Research Article

Existence of Nonoscillatory Solutions of Higher-Order Neutral Differential Equations with Distributed Coefficients and Delays

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We consider the existence of nonoscillatory solutions of higher-order neutral differential equations with distributed coefficients. We use the *Banach* contraction principle to obtain new sufficient condition for the existence of nonoscillatory solutions.

1. Introduction

In this paper, we consider the higher-order nonlinear neutral differential equation with distributed deviating arguments:

$$\begin{aligned} & \left[\Phi \left[r(t)x(t) + \int_a^b p(t, \theta)x(t-\theta)d\theta \right] \right]^{(n)} \\ & + \int_c^d q_1(t, \tau)g_1(x(t-\tau))d\tau \\ & - \int_e^f q_2(t, \sigma)g_2(x(t-\sigma))d\sigma = h(t). \end{aligned} \quad (1)$$

Throughout this paper, the following conditions are assumed to hold:

- (1) where $n \geq 1$ is a positive integer, $r \in C([t_0, \infty), R^+)$, $0 < a < b$, $0 < c < d$, $0 < e < f$;
- (2) $p \in C([t_0, \infty) \times [a, b], R)$, $q_1 \in C([t_0, \infty) \times [c, d], R^+)$, $q_2 \in C([t_0, \infty) \times [e, f], R^+)$, $h \in C([t_0, \infty), R)$;
- (3) $\Phi(u)$ are continuously increasing real odd function with respect to u defined on R , and $\Phi^{-1}(u)$ satisfy the local *Lipschitz* condition;
- (4) $g_i \in C(R, R)$, $g_i(u)$ satisfy the local *Lipschitz* condition and $ug_i(u) > 0$, $i = 1, 2$, for $u \neq 0$.

Recently there have been a lot of activities concerning the existence of nonoscillatory solutions for neutral differential equations with positive and negative coefficient. In 2002,

Zhou and Zhang [1] studied higher-order linear neutral delay differential equation:

$$\begin{aligned} & \frac{d^n}{dt^n} [x(t) + cx(t-\tau)] \\ & + (-1)^{n+1} [P(t)x(t-\sigma) - Q(t)x(t-\delta)] = 0. \end{aligned} \quad (2)$$

In 2005, the existence of nonoscillatory solutions of first-order linear neutral delay differential equations of the form

$$\begin{aligned} & \frac{d}{dt} [x(t) + P(t)x(t-\tau)] + Q_1(t)x(t-\sigma_1) \\ & - Q_2(t)x(t-\sigma_2) = 0 \end{aligned} \quad (3)$$

was investigated by Zhang et al. [2] and, in the same year, Yu and Wang [3] studied nonoscillatory solutions of second-order nonlinear neutral differential equations of the form

$$\begin{aligned} & [r(t)[x(t) + P(t)x(t-\tau)]']' + Q_1(t)f(x(t-\sigma_1)) \\ & - Q_2(t)g(x(t-\sigma_2)) = 0. \end{aligned} \quad (4)$$

In 2010, Candan and Dahiya [4] studied nonoscillatory solutions of first-order and second-order nonlinear neutral differential equations with positive and negative coefficients:

$$\begin{aligned} & \frac{d^k}{dt^k} [x(t) + P(t)x(t-\tau)] + \int_a^b q_1(t, \xi)x(t-\xi)d\xi \\ & - \int_c^d q_2(t, \mu)x(t-\mu)d\mu = 0, \quad k = 1, 2. \end{aligned} \quad (5)$$

In 2012, Candan [5] study higher-order nonlinear differential equation:

$$\begin{aligned} & \left[r(t) [x(t) + P(t) x(t - \tau)]^{(n-1)} \right]' + (-1)^n \\ & \cdot [Q_1(t) g_1(x(t - \sigma_1)) - Q_2(t) g_2(x(t - \mu))] \\ & - f(t) = 0. \end{aligned} \quad (6)$$

