) \) in (4), we get

\[
  y b_1 + (\alpha + \beta + y \theta) b_2 = y \int_0^\theta \frac{(\theta-s)^{\gamma-1}}{\Gamma(\gamma)} g(s) \, ds
\]

\[
  + \beta \int_0^1 \frac{(1-s)^{\gamma-2}}{\Gamma(\gamma-q-1)} g(s) \, ds. \quad (6)
\]

Solving system of (5) and (6) for \( b_1 \) and \( b_2 \), it is found that

\[
  b_1 = \frac{1}{\gamma (1-2\theta) - 2(\alpha + \beta)} \left[ - (\theta y + \alpha + \beta) \right]
\]

\[
  + y \int_0^\theta \frac{(\theta-s)^{\gamma-1}}{\Gamma(\gamma)} g(s) \, ds,
\]

\[
  b_2 = \frac{1}{\gamma (1-2\theta) - 2(\alpha + \beta)} \left[ \gamma \int_0^1 \frac{(1-s)^{\gamma-1}}{\Gamma(\gamma)} g(s) \, ds
\]

\[
  - 2\beta \int_0^1 \frac{(1-s)^{\gamma-2}}{\Gamma(\gamma-q-1)} g(s) \, ds
\]

\[
  - 2y \int_0^\theta \frac{(\theta-s)^{\gamma-1}}{\Gamma(\gamma)} g(s) \, ds. \quad (7)
\]

Substituting these values of \( b_1 \) and \( b_2 \) in (4) completes solution (3).

Let \( \mathcal{D} = C([0,1], \mathbb{R}) \) denote the Banach space of all continuous functions from \([0,1]\) into \( \mathbb{R} \) endowed with the usual supremum norm defined by \( ||x|| = \sup_{t \in [0,1]} |x(t)| \).

In view of Lemma 1, we consider a fixed point problem equivalent to the nonlinear antisymmetric problem (1) as follows:

\[
  \mathcal{H} x = x, \quad (8)
\]

where the operator \( \mathcal{H} : \mathcal{D} \to \mathcal{D} \) is defined as

\[
  (\mathcal{H} x)(t) = - \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} f(s, x(s)) \, ds
\]

\[
  + \frac{1}{\gamma (1-2\theta) - 2(\alpha + \beta)} \left[ (\gamma (t-\theta) - \alpha - \beta) \right]
\]

\[
  \cdot \int_0^1 \frac{(1-s)^{\gamma-1}}{\Gamma(\gamma)} f(s, x(s)) \, ds + (1-2t)
\]

\[
  \cdot \left[ \gamma \int_0^\theta \frac{(\theta-s)^{\gamma-1}}{\Gamma(\gamma)} f(s, x(s)) \, ds
\]

\[
  + \beta \int_0^1 \frac{(1-s)^{\gamma-2}}{\Gamma(\gamma-q-1)} f(s, x(s)) \, ds \right]. \quad (9)
\]

Next we set

\[
  \sigma = \frac{1}{\Gamma(q+1)} \left( 1 + \frac{\gamma (1-2\theta) - 2(\alpha + \beta) + \beta q + \gamma \theta^q}{\gamma (1-2\theta) - 2(\alpha + \beta)} \right). \quad (10)
\]

\( \square \)

3. Main Results

In this section, we present our main results. The first result relies on classical Banach's contraction mapping principle.
Theorem 2. Let \( f : [0, 1] \times \mathbb{R} \to \mathbb{R} \) be a continuous function satisfying the Lipschitz condition; that is, there exists \( \ell > 0 \) such that

\[
(H_1) \, |f(t, x) - f(t, y)| \leq \ell |x - y|, \forall t \in [0, 1], x, y \in \mathbb{R}.
\]

Then problem (1) has a unique solution if \( \ell \sigma < 1 \), where \( \sigma \) is given by (10).

