Research Article

A Time Discontinuous Galerkin Finite Element Method for Quasi-Linear Sobolev Equations

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We present a time discontinuous Galerkin finite element scheme for quasi-linear Sobolev equations. The approximate solution is sought as a piecewise polynomial of degree in time variable at most $q - 1$ with coefficients in finite element space. This piecewise polynomial is not necessarily continuous at the nodes of the partition for the time interval. The existence and uniqueness of the approximate solution are proved by use of Brouwer's fixed point theorem. An optimal $L^\infty(0, T; H^1(\Omega))$ -norm error estimate is derived. Just because of a damping term u_{xx} included in quasi-linear Sobolev equations, which is the distinct character different from parabolic equation, more attentions are paid to this term in the study. This is the significance of this paper.

1. Introduction

The time discontinuous Galerkin method was proposed by Delfour in 1981 [1] to analyze the ordinary differential equations. As well known, the fully discrete scheme for unsteady partial differential equations is normally derived by first discretizing in the space variables by the Galerkin finite element method and then replacing the resulting system of ordinary differential equation with respect to time by the finite difference method in the time variable. In [2], the time discontinuous Galerkin method was presented to approximate the heat conduction equation. By applying the Galerkin method to time variable, the purpose of this method was to find the approximate solution as a piecewise polynomial of degree in time variable t at most $q - 1$ with coefficients in finite element space. The piecewise polynomial was not necessarily continuous at the nodes of the partition for the time interval. So, a similar processing method can be used to analyze the space and time variables in the definition and analysis. Unifying the space and time variables, time discontinuous finite element method overcomes the low order accuracy in traditional finite element method caused by the difference discretization in time. This method has high

order accuracy in space and time directions, good dissipation on unstructured mesh, and unconditional stability. Thus, it becomes an efficiency method for the problems dependent on time. It has been successfully used in the fields of fluid mechanics, heat conduction, elastic dynamics, and structural mechanics.

Prior to the work of [2], Lesaint and Raviart [3] and Jamet [4] used such time discontinuous Galerkin finite element method as mentioned above to study the partial differential equations, respectively. After that, many works were analyzed about both the theoretical results and use of the time discontinuous Galerkin finite element method, such as the works of Eriksson, Johnson, Thomée, and Babuska (cf. [5–9]).

Sobolev equations are one of the important partial differential equations in practical use. They arise in the percolation theory of fluid through the crack of rock [10], the problem of moisture migration in soil [11], the problem of heat conduction between different media [12, 13], and so on. The existence and uniqueness of the solution of the Sobolev equation were discussed in [14, 15]. Numerical treatments for Sobolev equations can be found in many papers. The finite element method was discussed for the semilinear Sobolev equations in [16] and the nonlinear Sobolev equations in

[17–20]. The differences-streamline diffusion method and DG finite element method were also discussed for Sobolev equations in [21–23], respectively.

In this paper, we will establish a time discontinuous Galerkin finite element scheme for the quasi-linear Sobolev equations. That is to say, the approximate solution will be sought as a piecewise polynomial in time variable of degree at most $q - 1$ with coefficients in finite element space, which is not necessarily continuous at the nodes of the partition for the time interval. More attentions will be paid to treating a damping term u_{xx} , which is a distinct character of Sobolev equations different from parabolic equation. To our knowledge, this paper appears to be the first trial to approximate quasi-linear Sobolev equations by using the time discontinuous Galerkin finite element method.

The rest of this paper is organized as follows. In Section 2, we briefly introduce the quasi-linear Sobolev equations and some assumptions. In Section 3, we present the time discontinuous Galerkin finite element scheme. In Section 4, we prove the existence and uniqueness of the approximate solution by use of Brouwer's fixed point theorem. In Section 5, an optimal $L^\infty(0, T; H^1(\Omega))$ -norm error estimate is derived. Finally, we describe conclusions and perspectives in Section 6.

2. Mathematical Model

We consider the following quasi-linear Sobolev equations: to find $u = u(x, t)$ such that

$$\begin{aligned} \frac{\partial u}{\partial t} - \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left(a_{ij}(x, t, u) \frac{\partial u}{\partial x_j} + b_{ij}(x) \frac{\partial u_t}{\partial x_j} \right) \\ = f(x, t, u), \quad (x, t) \in \Omega \times (0, T], \\ u = 0, \quad (x, t) \in \partial\Omega \times [0, T], \\ u(x, 0) = u_0(x), \quad x \in \Omega, \quad t = 0, \end{aligned} \quad (1)$$

where Ω is a convex domain in R^d ($d \leq 3$) with a smooth boundary $\partial\Omega$.

We consider the coefficients in (1) with the following conditions (I):

$$(i) \quad f \in L^2(0, T; L^2(\Omega)), \quad u_0 \in H_0^1(\Omega).$$

$$(ii) \quad \text{Assume } a_{ij}(x, t, u) = a_{ji}(x, t, u), \quad b_{ij}(x) = b_{ji}(x), \\ \forall i, j = 1, \dots, d, \quad x \in \Omega, \quad t \in (0, T]. \text{ There exist positive constants } a_0, a_1, b_0, \text{ and } b_1 \text{ satisfying}$$

$$\begin{aligned} a_0 \sum_{i=1}^d \xi_i^2 \leq \sum_{i,j=1}^d a_{ij}(x, t, u) \xi_i \xi_j \leq a_1 \sum_{i=1}^d \xi_i^2, \\ \forall \xi \in \mathcal{R}^d, \quad x \in \Omega, \quad t \in (0, T], \\ b_0 \sum_{i=1}^d \xi_i^2 \leq \sum_{i,j=1}^d b_{ij}(x) \xi_i \xi_j \leq b_1 \sum_{i=1}^d \xi_i^2, \\ \forall \xi \in \mathcal{R}^d, \quad x \in \Omega, \quad t \in (0, T]. \end{aligned} \quad (2)$$

(iii) Assume that $a_{ij}(x, t, z)$ and $f(x, t, z)$ satisfy the uniform Lipschitz condition with respect to the variable z . Let $a_{ij}(x, t, z)$ have uniform bounded derivatives of order up to 1 and 2 with respect to the variable t and variable z , respectively.

For the purpose of theoretical analyses, we need the following regularity assumptions on the solution of (1):

(II)

$$\begin{aligned} u &\in L^\infty(0, T; H_0^1(\Omega)) \cap L^2(0, T; H^{r+1}(\Omega)), \\ &\quad r \geq 1, \\ u_t &\in L^\infty(0, T; H^{r+1}(\Omega)) \cap L^2(0, T; H^{r+1}(\Omega)), \\ D_t^q u &\in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^1(\Omega)), \end{aligned} \quad (3)$$

where $u_t = \partial u / \partial t$ and D_t^q denotes the q th-order derivative with respect to t , $q \geq 1$.

3. Time Discontinuous Galerkin Finite Element Scheme

In this section, we establish the approximate scheme based on the time discontinuous Galerkin method for (1).