In 2015, Liu et al. [6] study higher-order neutral differential equation with distributed deviating arguments:

$$\begin{aligned} & [r(t) x(t) + P(t) x(t - \theta)]^{(n)} \\ & + \int_c^d q_1(t, \tau) g_1(x(t - \tau)) d\tau \\ & - \int_e^f q_2(t, \sigma) g_2(x(t - \sigma)) d\sigma = f(t). \end{aligned} \quad (7)$$

As can be seen from the development process of the above equations, the delay of neutral part in the discussed differential equations was all constant delays, and the main thought in the employed verification method also kept the same in which the coefficient neutral part in the neutral was all discussed in four cases, that is, $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, and $(1, +\infty)$, and then was verified by constructing the corresponding operator. However, the case for distributed deviating arguments is rather rare. In 2013, Candan [7] studied first-order neutral differential equation with distributed deviating arguments:

$$\begin{aligned} & \left[\left[x(t) - \int_a^b p_2(t, \xi) x(t - \xi) d\xi \right]^\gamma \right]' \\ & + \int_c^d Q_2(t, \xi) G(x(t - \xi)) d\xi = 0, \end{aligned} \quad (8)$$

where γ is a ratio of odd positive integers; however the discussion only covered the condition for coefficient being $0 < \int_a^b p_2(t, \xi) d\xi < 1$, while being without the other three conditions, which might be caused by the difficulty in establishing feasible operator. In view of the above, here, in this paper, the difficulty of operator establishment was settled and sufficient condition for the existence of nonoscillatory solutions of differential equation with coefficient of $\int_a^b p_2(t, \xi) d\xi$ in the four cases was obtained. Thus, this paper may present its theoretical value as well as practical application value. For related work, we refer the reader to the references [8–12].

As usual, a solution of (1) is said to be oscillatory if it has arbitrarily large zeros. Otherwise the solution is said to be nonoscillatory.

A solution of (1) is a continuous function $x(t)$ defined on $[t_1 - \mu, \infty)$, for some $t_1 > t_0$, such that $\Phi(r(t)x(t) + \int_a^b p(t, \theta)x(t - \theta)d\theta)$ is n times continuously differentiable and (1) holds for all $n \geq 1$. Here, $\mu = \max\{b, d, f\}$.

Let L_i , $i = 1, 2$ denote the Lipschitz constants of functions g_i , $i = 1, 2$ on the set A , K denote the Lipschitz constants of functions $\Phi^{-1}(u)$, respectively, and $L = \max\{L_1, L_2\}$, $\beta_i = \max_{x \in A} g_i(x)$, $i = 1, 2$.

2. The Main Results

Theorem 1. Assume that $0 \leq \int_a^b p(t, \theta)d\theta \leq p_1 < 1$, $1 \leq r(t) \leq 1/p_1$, and

$$\begin{aligned} & \int_t^\infty s^{n-1} \int_c^d q_1(s, \tau) d\tau ds < \infty, \\ & \int_t^\infty s^{n-1} \int_e^f q_2(s, \sigma) d\sigma ds < \infty, \\ & \int_t^\infty s^{n-1} h(s) ds < \infty. \end{aligned} \quad (9)$$

Then (1) has a bounded nonoscillatory solution.

Proof. Let Λ be the set of all continuous and bounded functions on $[t_0, \infty)$ and the norm be $\|x(t)\| = \sup_{t_0 \leq t < +\infty} x(t)$. Set $A = \{x \in \Lambda, M_1 \leq x(t) \leq M_2, t \geq t_0\}$, where M_1, M_2 are two positive constants such that $p_1 M_2 + M_1/p_1 < \alpha < M_2$. From (9), one can choose $t_1 \geq t_0 + \mu$ and sufficiently large $t \geq t_1$ such that

$$\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\beta_1 \int_c^d q_1(s, \tau) d\tau + |h(s)| \right] ds \quad (10)$$