Proof. In the first step, it will be shown that \( \mathcal{H} B \subset B \), where \( B = \{ x \in \mathcal{D} : |x| \leq r \} \) with \( r > \sigma \mu \left( \frac{1}{\sigma} - \frac{1}{\ell} \right) \), and \( \sup_{t \in [0, 1]} |f(t, 0)| = \mu \). For \( x \in B, t \in [0, 1] \), using \( |f(s, x(s))| = |f(s, x(s)) - f(s, 0) + f(s, 0)| \leq \ell r + \mu \), we get

\[
\left\| (Hx)(t) \right\| \leq \sup_{t \in [0, 1]} \left\{ \int_{0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} \left| f(s, x(s)) - f(s, 0) + f(s, 0) \right| ds \right. \\
+ \frac{1}{|y(1-2\theta)| - 2(\alpha + \beta)} \left[ \left| y(t-\theta) - \alpha - \beta \right| \int_{0}^{1} \frac{(1-s)^{q-1}}{\Gamma(q)} \left| f(s, x(s)) - f(s, 0) + f(s, 0) \right| ds \right] \\
+ \left| 1 - 2\ell \left( \int_{0}^{\theta} \frac{(\theta-s)^{q-1}}{\Gamma(q)} ds + \int_{0}^{1} \frac{(1-s)^{q-2}}{\Gamma(q-1)} ds \right) \right| \right\} \leq \ell r \sigma \leq r,
\]

which implies that \( \mathcal{H} B \subset B \), where we have used (10). Now, for \( x, y \in \mathcal{D} \), and for each \( t \in [0, 1] \), we obtain

\[
\left\| (\mathcal{H} x) - (\mathcal{H} y) \right\| \leq \sup_{t \in [0, 1]} \left\{ \int_{0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} \left| f(s, x(s)) - f(s, y(s)) \right| ds \right. \\
+ \frac{1}{|y(1-2\theta)| - 2(\alpha + \beta)} \left[ \left| y(t-\theta) - \alpha - \beta \right| \int_{0}^{1} \frac{(1-s)^{q-1}}{\Gamma(q)} \left| f(s, x(s)) - f(s, y(s)) \right| ds \right] \\
+ \left| 1 - 2\ell \left( \int_{0}^{\theta} \frac{(\theta-s)^{q-1}}{\Gamma(q)} ds + \int_{0}^{1} \frac{(1-s)^{q-2}}{\Gamma(q-1)} ds \right) \right| \leq \ell \sigma \| x - y \|.
\]

Since \( \ell \sigma < 1 \) by the given assumption, the operator \( \mathcal{H} \) is a contraction. Thus, by Banach’s contraction mapping principle, there exists a unique solution for problem (1). This completes the proof. \( \square \)

The next existence result is based on the following Schaefer’s fixed point theorem [18], Th. 4.3.2.

Theorem 3. Let \( X \) be a Banach space. Assume that \( T : X \to X \) is completely continuous operator and the set

\[
V = \{ u \in X \mid u = vTu, \ 0 < v < 1 \}
\]

is bounded. Then \( T \) has a fixed point in \( X \).

Theorem 4. Assume that there exists a positive constant \( M \) such that \( |f(t, x)| \leq M \) for \( t \in [0, 1], x \in \mathbb{R} \). Then problem (1) has at least one solution.

Proof. We first show that the operator \( \mathcal{H} \) is completely continuous. Obviously continuity of the operator \( \mathcal{H} \) follows from continuity of \( f \). Let \( \mathcal{B} \subset \mathcal{D} \) be a bounded set. By the assumption \( |f(t, x)| \leq M \), for \( x \in \mathcal{B} \), we have

\[
|\mathcal{H} x(t)| \leq \int_{0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s, x(s))| ds \\
+ \frac{1}{|y(1-2\theta)| - 2(\alpha + \beta)} \left[ \left| y(t-\theta) - \alpha - \beta \right| \int_{0}^{1} \frac{(1-s)^{q-1}}{\Gamma(q)} |f(s, x(s))| ds \right]
\]
\[
\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} |f(s, x(s))| \, ds + |1 - 2t| \\
\left( \gamma \int_0^\theta \frac{(\theta - s)^{q-1}}{\Gamma(q)} |f(s, x(s))| \, ds + \beta \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} |f(s, x(s))| \, ds \right) \leq M_\sigma = M_1,
\]

which implies that \( \|\mathcal{H}x\| \leq M_1 \). Further, we find that
\[
\|\mathcal{H}x\| \leq \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} |f(s, x(s))| \, ds \\
+ \left( \gamma \int_0^\theta \frac{(\theta - s)^{q-1}}{\Gamma(q)} |f(s, x(s))| \, ds + \beta \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} |f(s, x(s))| \, ds \right) \leq M_\sigma = M_1,
\]