First, we introduce two bilinear forms

$$\begin{aligned} A(u; w, v) &= \sum_{i,j=1}^d \left(a_{ij}(x, t, u) \frac{\partial w}{\partial x_i} \frac{\partial v}{\partial x_j} \right), \\ B(w, v) &= \sum_{i,j=1}^d \left(b_{ij}(x) \frac{\partial w}{\partial x_i} \frac{\partial v}{\partial x_j} \right), \end{aligned} \quad \begin{aligned} \forall u, w, v \in H_0^1(\Omega), \\ \forall u, w, v \in H_0^1(\Omega). \end{aligned} \quad (4)$$

With the above notations, the variational form of (1) can be rewritten as follows: to find $u(t) \in H_0^1(\Omega)$ such that

$$\begin{aligned} \int_0^T [(u_t, v) + A(u; u, v) + B(u_t, v)] dt \\ = \int_0^T (f(u), v) dt, \quad \forall v(t) \in H_0^1(\Omega), \\ u(x, 0) = u_0(x), \quad \forall x \in \Omega. \end{aligned} \quad (5)$$

Let $0 = t^0 < t^1 < \dots < t^N = T$ be an unnecessarily uniform subdivision of $[0, T]$, and

$$\begin{aligned} I_n &= (t^n, t^{n+1}], \\ t^{n+1} - t^n &= \tau_n, \\ \tau &= \max_n \tau_n, \end{aligned} \quad (6)$$

$$\forall 0 \leq n \leq N - 1.$$

Let $\mathcal{T}_h = \{K\}$ be a partition of Ω , let h_K be the diameter of element K , and $h = \max_{K \in \mathcal{T}_h} h_K$. Define $S_{h,0} \subset H_0^1(\Omega)$ as a finite element space with index r (cf. [24]). We want to find an approximate solution of (1) belonging to the space

$$V_{h,\tau} = \left\{ v_h : v_h|_{I_n} = \sum_{j=0}^{q-1} \omega_{h,j} t^j, \omega_{h,j} \in S_{h,0} \right\}. \quad (7)$$

In other words, a function in $V_{h,\tau}$ reduces to a polynomial of degree at most $q-1$ in t on each interval I_n with coefficients in $S_{h,0}$. Note that these functions in $V_{h,\tau}$ are allowed to be discontinuous at the nodes t^n , $n = 0, \dots, N-1$, and are taken to be continuous to the left side of the nodes. For $U \in V_{h,\tau}$, we denote U^n and U_+^n as the value of U and its limit from above t^n , respectively. We write $V_{h,\tau}^n$ for the restrictions to I_n of the function in $V_{h,\tau}$.

Our time discontinuous Galerkin finite element scheme is to find $U \in V_{h,\tau}$ such that

$$\begin{aligned} & \sum_{n=0}^{N-1} \left[(U^{n+1}, v^{n+1}) - (U^n, v_+^n) - \int_{I_n} (U, v_t) dt \right. \\ & \quad + \int_{I_n} A(U; U, v) dt + B(U^{n+1}, v^{n+1}) - B(U^n, v_+^n) \\ & \quad \left. - \int_{I_n} B(U, v_t) dt \right] = \sum_{n=0}^{N-1} \int_{I_n} (f(U), v) dt, \end{aligned} \quad (8)$$

$\forall v \in V_{h,\tau},$

with an initial value

$$U^0 = u_{0,h}. \quad (9)$$

Here, $u_{0,h}$ is a certain approximation to u_0 in $S_{h,0}$.

Noticing that there exist

$$\begin{aligned} & - \sum_{n=0}^{N-1} \int_{I_n} (U, v_t) dt = \sum_{n=0}^{N-1} \left[\int_{I_n} (U_t, v) dt - (U^{n+1}, v^{n+1}) \right. \\ & \quad \left. + (U_+^n, v_+^n) \right], \\ & - \sum_{n=0}^{N-1} \int_{I_n} B(U, v_t) dt = \sum_{n=0}^{N-1} \left[\int_{I_n} B(U_t, v) dt \right. \\ & \quad \left. - B(U^{n+1}, v^{n+1}) + B(U_+^n, v_+^n) \right], \end{aligned} \quad (10)$$

we can rewrite (8) as

$$\begin{aligned} & \sum_{n=0}^{N-1} \left\{ \int_{I_n} [(U_t, v) + A(U; U, v) + B(U_t, v)] dt \right. \\ & \quad \left. + (U_+^n - U^n, v_+^n) + B(U_+^n - U^n, v_+^n) \right\} \\ & = \sum_{n=0}^{N-1} \int_{I_n} (f(U), v) dt, \end{aligned} \quad (11)$$

$$\forall v \in V_{h,\tau}^n, \quad n = 0, 1, \dots, N-1.$$

Hence, on each time interval I_n , we have a local scheme: to find $U \in V_{h,\tau}^n$, $n = 0, 1, \dots, N-1$, such that

$$\begin{aligned} & \int_{I_n} [(U_t, v) + A(U; U, v) + B(U_t, v)] dt + (U_+^n, v_+^n) \\ & \quad + B(U_+^n, v_+^n) \\ & = (U^n, v_+^n) + B(U^n, v_+^n) + \int_{I_n} (f(U), v) dt, \end{aligned} \quad (12)$$

$\forall v \in V_{h,\tau}^n.$

Remark 1. From scheme (12), we can notice that the solution on I_n can be determined once U^n and U_+^n are given. In fact, the solution is defined on I_n by q elements in $S_{h,0}$, or qN_h scalars if the dimension of $S_{h,0}$ is N_h , and the number of equations, which is equal to the dimension of $V_{h,\tau}^n$, is also qN_h . Therefore, the main advantage of our scheme is that the approximate solution can be solved in each time interval gradually.

4. Existence and Uniqueness of the Solution

In this section, we prove the existence and uniqueness of solution (12). We can obtain the following theorem.

Theorem 2. *When τ_n is small enough, the solution of (12) exists uniquely.*

Proof. Let us introduce a projection $T : V_{h,\tau}^n \mapsto V_{h,\tau}^n$ by

$$\begin{aligned} & \int_{I_n} [(Y_t, v) + A(X; Y, v) + B(Y_t, v)] dt \\ & \quad + (Y_+^n - U^n, v_+^n) + B(Y_+^n - U^n, v_+^n) \\ & = \int_{I_n} (f(X), v) dt, \quad \forall v \in V_{h,\tau}^n, \end{aligned} \quad (13)$$

where $Y = TX$, $X \in V_{h,\tau}^n$.

Obviously, for any $X \in V_{h,\tau}^n$, the above equation has a unique solution so that the projection T is uniquely identified in $V_{h,\tau}^n$. If T has a fixed point, $X^* = TX^*$, then $U = X^*$ is the solution of (12). If T has a unique fixed point, the solution of (12) is unique.

Now, we begin to prove that T has a unique fixed point by three steps. For a real number $R > 0$, we define a set

$$\begin{aligned} D_R & = \left\{ X \right. \\ & \quad \left. \in V_{h,\tau}^n, \int_{I_n} \{ \|X - U^n\|^2 + b_0 \| \nabla (X - U^n) \|^2 \} dt \right. \\ & \quad \left. \leq R \right\}. \end{aligned} \quad (14)$$

It is easy to see that D_R is a closed set in $V_{h,\tau}^n$.

Step 1 (we prove that when τ_n is small enough, T maps D_R onto itself). (1) Taking $v = Y - U^n$ in (13), we can obtain

$$\begin{aligned}
& \frac{1}{2} \int_{I_n} \frac{d}{dt} \|Y - U^n\|^2 dt + \int_{I_n} A(X; Y - U^n, Y - U^n) dt \\
& + \int_{I_n} A(X; U^n, Y - U^n) dt \\
& + \frac{1}{2} \int_{I_n} \frac{d}{dt} \|\nabla(Y - U^n)\|_b^2 dt + \|Y_+^n - U^n\|^2 \\
& + \|\nabla(Y_+^n - U^n)\|_b^2 \\
& = \int_{I_n} (f(X) - f(U^n), Y - U^n) dt \\
& + \int_{I_n} (f(U^n), Y - U^n) dt,
\end{aligned} \tag{15}$$

where the norm $\|v\|_b^2 := (b(x)v, v)$, $\forall v \in V_{h,\tau}^n$.