$$\leq \Phi(M_2) - \Phi(\alpha),$$

$$\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\beta_2 \int_e^f q_2(s, \sigma) d\sigma + |h(s)| \right] ds \quad (11)$$

$$\leq \Phi(\alpha) - \Phi\left(p_1 M_2 + \frac{M_1}{p_1}\right),$$

$$\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\int_c^d q_1(s, \tau) d\tau + \int_e^f q_2(s, \sigma) d\sigma \right] ds \quad (12)$$

$$\leq \frac{1-p_1}{KL},$$

and define an operator T on A as follows:

$$\begin{aligned} & (Tx)(t) \\ & = \begin{cases} \frac{1}{r(t)} \left\{ - \int_a^b p(t, \theta) x(t - \theta) d\theta + \Phi^{-1} \left[\Phi(\alpha) + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\int_c^d q_1(s, \tau) g_1(x(s - \tau)) d\tau - \int_e^f q_2(s, \sigma) g_2(x(s - \sigma)) d\sigma - h(s) \right] ds \right] \right\}, & t \geq t_1, \\ (Tx)(t_1), & t_0 \leq t \leq t_1. \end{cases} \end{aligned} \quad (13)$$

It is easy to see that T is continuous, for $t \geq t_1$, $x \in A$. By using (10), we have

$$\begin{aligned} (Tx)(t) &\leq \frac{1}{r(t)} \left\{ \Phi^{-1} \left[\Phi(\alpha) \right. \right. \\ &\quad \left. \left. + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\int_c^d q_1(s, \tau) g_1(x(s-\tau)) d\tau - h(s) \right] ds \right] \right\} \\ &\leq \Phi^{-1} \left[\Phi(\alpha) \right] \\ &\quad + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left(\beta_1 \int_c^d q_1(s, \tau) d\tau + |h(s)| \right) ds \\ &\leq \Phi^{-1} [\Phi(\alpha) + \Phi(M_2) - \Phi(\alpha)] = M_2, \end{aligned} \quad (14)$$

and, taking (11) into account, we have

$$(Tx)(t) \geq \frac{1}{r(t)} \left\{ - \int_a^b p(t, \theta) x(t-\theta) d\theta - \Phi^{-1} \left[\Phi(\alpha) \right] \right\}$$

$$\begin{aligned} &- \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\int_e^f q_2(s, \sigma) g_2(x(s-\sigma)) d\sigma + h(s) \right] ds \Bigg\} \\ &\geq p_1 \left\{ -p_1 M_2 + \Phi^{-1} \left[\Phi(\alpha) \right. \right. \\ &\quad \left. \left. - \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left(\beta_2 \int_e^f q_2(s, \sigma) d\sigma + |h(s)| \right) ds \right] \right\} \\ &\geq p_1 \left\{ -p_1 M_2 + \Phi^{-1} \left[\Phi(\alpha) - \Phi(\alpha) + \Phi \left(p_1 M_2 + \frac{M_1}{p_1} \right) \right] \right\} \\ &= M_1. \end{aligned} \quad (15)$$

These show that $TA \subset A$. Since A is bounded, close, convex subset of Λ , in order to apply the contraction principle we have to show that T is a contraction mapping on A . For $\forall x_1, x_2 \in A$, and $t \geq t_1$,

$$\begin{aligned} |(Tx_1)(t) - (Tx_2)(t)| &\leq \frac{1}{r(t)} \left\{ \int_a^b p(t, \theta) |x_1(t-\theta) - x_2(t-\theta)| d\theta + \Phi^{-1} \left[\Phi(\alpha) \right. \right. \\ &\quad \left. \left. + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left| \int_c^d q_1(s, \tau) g_1(x_1(s-\tau)) d\tau - \int_e^f q_2(s, \sigma) g_2(x_1(s-\sigma)) d\sigma - h(s) \right| ds \right] - \Phi^{-1} \left[\Phi(\alpha) \right. \right. \\ &\quad \left. \left. + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left| \int_c^d q_1(s, \tau) g_1(x_2(s-\tau)) d\tau - \int_e^f q_2(s, \sigma) g_2(x_2(s-\sigma)) d\sigma - h(s) \right| ds \right] \right\} \leq \frac{1}{r(t)} \left\{ p_1 |x_1 - x_2| \right. \\ &\quad \left. + K \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\int_c^d q_1(s, \tau) |g_1(x_1(s-\tau)) - g_1(x_2(s-\tau))| d\tau \right. \right. \\ &\quad \left. \left. + \int_e^f q_2(s, \sigma) |g_2(x_1(s-\sigma)) - g_2(x_2(s-\sigma))| d\sigma \right] ds \right\}. \end{aligned} \quad (16)$$