Hence, for \( t_1, t_2 \in [0, T] \), we have
\[
\|\mathcal{H}x(t_2) - \mathcal{H}x(t_1)\| \leq \int_{t_1}^{t_2} \|\mathcal{H}x\| \, ds \leq M_2 |t_2 - t_1|.
\]

This implies that \( \mathcal{H} \) is equicontinuous on \([0, 1]\). Thus, by the Arzelà-Ascoli theorem, the operator \( \mathcal{H} : D \to D \) is completely continuous.

Next, we consider the set
\[
\mathcal{A} = \{ x \in D : x = \gamma \mathcal{H}x, \ 0 < \gamma < 1 \},
\]
and show that the set \( \mathcal{A} \) is bounded. Let \( x \in \mathcal{A} \), and then \( x = \gamma \mathcal{H}x \), \( 0 < \gamma < 1 \). For any \( t \in [0, 1] \), we have
\[
x(t) = -\int_0^t (t-s)^{q-1} \frac{1}{\Gamma(q)} f(s, x(s)) \, ds \\
+ \frac{1}{\gamma(1-2\theta) - 2(\alpha + \beta)} \left[ (\gamma(t-\theta) - \alpha - \beta) \right] \\
\cdot \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s, x(s)) \, ds \\
+ \frac{1}{\gamma(1-2\theta) - 2(\alpha + \beta)} \left[ \gamma(t-\theta) - \alpha - \beta \right] \\
\cdot \left( \gamma \int_0^\theta \frac{(\theta - s)^{q-1}}{\Gamma(q)} |f(s, x(s))| \, ds + |1 - 2t| \right)
\]
and \( |x| = |\gamma \mathcal{H}x| \leq |(\mathcal{H}x)(t)| \leq M_\sigma = M_1 \) for any \( t \in [0, 1] \). So the set \( \mathcal{A} \) is bounded. Thus, the conclusion of Theorem 3 applies and the operator \( \mathcal{H} \) has at least one fixed point. This, in turn, implies that problem (1) has at least one solution on \([0, 1]\).

Now we show the existence of solutions for problem (1) by means of Leray-Schauder degree theory.

**Theorem 5.** Suppose that there exist constants \( 0 \leq \varphi < 1/\sigma \) and \( N > 0 \) such that \( |f(t, x)| \leq \varphi|\|x|N \) for all \( t \in [0, 1], x \in \mathcal{D} \). Then problem (1) has at least one solution.

**Proof.** Define a ball \( B_R \subset \mathcal{D} \) with radius \( R > 0 \) by
\[
B_R = \left\{ x \in \mathcal{D} : \max_{t \in [0, 1]} |x(t)| < R \right\},
\]
where \( R \) will be fixed later. Then, it is enough to show that the operator \( \mathcal{H} : \mathcal{H} \to \mathcal{D} \) (given by (10)) is such that
\[
x \neq \lambda \mathcal{H}x, \quad \forall x \in \partial B_R, \ \forall \lambda \in [0, 1].
\]

Now we set
\[
\mathcal{H}(\lambda, x) = \lambda \mathcal{H}x, \quad \lambda \in [0, 1].
\]