Applying the ε -inequality, we find

$$\begin{aligned}
& \int_{I_n} A(X; U^n, Y - U^n) dt \leq \frac{1}{2a_0} \int_{I_n} a_1^2 \|\nabla U^n\|^2 dt \\
& + \frac{a_0}{2} \int_{I_n} \|\nabla(Y - U^n)\|^2 dt \leq \frac{\tau_n a_1^2}{2a_0} \|\nabla U^n\|^2 \\
& + \frac{a_0}{2} \int_{I_n} \|\nabla(Y - U^n)\|^2 dt, \\
& \int_{I_n} (f(X) - f(U^n), Y - U^n) dt \\
& + \int_{I_n} (f(U^n), Y - U^n) dt \leq 2\varepsilon \int_{I_n} \|Y - U^n\|^2 dt \\
& + \frac{1}{4\varepsilon} \int_{I_n} \{M^2 \|X - U^n\|^2 + \|f(U^n)\|^2\} dt \\
& \leq 2\varepsilon \int_{I_n} \|Y - U^n\|^2 dt + \frac{M^2}{4\varepsilon} \int_{I_n} \|X - U^n\|^2 dt \\
& + \frac{\tau_n}{4\varepsilon} \|f(U^n)\|^2,
\end{aligned} \tag{16}$$

where M is the Lipschitz constant of $f(x, t, z)$ with respect to z .

Combining (15) with (16) shows

$$\begin{aligned}
& \frac{1}{2} \{ \|Y^{n+1} - U^n\|^2 + \|Y_+^n - U^n\|^2 \} \\
& + \frac{a_0}{2} \int_{I_n} \|\nabla(Y - U^n)\|^2 dt \\
& + \frac{1}{2} \{ \|\nabla(Y^{n+1} - U^n)\|_b^2 + \|\nabla(Y_+^n - U^n)\|_b^2 \}
\end{aligned}$$

$$\begin{aligned}
& \leq \frac{\tau_n a_1^2}{2a_0} \|\nabla U^n\|^2 + \frac{M^2}{4\varepsilon} \int_{I_n} \|X - U^n\|^2 dt \\
& + \frac{\tau_n}{4\varepsilon} \|f(U^n)\|^2 + 2\varepsilon \int_{I_n} \|Y - U^n\|^2 dt.
\end{aligned} \tag{17}$$

(2) Taking $v = (t - t_n)Y_t$ in (13), since $v_+^n = 0$, we have

$$\begin{aligned}
& \int_{I_n} (t - t_n) \left[\|Y_t\|^2 + \frac{a_0}{2} \frac{d}{dt} \|\nabla Y\|^2 + \|\nabla Y_t\|_b^2 \right] dt \\
& \leq \int_{I_n} (f(X), (t - t_n)Y_t) dt.
\end{aligned} \tag{18}$$

Notice that there exists

$$\begin{aligned}
& \int_{I_n} (t - t_n) \frac{d}{dt} \|\nabla Y\|^2 dt \\
& = \tau_n \|\nabla Y^{n+1}\|^2 - \int_{I_n} \|\nabla Y\|^2 dt.
\end{aligned} \tag{19}$$

Hence, inserting (19) into (18) yields

$$\begin{aligned}
& \int_{I_n} (t - t_n) \left[\|Y_t\|^2 + \|\nabla Y_t\|_b^2 \right] dt + \frac{a_0 \tau_n}{2} \|\nabla Y^{n+1}\|^2 \\
& \leq \frac{a_0}{2} \int_{I_n} \|\nabla Y\|^2 dt \\
& + \int_{I_n} (f(X) - f(U^n), (t - t_n)Y_t) dt \\
& + \int_{I_n} (f(U^n), (t - t_n)Y_t) dt.
\end{aligned} \tag{20}$$

Using the ε -inequality to (20), we obtain

$$\begin{aligned}
& \int_{I_n} (t - t_n) \left[\|Y_t\|^2 + b_0 \|\nabla Y_t\|^2 \right] dt + \frac{a_0 \tau_n}{2} \|\nabla Y^{n+1}\|^2 \\
& \leq \frac{a_0}{2} \int_{I_n} \|\nabla Y\|^2 dt + \varepsilon \int_{I_n} (t - t_n) \|Y_t\|^2 dt \\
& + \frac{M^2}{4\varepsilon} \int_{I_n} (t - t_n) \|X - U^n\|^2 dt \\
& + \varepsilon \int_{I_n} (t - t_n) \|Y_t\|^2 dt + \frac{\tau_n^2}{4\varepsilon} \|f(U^n)\|^2.
\end{aligned} \tag{21}$$

Therefore, by taking $\varepsilon = 1/4$ in (21), the following estimate holds:

$$\begin{aligned}
& \frac{1}{2} \int_{I_n} (t - t_n) \left[\|Y_t\|^2 + b_0 \|\nabla Y_t\|^2 \right] dt + \frac{a_0 \tau_n}{2} \|\nabla Y^{n+1}\|^2 \\
& \leq \frac{a_0}{2} \int_{I_n} \|\nabla Y\|^2 dt + M^2 \tau_n \int_{I_n} \|X - U^n\|^2 dt \\
& + \tau_n^2 \|f(U^n)\|^2.
\end{aligned} \tag{22}$$

(3) It is easy to see that there exists

$$\begin{aligned} & \int_{I_n} \|Y - U^n\|^2 dt \\ & \leq 2\tau_n \left(\|Y^{n+1} - U^n\|^2 + \tau_n \int_{I_n} \|Y_t\|^2 dt \right), \\ & \int_{I_n} \|\nabla(Y - U^n)\|^2 dt \\ & \leq 2\tau_n \left(\|\nabla(Y^{n+1} - U^n)\|^2 + \tau_n \int_{I_n} \|\nabla Y_t\|^2 dt \right). \end{aligned} \quad (23)$$

As $\|Y_t\|^2$ and $\|\nabla Y_t\|^2$ are both polynomials in time of degree $2(q-2)$ in I_n , and the norms are equivalent on finite dimensional space, we know that there exists a constant $C_0 > 0$, such that

$$\begin{aligned} \tau_n \int_{I_n} \|Y_t\|^2 dt & \leq C_0 \int_{I_n} (t - t_n) \|Y_t\|^2 dt, \\ \tau_n \int_{I_n} \|\nabla Y_t\|^2 dt & \leq C_0 \int_{I_n} (t - t_n) \|\nabla Y_t\|^2 dt. \end{aligned} \quad (24)$$

Then, it follows from (23) and (24) that

$$\begin{aligned} & \int_{I_n} (\|Y - U^n\|^2 + b_0 \|\nabla(Y - U^n)\|^2) dt \\ & \leq 2\tau_n \left(\|Y^{n+1} - U^n\|^2 + \tau_n \int_{I_n} \|Y_t\|^2 dt \right. \\ & \quad \left. + b_0 \|\nabla(Y^{n+1} - U^n)\|^2 + \tau_n b_0 \int_{I_n} \|\nabla Y_t\|^2 dt \right) \\ & \leq 2\tau_n \left(\|Y^{n+1} - U^n\|^2 + b_0 \|\nabla(Y^{n+1} - U^n)\|^2 \right. \\ & \quad \left. + C_0 \int_{I_n} (t - t_n) (\|Y_t\|^2 + b_0 \|\nabla Y_t\|^2) dt \right). \end{aligned} \quad (25)$$