Using (12),

$$\begin{aligned} |(Tx_1)(t) - (Tx_2)(t)| &\leq |x_1 - x_2| \left(p_1 \right. \\ &\quad \left. + KL \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\int_c^d q_1(s, \tau) d\tau \right. \right. \\ &\quad \left. \left. + \int_e^f q_2(s, \sigma) d\sigma \right] ds \right) < \rho |x_1 - x_2|. \end{aligned} \quad (17)$$

This implies with the sup norm that

$$\|Tx_1 - Tx_2\| < \rho \|x_1 - x_2\|, \quad (18)$$

where $\rho = 1$, which shows that T is a contraction mapping on A and therefore there exists a unique solution, obviously a bounded positive solution of (1) $x \in A$, such that $Tx = x$. The proof is complete. $\square$

Theorem 2. Assume that $1 < p_3 \leq \int_a^b p(t, \theta) d\theta \leq p_2 < 2p_3 < +\infty$, $2p_3 \leq r(t) \leq 2p_2$ and that (9) holds.

Then (1) has a bounded nonoscillatory solution.

Proof. Let Λ be the set of all continuous and bounded functions on $[t_0, \infty)$ and the norm be $\|x(t)\| = \sup_{t_0 \leq t < +\infty} x(t)$. Set $A = \{x \in \Lambda, M_3 \leq x(t) \leq M_4, t \geq t_0\}$, where M_3, M_4 are two positive constants such that $p_2 M_4 + 2p_2 M_3 < \alpha < 2p_3 M_4$. From (9), one can choose $t_1 \geq t_0 + \mu$ and sufficiently large $t \geq t_1$ such that

$$\begin{aligned} &\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\beta_1 \int_c^d q_1(s, \tau) d\tau + |h(s)| \right] ds \\ &\leq \Phi(2p_3 M_4) - \Phi(\alpha), \end{aligned} \quad (19)$$

$$\begin{aligned} &\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\beta_2 \int_e^f q_2(s, \sigma) d\sigma + |h(s)| \right] ds \\ &\leq \Phi(\alpha) - \Phi(p_2 M_4 - 2p_2 M_3), \end{aligned} \quad (20)$$

$$\begin{aligned} &\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\int_c^d q_1(s, \tau) d\tau + \int_e^f q_2(s, \sigma) d\sigma \right] ds \\ &\leq \frac{2p_3 - p_2}{KL}, \end{aligned} \quad (21)$$

and define an operator T on A as follows:

$$(Tx)(t) = \begin{cases} \frac{1}{r(t)} \left\{ - \int_a^b p(t, \theta) x(t - \theta) d\theta + \Phi^{-1} \left[\Phi(\alpha) + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\int_c^d q_1(s, \tau) g_1(x(s-\tau)) d\tau - \int_e^f q_2(s, \sigma) g_2(x(s-\sigma)) d\sigma - h(s) \right] ds \right] \right\}, & t \geq t_1, \\ (Tx)(t_1), & t_0 \leq t \leq t_1. \end{cases} \quad (22)$$