Then, by Arzelà-Ascoli theorem, \( \omega_1(x) = x - \mathcal{H}(\lambda, x) = x - \lambda \mathcal{H}x \) is completely continuous. If condition (20) holds true, then the following Leray-Schauder degrees are well defined and, by the homotopy invariance of topological degree, we have that
\[
\deg(\omega_1, B_R, 0) = \deg(I - \lambda \mathcal{H}, B_R, 0) = \deg(\omega_0, B_R, 0) = \deg(I, B_R, 0) = 1 \neq 0, \quad 0 \in B_R,
\]
where \( I \) denotes the unit operator. By the nonzero property of Leray-Schauder degree, \( \omega_1(t) = x - \lambda \mathcal{H}x = 0 \) for at least one \( x \in B_R \). In order to justify condition (20), it is assumed that \( x = \lambda \mathcal{H}x \) for some \( \lambda \in [0, 1] \) and for all \( t \in [0, 1] \) so that
\[
|x(t)| = |\lambda \mathcal{H}x(t)| \leq \int_0^t (t-s)^{q-1} \frac{1}{\Gamma(q)} |f(s, x(s))| \, ds \\
+ \frac{1}{\gamma(1-2\theta) - 2(\alpha + \beta)} \left[ \gamma(t-\theta) - \alpha - \beta \right] \\
\cdot \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} |f(s, x(s))| \, ds + |1 - 2t| \\
+ \frac{1}{\gamma(1-2\theta) - 2(\alpha + \beta)} \left[ \gamma(t-\theta) - \alpha - \beta \right] \\
\cdot \left( \gamma \int_0^\theta \frac{(\theta - s)^{q-1}}{\Gamma(q)} |f(s, x(s))| \, ds + \beta \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} |f(s, x(s))| \, ds \right) \leq (\varphi \|x\| + N)
\]
In view of condition (26), the operator $\mathcal{K}_2$ is a contraction. Continuity of the operator $\mathcal{K}_1$ follows from that of $f$. Also, $\|\mathcal{K}_1 x\| \leq \|\mu\|/\Gamma(q + 1)$ implies that $\mathcal{K}_1$ is uniformly bounded on $B_r$. Furthermore, with $\sup_{(t, x) \in [0, 1] \times \mathbb{R}} |f(t, x)| = f_0 < \infty$, we have

$$\|(\mathcal{K}_1 x)(t) - (\mathcal{K}_1 y)(t)\| \leq \frac{f_0}{\Gamma(q + 1)} (|t^q - t_1^q| + 2 |t_2 - t_1|^q) \to 0,$$

independent of $x$ as $t_1 \to t_2$. This shows that $\mathcal{K}_1$ is relatively compact on $B_r$. Hence, we infer by the Arzelà-Ascoli theorem that $\mathcal{K}_1$ is compact on $B_r$. Thus all the conditions of Krasnoselskii's fixed point theorem hold true. Hence, problem (1) has at least one solution on $[0, 1]$. This completes the proof.

Finally, we make use of Leray-Schauder nonlinear alternative to show the existence of solutions for problem (1).

**Lemma 7** (nonlinear alternative for single valued maps [19]). Let $E_1$ be a closed, convex subset of a Banach space $E$ and $V$ be an open subset of $E_1$ with $0 \in V$. Suppose that $\mathcal{V} : \overline{V} \to E_1$ is continuous, compact (i.e., $\mathcal{V}(\overline{V})$ is a relatively compact subset of $E_1$) map. Then either

(i) $\mathcal{V}$ has a fixed point in $\overline{V}$ or

(ii) there is $x \in \partial V$ (the boundary of $V$ in $E_1$) and $\kappa \in (0, 1)$ with $x = \kappa \mathcal{V}(x)$.

**Theorem 8.** Let $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that

(H3) there exist a function $p \in C([0, 1], \mathbb{R}^+)$ and a nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $|f(t, x)| \leq p(t)\psi(|x|), V(t, x) \in [0, 1] \times \mathbb{R};$

(H4) there exists a constant $M > 0$ such that

$$\frac{M}{\psi(M)} \|p\| \sigma > 1,$$

where $\sigma$ is given by (10).

Then problem (1) has at least one solution on $[0, 1]$.

**Proof.** As a first step, we show that the operator $\mathcal{K} : \mathcal{D} \to \mathcal{D}$ defined by (10) maps bounded sets into bounded sets in $\mathcal{D}$. For a positive number $r$, let $B_r = \{x \in \mathcal{D} : \|x\| \leq r\}$ be a bounded set in $\mathcal{D}$. Then, for $x \in B_r$, together with (H3) and (H4), we obtain (as before)

$$\|(\mathcal{K}_1 x)(t)\| \leq \|p\| \sigma \psi(M) < M.$$

\[\Box\]
Next, it will be shown that \( H \) maps bounded sets into equicontinuous sets of \( \mathcal{D} \). Let \( t_1, t_2 \in [0, 1] \) and \( x \in B \). Then

\[
|H(t_2) - H(t_1)| = \left| \int_{t_1}^{t_2} \left( \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s)) + \frac{(t_2 - s)^{\beta-1}}{\Gamma(\beta)} f(s, x(s)) \right) ds \right|
\]