Furthermore, (25) can be amplified as

$$\begin{aligned} & \int_{I_n} (\|Y - U^n\|^2 + b_0 \|\nabla(Y - U^n)\|^2) dt \\ & \leq 2\tau_n \left(\|Y^{n+1} - U^n\|^2 + b_0 \|\nabla(Y^{n+1} - U^n)\|^2 \right. \\ & \quad \left. + a_0 \int_{I_n} \|\nabla(Y - U^n)\|^2 dt \right. \\ & \quad \left. + C_0 \int_{I_n} (t - t_n) (\|Y_t\|^2 + b_0 \|\nabla Y_t\|^2) dt \right. \\ & \quad \left. + a_0 C_0 \tau_n \|\nabla Y^{n+1}\|^2 \right). \end{aligned} \quad (26)$$

Putting (17) and (22) into the right-hand side of (26) and using

$$\int_{I_n} \|\nabla Y\|^2 dt \leq 2 \int_{I_n} \|\nabla(Y - U^n)\|^2 dt + 2\tau_n \|\nabla U^n\|^2, \quad (27)$$

we have

$$\begin{aligned} & \int_{I_n} (\|Y - U^n\|^2 + b_0 \|\nabla(Y - U^n)\|^2) dt \\ & \leq 2\tau_n \left(4\epsilon \int_{I_n} \|Y - U^n\|^2 dt \right. \\ & \quad \left. + 2C_0 a_0 \int_{I_n} \|\nabla(Y - U^n)\|^2 dt \right) \\ & \quad + 2\tau_n \left\{ \left(\frac{a_1^2}{a_0} + 2C_0 a_0 \right) \tau_n \|\nabla U^n\|^2 \right. \\ & \quad \left. + \left(\frac{M^2}{2\epsilon} + 2C_0 M^2 \tau_n \right) \int_{I_n} \|X - U^n\|^2 dt \right. \\ & \quad \left. + \left(\frac{\tau_n}{2\epsilon} + 2C_0 \tau_n^2 \right) \|f(U^n)\|^2 \right\}. \end{aligned} \quad (28)$$

If τ_n is taken to be small enough to satisfy (i) $\max\{8\tau_n\epsilon, 4\tau_n(C_0 a_0/b_0)\} < 1/2$ and (ii) the last term on the right-hand side of (28) $< R/2$, we derive

$$\int_{I_n} (\|Y - U^n\|^2 + b_0 \|\nabla(Y - U^n)\|^2) dt \leq R; \quad (29)$$

that is, $T(D_R) \subset D_R$.

Step 2 (we prove that when τ_n is small enough, T is continuous on D_R). For any point $X_0 \in D_R$, $X \in D_R$, denote $Y_0 = TX_0$, $Y = TX$. Then, from (13) we have

$$\begin{aligned} & \int_{I_n} [((Y - Y_0)_t, v) + A(X; Y - Y_0, v) \\ & \quad + B((Y - Y_0)_t, v)] dt + ((Y - Y_0)_+^n, v_+^n) + B((Y - Y_0)_+^n, v_+^n) \\ & \quad = \int_{I_n} [(f(X) - f(X_0), v) \\ & \quad - A(X; Y_0, v) + A(X_0; Y_0, v)] dt. \end{aligned} \quad (30)$$

Taking $v = Y - Y_0$ in (30) and with similar analyses in Step 1, we can find that when τ_n is small enough, there exists

$$\lim_{X \rightarrow X_0} \int_{I_n} (\|Y - Y_0\|^2 + b_0 \|\nabla(Y - Y_0)\|^2) dt = 0. \quad (31)$$

This means T is continuous on D_R .

Step 3. The above proofs show that T is a contraction mapping on D_R . Using Brouwer's fixed point theorem, we know that T has a unique fixed point in D_R ; that is, $X^* = TX^*$. Therefore, the existence and uniqueness of the solution $U = X^*$ of (12) have been proved.

Thus, the proof of Theorem 2 is completed. $\square$

5. $L^\infty(0, T; H^1(\Omega))$ -Norm Error Estimate

In order to derive an optimal $L^\infty(0, T; H^1(\Omega))$ -norm error estimate, it is necessary to refer to a nonstandard elliptic

projection $P_1 : u \in [0, T] \rightarrow P_1 u \in S_{h,0}$ from [17] such that

$$\begin{aligned} A(u; P_1 u - u, v) + B((P_1 u - u)_t, v) \\ + \lambda ((P_1 u - u)_t, v) = 0, \quad \forall v \in S_{h,0}, \end{aligned} \quad (32)$$

where constant $\lambda > 0$. Also, we have the following lemmas about projection P_1 .

Lemma 3 (see [17]). *There exists a unique solution $P_1 u \in S_{h,0}$ for any $\lambda > 0$.*

Lemma 4 (see [17]). *Assume that $\|u_{0,h} - u_0\| + h\|u_{0,h} - u_0\|_1 \leq Ch^{r+1}\|u_0\|_{r+1}$ and $u, u_t \in H^{r+1}(\Omega)$, $u, \nabla u, u_t$, and ∇u_t are bounded. Then, there exists a positive constant $C = C(u)$ such that*

$$\begin{aligned} \|P_1 u - u\| + h\|\nabla(P_1 u - u)\| &\leq Ch^{r+1}\|u\|_{H^{r+1}}, \\ \|(P_1 u - u)_t\| + h\|\nabla(P_1 u - u)_t\| &\leq Ch^{r+1}\|u_t\|_{H^{r+1}}. \end{aligned} \quad (33)$$

Lemma 5 (see [17]). *Assume that $u, u_t \in H^{r+1}(\Omega)$ and $u, \nabla u, u_t$, and ∇u_t are bounded. There exists a positive constant $C = C(u)$ such that*

$$\begin{aligned} \|P_1 u\|_{L^\infty} + \|\nabla P_1 u\|_{L^\infty} + \|(P_1 u)_t\|_{L^\infty} + \|\nabla(P_1 u)_t\|_{L^\infty} \\ \leq C(u). \end{aligned} \quad (34)$$

Lemma 6 (see [17]). *Let $\pi = P_1 u - u$. There exists a positive constant C such that*

$$\begin{aligned} \|\pi\|_{L^2(0,T;L^2(\Omega))} &\leq Ch^{r+1}\|u\|_{L^2(0,T;H^{r+1}(\Omega))}, \\ \|\pi_t\|_{L^2(0,T;L^2(\Omega))} &\leq Ch^{r+1}\|u_t\|_{L^2(0,T;H^{r+1}(\Omega))}. \end{aligned} \quad (35)$$

Let $\tilde{U} \in V_{h,r}^n$ be the interpolation of $P_1 u$ with respect to t satisfying

$$\begin{aligned} \tilde{U}\left(t_{n+1} - \frac{j}{q-1}\tau_n\right) &= P_1 u\left(t_{n+1} - \frac{j}{q-1}\tau_n\right), \\ 0 \leq j &\leq q-1. \end{aligned} \quad (36)$$

By the Lagrange interpolation theory ([24]), the following lemma holds.

Lemma 7. *Let $\rho = \tilde{U} - P_1 u$, where $\tilde{U}$ satisfies (36). Then, there exist*

$$\begin{aligned} \sup_{I_n} \|D_t^i \rho\|_{H^s} &\leq C\tau_n^{j-i} \sup_{I_n} \|D_t^j u\|_{H^s}, \\ \int_{I_n} \|D_t^i \rho\|_{H^s}^2 dt &\leq C\tau_n^{2(j-i)} \int_{I_n} \|D_t^j u\|_{H^s}^2 dt, \end{aligned} \quad (37)$$

where $0 \leq i \leq q-1$, $i+1 \leq j \leq q$, and $-1 \leq s \leq 1$; D_t^i denotes the i th-order derivative with respect to t ; C is a positive constant depending only on q .

We begin to analyze the error estimate about $u - U$. From (5) and (12), we can obtain the error equation

$$\begin{aligned} \int_{I_n} [(U_t - u_t, v) + A(U; U, v) - A(u; u, v) \\ + B(U_t - u_t, v)] dt + (U_+^n, v_+^n) + B(U_+^n, v_+^n) = (U^n, \\ v_+^n) + B(U^n, v_+^n) + \int_{I_n} (f(U) - f(u), v) dt, \\ \forall v \in V_{h,\tau}^n. \end{aligned} \quad (38)$$

Defining $\theta = U - \tilde{U}$, $\pi = P_1 u - u$, then we have $U - u = \theta + \rho + \pi$. Taking them into (38), we derive

$$\begin{aligned} \int_{I_n} [(\theta_t + \rho_t + \pi_t, v) + A(U; \theta + \rho + \pi, v) \\ + A(U; u, v) - A(u; u, v) + B(\theta_t + \rho_t + \pi_t, v)] dt \\ + (\theta_+^n - \theta^n, v_+^n) + B(\theta_+^n - \theta^n, v_+^n) = \int_{I_n} (f(U) \\ - f(u), v) dt, \quad \forall v \in V_{h,\tau}^n. \end{aligned} \quad (39)$$

Applying elliptic projection (32), we find

$$\begin{aligned} A(U; \pi, v) + A(U; u, v) - A(u; u, v) + B(\pi_t, v) \\ = A(U; P_1 u, v) - A(u; u, v) + B(\pi_t, v) \\ = A(U; P_1 u, v) - A(u; P_1 u, v) + A(u; P_1 u - u, v) \\ + B(\pi_t, v) \\ = A(U; P_1 u, v) - A(u; P_1 u, v) - \lambda(\pi_t, v). \end{aligned} \quad (40)$$

Hence, (39) can be rewritten to

$$\begin{aligned} \int_{I_n} [(\theta_t, v) + A(U; \theta, v) + B(\theta_t, v)] dt + (\theta_+^n - \theta^n, v_+^n) \\ + B(\theta_+^n - \theta^n, v_+^n) = \int_{I_n} [(\rho, v_t) + (\lambda - 1)(\pi_t, v) \\ - A(U; \rho, v) + B(\rho, v_t)] dt + \int_{I_n} [A(u; P_1 u, v) \\ - A(U; P_1 u, v)] dt + \int_{I_n} (f(U) - f(u), v) dt, \\ \forall v \in V_{h,\tau}^n. \end{aligned} \quad (41)$$

From the two inequalities

$$\begin{aligned} \|\theta(t)\| &\leq \|\theta^{n+1}\| + \int_{I_n} \|\theta_t\| dt, \\ \|\nabla \theta(t)\| &\leq \|\nabla \theta^{n+1}\| + \int_{I_n} \|\nabla \theta_t\| dt, \end{aligned} \quad (42)$$

we know that

$$\begin{aligned}\|\theta(t)\|^2 &\leq 2 \left(\|\theta^{n+1}\|^2 + \tau_n \int_{I_n} \|\theta_t\|^2 dt \right), \\ \|\nabla \theta(t)\|^2 &\leq 2 \left(\|\nabla \theta^{n+1}\|^2 + \tau_n \int_{I_n} \|\nabla \theta_t\|^2 dt \right).\end{aligned}\quad (43)$$

So, it remains to estimate $\|\theta^{n+1}\|_1$ and $\int_{I_n} \|\nabla \theta_t\| dt$. From now on, we use notations $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_1 = \|\cdot\|_{H^1(\Omega)}$.

Lemma 8. Let $\xi \in V_{h,\tau}^n$, $\zeta, \zeta_t \in L^2(I_n, H^1(\Omega))$ satisfying $\zeta^n = \zeta^{n+1} = 0$, $\eta, \eta_t \in L^2(I_n, L^2(\Omega))$. Define $\xi + \zeta + \eta = U - u$ to satisfy

$$\begin{aligned}&\int_{I_n} [(\xi_t, v) + A(U; \xi, v) + B(\xi_t, v)] dt + (\xi_+^n - \xi^n, v_+^n) \\ &+ B(\xi_+^n - \xi^n, v_+^n) = \int_{I_n} [(\zeta, v_t) + (\lambda - 1)(\eta_t, v) \\ &- A(U; \zeta, v) + B(\zeta, v_t)] dt + \int_{I_n} [A(u; P_1 u, v) \\ &- A(U; P_1 u, v)] dt + \int_{I_n} (f(U) - f(u), v) dt, \\ &\forall v \in V_{h,\tau}^n.\end{aligned}\quad (44)$$

Then, we can have the following estimates:

$$\begin{aligned}&\int_{I_n} [\|\xi_t\|^2 + b_0 \|\nabla \xi_t\|^2] dt \\ &\leq C \int_{I_n} [\|\zeta_t\|_1^2 + \|\zeta\|_1^2 + \|\eta_t\|^2 + \|\xi\|_1^2 + \|\eta\|^2] dt,\end{aligned}\quad (45)$$

$$\begin{aligned}&\|\xi^{n+1}\|^2 + b_0 \|\nabla \xi^{n+1}\|^2 + a_0 \int_{I_n} \|\nabla \xi\|^2 dt \\ &\leq \|\xi^n\|^2 + b_1 \|\nabla \xi^n\|^2 \\ &+ C(\varepsilon) \int_{I_n} [\|\zeta_t\|_1^2 + \|\zeta\|_1^2 + \|\eta\|^2 + \|\eta_t\|^2] dt,\end{aligned}\quad (46)$$

$$\|\xi^{n+1}\|_1^2 \leq \|\xi^0\|_1^2 + \int_{I_n} [\|\zeta_t\|_1^2 + \|\zeta\|_1^2 + \|\eta\|^2 + \|\eta_t\|^2] dt. \quad (47)$$

Proof. (1) Taking $v = (t - t_n)\xi_t$ in (44), since $v_+^n = 0$, we can get

$$\begin{aligned}&\int_{I_n} (t - t_n) [(\xi_t, \xi_t) + A(U; \xi, \xi_t) + B(\xi_t, \xi_t)] dt \\ &= \int_{I_n} (t - t_n) [-(\zeta_t, \xi_t) + (\lambda - 1)(\eta_t, \xi_t) \\ &- A(U; \zeta, \xi_t) - B(\zeta_t, \xi_t)] dt + \int_{I_n} (t - t_n) \\ &\cdot [A(u; P_1 u, \xi_t) - A(U; P_1 u, \xi_t)] dt + \int_{I_n} (t \\ &- t_n) (f(U) - f(u), \xi_t) dt.\end{aligned}\quad (48)$$

Further, we have

$$\begin{aligned}&\int_{I_n} (t - t_n) [\|\xi_t\|^2 + \|\nabla \xi_t\|_b^2] dt \leq \frac{1}{2} \int_{I_n} (t - t_n) \\ &\cdot [\|\xi_t\|^2 + \|\nabla \xi_t\|_b^2] dt + C \int_{I_n} (t - t_n) \\ &\cdot [\|\zeta_t\|_1^2 + \|\nabla \zeta\|^2 + \|\eta_t\|^2 + \|\xi\|_1^2 + \|\zeta\|^2 + \|\eta\|^2] dt.\end{aligned}\quad (49)$$