Let \( t \in [0, 1] \) and \( \lambda \in (0, 1) \), as before, we obtain

\[
\|x\| = \sup_{t \in [0, 1]} |x(t)| \leq \psi (\|x\|)\|p\|\|\sigma\|.
\]

In view of (H4), there exists a positive constant \( M \) such that \( \|x\| \neq M \). Let us choose \( \mathcal{P} = \{x \in \mathcal{D} : \|x\| < M \} \).

Notice that the operator \( H : \mathcal{P} \to \mathcal{D} \) is continuous and completely continuous. From the choice of \( \mathcal{P} \), there is no \( x \in \partial \mathcal{P} \) such that \( x = \lambda H(x) \) for some \( \lambda \in (0, 1) \). Consequently, the conclusion of Lemma 7 applies and hence the operator \( H \) has a fixed point \( x \in \mathcal{P} \) which is a solution of problem (1). This completes the proof.

3.1. Examples. Consider a three-point boundary value problem of nonlinear fractional differential equations given by

\[
\begin{align*}
\frac{d^\alpha}{dt^\alpha} x(t) + f(t, x(t)) &= 0, \quad t \in [0, 1], \\
x(0) &= -x(1), \\
x'(0) + x'(1) &= -x \left( \frac{1}{2} \right).
\end{align*}
\]

Here \( q = 3/2, \alpha = \beta = \gamma = 1, \) and \( \theta = 1/2 \). With the given values, it is found that

\[
\begin{align*}
\sigma &= \frac{1}{\Gamma(q+1)} \left( 1 + \frac{|y(1-\theta) - \alpha - \beta + \beta q + \gamma q\theta|}{|y(1-\theta) - 2(\alpha + \beta)|} \right) \\
&= \frac{14\sqrt{2} + 1}{6\sqrt{2}\pi} = 1.3829.
\end{align*}
\]

(a) In (34), let us choose a continuous function

\[
\begin{align*}
f(t, x(t)) &= \frac{e^{-t}}{\sqrt{49 + t^2}} \left( \frac{\pi}{2} + x + \tan^{-1} x \right), \\
t &\in [0, 1].
\end{align*}
\]

Clearly \( |f(t, x) - f(t, y)| \leq (2/7)|x - y|, \forall t \in [0, 1], x, y \in \mathbb{R}, \) and \( \ell/\sigma < 1, \) where \( \ell = 2/7 \) and \( \sigma \) is given by (36). Thus problem (34)-(35) with \( f(t, x) \) given by (37) has a unique solution on \([0, 1]\) by Theorem 2.

(b) Taking

\[
f(t, x) = \frac{(t + 1)^2}{16} \left( ((e^{-2t} + 1) \sin|x| + 3\cos^2 t) \right)
\]

in (34), one can notice that \( |f(t, x)| \leq (1/2)|x| + 3/4. \) Thus, with \( q = 1/2, N = 3/4, \) the assumptions of Theorem 5 are satisfied and consequently its conclusion applies to problem (34)-(35) with \( f(t, x) \) given by (38).

(c) Choosing

\[
f(t, x) = \frac{\cos \left( \frac{t^2 + 1}{\sqrt{4 + t}} \right)}{\sqrt{4 + t}} \left( \frac{|x|^3}{1 + |x|} + |x| + \frac{1}{2} \right)
\]

in (34), we find that \( |f(t, x)| \leq p(t)\psi (\|x\|), \forall (t, x) \in [0, 1] \times \mathbb{R}, \) where

\[
p(t) = \frac{\cos \left( \frac{t^2 + 1}{\sqrt{4 + t}} \right)}{\sqrt{4 + t}},
\]

\[
\psi(\|x\|) = \|x\| + \frac{3}{2}.
\]

Using condition (H4), that is, \( M\psi(M)\|p\|\sigma > 1, \) we get \( M > M_1, M_1 = 1.1205. \) Clearly all the conditions of Theorem 8 are satisfied. Hence there exists a solution of problem (34)-(35) with \( f(t, x) \) given by (39).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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