Because $\|\xi_t\|^2$ and $\|\nabla \xi_t\|^2$ are both polynomials in t of degree at most $2(q - 2)$, we have two inverse inequalities

$$\tau_n \int_{I_n} \|\xi_t\|^2 dt \leq C \int_{I_n} (t - t_n) \|\xi_t\|^2 dt, \quad (50)$$

$$\tau_n \int_{I_n} \|\nabla \xi_t\|^2 dt \leq C \int_{I_n} (t - t_n) \|\nabla \xi_t\|^2 dt. \quad (51)$$

Combining the above inequalities with (49), we obtain

$$\begin{aligned}&\int_{I_n} [\|\xi_t\|^2 + b_0 \|\nabla \xi_t\|^2] dt \\ &\leq C \int_{I_n} [\|\zeta_t\|_1^2 + \|\zeta\|_1^2 + \|\eta_t\|^2 + \|\xi\|_1^2 + \|\eta\|^2] dt.\end{aligned}\quad (52)$$

This is just estimate (45).

(2) Now, we go to prove estimate (46). Taking $v = \xi$ into (44), we have

the left-hand side of (44)

$$\begin{aligned}&\geq \int_{I_n} \left\{ \frac{d}{2dt} \|\xi\|^2 + a_0 \|\nabla \xi\|^2 + \frac{d}{2dt} \|\nabla \xi\|_b^2 \right\} dt \\ &+ \frac{1}{2} (\|\xi_+^n\|^2 - \|\xi^n\|^2) + \frac{1}{2} (\|\nabla \xi_+^n\|_b^2 - \|\nabla \xi^n\|_b^2) \\ &= \frac{1}{2} (\|\xi^{n+1}\|^2 - \|\xi^n\|^2) + a_0 \int_{I_n} \|\nabla \xi\|^2 dt \\ &+ \frac{1}{2} (\|\nabla \xi^{n+1}\|_b^2 - \|\nabla \xi^n\|_b^2),\end{aligned}\quad (53)$$

the right-hand side of (44)

$$\begin{aligned}&\leq \varepsilon \int_{I_n} [\|\xi_t\|^2 + \|\nabla \xi\|^2] dt \\ &+ C(\varepsilon) \int_{I_n} [\|\zeta\|_1^2 + \|\eta\|^2 + \|\eta_t\|^2 + \|\xi\|^2] dt.\end{aligned}$$

From (53), we conclude that

$$\begin{aligned}&\frac{1}{2} (\|\xi^{n+1}\|^2 - \|\xi^n\|^2) + a_0 \int_{I_n} \|\nabla \xi\|^2 dt \\ &+ \frac{1}{2} (\|\nabla \xi^{n+1}\|_b^2 - \|\nabla \xi^n\|_b^2) \\ &\leq \varepsilon \int_{I_n} [\|\xi_t\|^2 + \|\nabla \xi\|^2] dt \\ &+ C(\varepsilon) \int_{I_n} [\|\zeta\|_1^2 + \|\eta\|^2 + \|\eta_t\|^2 + \|\xi\|^2] dt.\end{aligned}\quad (54)$$

Combining (54) with (45), taking ε to be small enough, and using the Gronwall lemma, we can derive estimate (46).

(3) Finally, we prove estimate (47). Define a bilinear form: $\forall U, w, v \in V_{h,\tau}$,

$$\begin{aligned} B_n(U; w, v) &= \sum_{j=0}^n \left\{ \int_{I_j} [(w_t, v) + A(U; w, v) + B(w_t, v)] dt \right. \\ &\quad \left. + (w_+^j - w_-^j, v_+^j) + (b \nabla (w_+^j - w_-^j), \nabla v_+^j) \right\} + (w_+^0, \\ &\quad v_+^0) + (b \nabla w_+^0, \nabla v_+^0). \end{aligned} \quad (55)$$

From (44), we know that ξ, ζ , and η defined in Lemma 8 satisfy $\forall v \in V_{h,\tau}$

$$\begin{aligned} B_n(U; \xi, v) &= \sum_{j=0}^n \left\{ \int_{I_j} [(\xi_t, v) + A(U; \xi, v) + B(\xi_t, v)] dt \right. \\ &\quad \left. + (\xi_+^j - \xi_-^j, v_+^j) + (b \nabla (\xi_+^j - \xi_-^j), \nabla v_+^j) \right\} + (\xi_+^0, v_+^0) \\ &\quad + (b \nabla \xi_+^0, \nabla v_+^0) \\ &= \sum_{j=0}^n \left\{ \int_{I_j} [(\zeta, v_t) + (\lambda - 1)(\eta_t, v) - A(U; \zeta, v)] dt \right. \\ &\quad \left. + \int_{I_j} [A(u; P_1 u, v) - A(U; P_1 u, v)] dt \right. \\ &\quad \left. + \int_{I_j} (f(U) - f(u), v) dt \right\} + (\xi_+^0, v_+^0) + (b \nabla \xi_+^0, \\ &\quad \nabla v_+^0). \end{aligned} \quad (56)$$

Let $v_h \in S_{h,0}$. We introduce an auxiliary function $\beta \in V_{h,\tau}$, which satisfies $\beta^0 = v_h$ and

$$B_n(U; \beta, v) = (c v_h, v_+^0) + (b \nabla v_h, \nabla v_+^0), \quad \forall v \in V_{h,\tau}. \quad (57)$$

Using (55), we know

$$\begin{aligned} &\int_{I_j} [(\beta_t, v) + A(U; \beta, v) + B(\beta_t, v)] dt \\ &\quad + (\beta_+^n - \beta_-^n, v_+^n) + (b \nabla (\beta_+^n - \beta_-^n), \nabla v_+^n) = 0, \end{aligned} \quad (58)$$

where β satisfies the conditions in Lemma 8.

Hence, from (45) and (46), we can derive the estimate

$$\begin{aligned} \|\beta^{n+1}\|^2 &+ \sum_{j=0}^n \int_{I_j} [\|\beta_t\|^2 + a_0 \|\nabla \beta\|^2 + b_0 \|\nabla \beta_t\|^2] dt \\ &+ b_0 \|\nabla \beta^{n+1}\|^2 \leq \|v_h\|^2 + b_1 \|\nabla v_h\|^2. \end{aligned} \quad (59)$$

Now, we consider an adjoint problem of (1): to find $\beta^* \in V_{h,\tau}$ such that

$$B_n(u; w, \beta^*) = (v_h, w^{n+1}) + (b \nabla v_h, \nabla w^{n+1}). \quad (60)$$

In other words, that β^* is the finite element solution in $V_{h,\tau}$ of

$$\begin{aligned} -\frac{\partial \beta^*}{\partial t} - \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left(a_{ij}(x, t, u) \frac{\partial \beta^*}{\partial x_j} - b_{ij}(x) \frac{\partial \beta_t^*}{\partial x_j} \right) &= 0, \\ x \in \Omega, \quad 0 < t < t_{n+1}, \end{aligned} \quad (61)$$

$$\beta^*|_{t_{n+1}} = v_h,$$

$$\beta^* = 0, \quad x \in \partial\Omega.$$

Similar to the derivation of result (59), we can bound the estimate

$$\begin{aligned} \|\beta_+^{*0}\|^2 &+ \sum_{j=0}^n \int_{I_j} [\|\beta_t^*\|^2 + b_0 \|\nabla \beta_t^*\|^2 + a_0 \|\nabla \beta^*\|^2] dt \\ &+ b_0 \|\nabla \beta_+^{*0}\|^2 \leq \|v_h\|^2 + b_1 \|\nabla v_h\|^2. \end{aligned} \quad (62)$$

Taking $w = \xi$ in (60), we have

$$\begin{aligned} (\xi^{n+1}, v_h) + (b \nabla \xi^{n+1}, \nabla v_h) &= B_n(u; \xi, \beta^*) \\ &= B_n(U; \xi, \beta^*) + B_n(u; \xi, \beta^*) - B_n(U; \xi, \beta^*). \end{aligned} \quad (63)$$

We analyze the terms on the right-hand side of (63) term by term. Using (46) and (62), we have

$$\begin{aligned} B_n(u; \xi, \beta^*) - B_n(U; \xi, \beta^*) &= \sum_{j=0}^n \int_{I_j} [A(u; \xi, \beta^*) - A(U; \xi, \beta^*)] dt \\ &\leq C a_1 \int_0^{t_{n+1}} \|\nabla \xi\| \|\nabla \beta^*\| dt \leq \left\{ \|\xi^0\|^2 + b_1 \|\nabla \xi^0\|^2 \right. \\ &\quad \left. + C(\varepsilon) \sum_{j=0}^n \int_{I_j} [\|\zeta_t\|^2 + \|\zeta\|_1^2 + \|\eta\|^2 + \|\eta_t\|^2] dt \right\}^{1/2} \\ &\quad \cdot \left\{ \|v_h\|^2 + b_1 \|\nabla v_h\|^2 \right\}^{1/2}. \end{aligned} \quad (64)$$

Using (56) and (62), we have

$$\begin{aligned} B_n(U; \xi, \beta^*) &\leq \left\{ \|\xi^0\|^2 + b_1 \|\nabla \xi^0\|^2 \right. \\ &\quad \left. + C(\varepsilon) \sum_{j=0}^n \int_{I_j} [\|\zeta_t\|^2 + \|\zeta\|_1^2 + \|\eta\|^2 + \|\eta_t\|^2] dt \right\}^{1/2} \end{aligned}$$

$$\begin{aligned}
& \cdot \left\{ \|\beta_+^{*0}\|^2 + b_0 \|\nabla \beta_+^{*0}\|^2 \right. \\
& \left. + \sum_{j=0}^n \int_{I_j} [\|\beta_t^*\|^2 + b_0 \|\nabla \beta_t^*\|^2 + a_0 \|\nabla \beta^*\|^2] dt \right\}^{1/2} \\
& \leq \left\{ \|\xi^0\|^2 + b_1 \|\nabla \xi^0\|^2 \right. \\
& \left. + C(\varepsilon) \sum_{j=0}^n \int_{I_j} [\|\zeta_t\|^2 + \|\zeta\|_1^2 + \|\eta\|^2 + \|\eta_t\|^2] dt \right\}^{1/2} \\
& \cdot \left\{ \|v_h\|^2 + b_1 \|\nabla v_h\|^2 \right\}^{1/2}.
\end{aligned} \tag{65}$$

Substituting (64) and (65) into (63) yields

$$\begin{aligned}
& (\xi^{n+1}, v_h) + (b \nabla \xi^{n+1}, \nabla v_h) = B_n(U; \xi, \beta^*) \\
& \leq C \left[\|\xi^0\|_1^2 \right. \\
& \left. + \sum_{j=0}^n \int_{I_j} \{\|\zeta_t\|_1^2 + \|\zeta\|_1^2 + \|\eta\|^2 + \|\eta_t\|^2\} dt \right]^{1/2} \\
& \cdot \|v_h\|_1.
\end{aligned} \tag{66}$$

By the arbitrariness of v_h , taking $v_h = \xi^{n+1}$ into (66), we get estimate (47). $\square$

In order to obtain the optimal order estimate with respect to t , we still need the following lemma (Lemma 9).

Lemma 9. Under the conditions in Lemma 8, one has

$$\begin{aligned}
& \|\xi^{n+1}\|^2 + b_0 \|\nabla \xi^{n+1}\|^2 + a_0 \int_{I_n} \|\nabla \xi\|^2 dt \leq \|\xi^n\|^2 \\
& + b_1 \|\nabla \xi^n\|^2 + C \int_{I_n} \{\tau_n [\|\zeta_t\|_1^2 + \|\nabla \zeta\|^2 + \|\eta\|^2 \\
& + \|\eta_t\|^2] + \tau_n^{-1} [\|\zeta\|_1^2 + \|\eta\|^2 + \|\eta_t\|^2]\} dt. \\
& \tau_n \int_{I_n} [\|\xi_t\|^2 + b_0 \|\nabla \xi_t\|^2] dt \leq C \left\{ \|\xi^n\|^2 + b_1 \|\nabla \xi^n\|^2 \right. \\
& + \int_{I_n} \{\tau_n [\|\zeta_t\|_1^2 + \|\nabla \zeta\|^2 + \|\eta\|^2 + \|\eta_t\|^2] \\
& \left. + \tau_n^{-1} [\|\zeta\|_1^2 + \|\eta\|^2 + \|\eta_t\|^2]\} dt \right\}.
\end{aligned} \tag{67}$$

$$\tag{68}$$

Proof. (1) Taking $v = (t - t_n)\xi_t$ in (44), since $v_+^n = 0$, we can get

$$\begin{aligned}
& \int_{I_n} (t - t_n) [(\xi_t, \xi_t) + B(\xi_t, \xi_t)] dt = - \int_{I_n} (t - t_n) \\
& \cdot A(U; \xi, \xi_t) dt + \int_{I_n} (t - t_n) \\
& \cdot [- (\zeta_t, \xi_t) + (\lambda - 1)(\eta_t, \xi_t) - A(U; \zeta, \xi_t)] dt \\
& + \int_{I_n} (t - t_n) [A(u; P_1 u, \xi_t) - A(U; P_1 u, \xi_t)] dt \\
& + \int_{I_n} (t - t_n) (f(U) - f(u), \xi_t) dt.
\end{aligned} \tag{69}$$

Noticing that

$$\begin{aligned}
& \int_{I_n} (t - t_n) A(U; \xi, \xi_t) dt \\
& = \frac{1}{2} \tau_n A(U^{n+1}; \xi^{n+1}, \xi^{n+1}) - \frac{1}{2} \int_{I_n} A(U; \xi, \xi) dt \\
& - \frac{1}{2} \int_{I_n} (t - t_n) A(U_t; \xi, \xi) dt,
\end{aligned} \tag{70}$$

similar to the proof in step (1) of Lemma 8, we can derive

$$\begin{aligned}
& \int_{I_n} (t - t_n) [(\xi_t, \xi_t) + B(\xi_t, \xi_t)] dt \\
& + \tau_n A(U^{n+1}; \xi^{n+1}, \xi^{n+1}) \leq \int_{I_n} A(U; \xi, \xi) dt \\
& + C \int_{I_n} (t - t_n) \\
& \cdot [\|\zeta_t\|_1^2 + \|\nabla \zeta\|^2 + \|\eta\|^2 + \|\eta_t\|^2 + \|\nabla \xi\|^2] dt.
\end{aligned} \tag{71}$$

Combining inequalities (50), (51), and (71) and using the Gronwall lemma, we obtain

$$\begin{aligned}
& \tau_n \int_{I_n} [\|\xi_t\|^2 + b_0 \|\nabla \xi_t\|^2] dt + \tau_n a_0 \|\nabla \xi^{n+1}\|^2 \\
& \leq C \int_{I_n} \{\|\nabla \xi\|^2 \\
& + \tau_n [\|\zeta_t\|_1^2 + \|\nabla \zeta\|^2 + \|\eta\|^2 + \|\eta_t\|^2]\} dt.
\end{aligned} \tag{72}$$

(2) Taking $v = \xi$ into (44), similar to the analysis of (53) and (54), we also have

$$\begin{aligned}
& \frac{1}{2} (\|\xi^{n+1}\|^2 - \|\xi^n\|^2) + a_0 \int_{I_n} \|\nabla \xi\|^2 dt \\
& + \frac{1}{2} (\|\nabla \xi^{n+1}\|_b^2 - \|\nabla \xi^n\|_b^2) \\
& \leq \varepsilon \int_{I_n} \tau_n [\|\xi_t\|^2 + \|\nabla \xi\|^2] dt \\
& + C(\varepsilon) \int_{I_n} \tau_n^{-1} [\|\zeta\|_1^2 + \|\eta\|^2 + \|\eta_t\|^2] dt.
\end{aligned} \tag{73}$$

Combining estimates (73) and (72) together, letting ε be small enough, we derive estimate (67). Then, we insert the estimate of $\int_{I_n} \|\nabla \xi\|^2 dt$ of (67) into (72) to get (68). $\square$

Let $\xi = \theta$, $\zeta = \rho$, and $\eta = \pi_t$. Using Lemmas 8 and 9, we deduce the following lemma.

Lemma 10. *Under the conditions in Lemmas 8 and 9, there exist*

$$\|\theta^{n+1}\|_1^2 \leq C \left\{ \|\theta^0\|_1^2 + \tau^{2q} \int_0^{t_{n+1}} \|D_t^q u\|_1^2 dt + h^{2(r+1)} \int_0^{t_{n+1}} \|D_t u\|_{H^{r+1}}^2 dt \right\}, \quad (74)$$

$$\begin{aligned} \|\theta\|_{L^\infty(0,T;H^1(\Omega))}^2 &\leq C \left\{ \|\theta_0\|_{H^1(\Omega)}^2 + h^{2(r+1)} \left(\|D_t u\|_{L^2(0,T;H^{r+1}(\Omega))}^2 \right. \right. \\ &\quad \left. \left. + \|D_t u\|_{L^\infty(0,T;H^{r+1}(\Omega))}^2 \right) + \tau^{2q} \left(\|D_t^q u\|_{L^2(0,T;H^1(\Omega))}^2 \right. \right. \\ &\quad \left. \left. + \|D_t^q u\|_{L^\infty(0,T;H^1(\Omega))}^2 + \|D_t u\|_{L^2(0,T;H^{r+1}(\Omega))}^2 \right) \right\}. \end{aligned} \quad (75)$$

Proof. It is easy to see that (74) follows directly from (68), Lemmas 4 and 7.

Notice that, $\forall t \in I_n$, there exists

$$\|\theta(t)\|_1^2 \leq 2 \left\{ \|\theta_{n+1}\|_1^2 + \tau_n \int_{I_n} \|\theta_t\|_1^2 dt \right\}. \quad (76)$$

So combining the above estimate with (68) yields

$$\begin{aligned} \|\theta(t)\|_1^2 &\leq C \left\{ \|\theta_{n+1}\|_1^2 + \|\theta_n\|_1^2 + \int_{I_n} \left[\tau_n^{-1} \|\rho\|_1^2 + \|\nabla \rho\|^2 \right. \right. \\ &\quad \left. \left. + \tau_n^{-1} \|\pi_t\|^2 + \tau_n \|\rho_t\|_1^2 + \tau_n \|\nabla \rho\|^2 + \tau_n \|\pi_t\|^2 \right] dt \right\}. \end{aligned} \quad (77)$$

Using Lemmas 4 and 7, we have the following estimates:

$$\begin{aligned} \int_{I_n} \left[\tau_n^{-1} \|\rho\|_1^2 + \tau_n \|\rho_t\|_1^2 \right] dt &\leq C \tau_n^{2q} \sup_{I_n} \|D_t^q u\|_1^2, \\ \int_{I_n} \left[\|\nabla \rho\|^2 + \tau_n \|\nabla \rho\|^2 \right] dt &\leq C \tau_n^{2q} \|D_t^q u\|_{L^2(I_n;H^1(\Omega))}^2, \\ \int_{I_n} \tau_n^{-1} \|\pi_t\|^2 dt &\leq C h^{2(r+1)} \sup_{I_n} \|u_t\|_{H^{r+1}(\Omega)}^2, \\ \int_{I_n} \tau_n \|\pi_t\|^2 dt &\leq C \int_{I_n} \tau_n h^{2(r+1)} \|u_t\|_{H^{r+1}(\Omega)}^2 dt \\ &\leq C \left(\tau_n^{2q} + h^{2(r+1)(2q/(2q-1))} \right) \|u_t\|_{L^2(I_n;H^{r+1}(\Omega))}^2. \end{aligned} \quad (78)$$

Inserting the above estimates into (77) and using (74) again, we derive (75). $\square$

Finally, using the triangle inequality $\|u - U\|_1 \leq \|\theta\|_1 + \|\rho\|_1 + \|\pi\|_1$, Lemmas 4 and 5, we can establish the following theorem.

Theorem 11. *Let u be the solution of (1) and let U be the solution of (12), respectively. Assume that u is smooth enough to satisfy the required regularities in the analyses. Then, the following error estimate holds:*

$$\begin{aligned} \|u - U\|_{L^\infty(0,T;H^1(\Omega))} &\leq C \left\{ \|u_0 - u_{0,h}\|_{H^1(\Omega)} \right. \\ &\quad \left. + h^r \|u\|_{L^\infty(0,T;H^{r+1}(\Omega))} + h^{(r+1)} \left(\|D_t u\|_{L^2(0,T;H^{r+1}(\Omega))} \right. \right. \\ &\quad \left. \left. + \|D_t u\|_{L^\infty(0,T;H^{r+1}(\Omega))} \right) + \tau^q \left(\|D_t^q u\|_{L^2(0,T;H^1(\Omega))} \right. \right. \\ &\quad \left. \left. + \|D_t^q u\|_{L^\infty(0,T;H^1(\Omega))} + \|D_t u\|_{L^2(0,T;H^{r+1}(\Omega))} \right) \right\}. \end{aligned} \quad (79)$$

Furthermore, if an initial assumption $\|u_0 - u_{0,h}\|_{H^1(\Omega)} \leq Ch^r \|u_0\|_{H^{r+1}(\Omega)}$ exists, we obtain

$$\|u - U\|_{L^\infty(0,T;H^1(\Omega))} \leq C (h^r + \tau^q). \quad (80)$$

In other words, we get higher accuracy for both the space variable and the time variable. The results are consistent with [25].

6. Conclusions

The quasi-linear Sobolev equations (1) are solved by the time discontinuous Galerkin finite element methods. We presented the approximate scheme and proved the existence and uniqueness of the approximate solution by use of Brouwer's fixed point theorem. We made the error analysis and derived the optimal $L^\infty(0,T;H^1(\Omega))$ -norm error estimates. We got higher accuracy for both the space variable and the time variable in this paper. In the forthcoming work, some numerical experiments will be considered to verify our analysis for the schemes.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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