It is easy to see that T is continuous, for $t \geq t_1$, $x \in A$. By using (19), we have

$$\begin{aligned} (Tx)(t) &\leq \frac{1}{r(t)} \left\{ \Phi^{-1} \left[\Phi(\alpha) \right. \right. \\ &\quad \left. \left. + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\int_c^d q_1(s, \tau) g_1(x(s-\tau)) d\tau - h(s) \right] ds \right] \right\} \\ &\leq \frac{1}{2p_3} \left\{ \Phi^{-1} \left[\Phi(\alpha) \right. \right. \\ &\quad \left. \left. + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\beta_1 \int_c^d q_1(s, \tau) d\tau + |h(s)| \right] ds \right] \right\} \leq M_4, \end{aligned} \quad (23)$$

and, taking (20) into account, we have

$$\begin{aligned} (Tx)(t) &\geq \frac{1}{r(t)} \left\{ - \int_a^b p(t, \theta) x(t - \theta) d\theta + \Phi^{-1} \left[\Phi(\alpha) \right. \right. \\ &\quad \left. \left. - \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\int_e^f q_2(s, \sigma) g_2(x(s-\sigma)) d\sigma + h(s) \right] ds \right] \right\} \\ &\geq \frac{1}{2p_2} \left\{ -p_2 M_4 + \Phi^{-1} \left[\Phi(\alpha) \right. \right. \\ &\quad \left. \left. - \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left(\beta_2 \int_e^f q_2(s, \sigma) d\sigma + |h(s)| \right) ds \right] \right\} \geq M_3. \end{aligned} \quad (24)$$

These show that $TA \subset A$. Since A is bounded, close, convex subset of Λ , in order to apply the contraction principle, we have to show that T is a contraction mapping on A . For $\forall x_1, x_2 \in A$, and $t \geq t_1$,

$$\begin{aligned} |(Tx_1)(t) - (Tx_2)(t)| &\leq \frac{1}{r(t)} \left\{ \int_a^b p(t, \theta) |x_1(t - \theta) - x_2(t - \theta)| d\theta + \Phi^{-1} \left[\Phi(\alpha) \right. \right. \\ &\quad \left. \left. + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left| \int_c^d q_1(s, \tau) g_1(x_1(s-\tau)) d\tau - \int_e^f q_2(s, \sigma) g_2(x_1(s-\sigma)) d\sigma - h(s) \right| ds \right] - \Phi^{-1} \left[\Phi(\alpha) \right. \right. \\ &\quad \left. \left. + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left| \int_c^d q_1(s, \tau) g_1(x_2(s-\tau)) d\tau - \int_e^f q_2(s, \sigma) g_2(x_2(s-\sigma)) d\sigma - h(s) \right| ds \right] \right\} \leq \frac{1}{r(t)} \left\{ p_2 |x_1 - x_2| \right. \\ &\quad \left. + K \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\int_c^d q_1(s, \tau) |g_1(x_1(s-\tau)) - g_1(x_2(s-\tau))| d\tau \right. \right. \\ &\quad \left. \left. + \int_e^f q_2(s, \sigma) |g_2(x_1(s-\sigma)) - g_2(x_2(s-\sigma))| d\sigma \right] ds \right\}. \end{aligned} \quad (25)$$

Or, using (21),

$$\begin{aligned} |(Tx_1)(t) - (Tx_2)(t)| &\leq |x_1 - x_2| \frac{1}{2p_3} \left\{ p_2 \right. \\ &\quad \left. + KL \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\int_c^d q_1(s, \tau) d\tau \right. \right. \\ &\quad \left. \left. + \int_e^f q_2(s, \sigma) d\sigma \right] ds \right\} < \rho |x_1 - x_2|. \end{aligned} \quad (26)$$

This implies with the sup norm that

$$\|Tx_1 - Tx_2\| < \rho \|x_1 - x_2\|, \quad (27)$$

where $\rho = 1$, which shows that T is a contraction mapping on A and therefore there exists a unique solution, obviously a bounded positive solution of (1) $x \in A$, such that $Tx = x$. The proof is complete. $\square$

Theorem 3. Assume that $-1 < p_4 \leq \int_a^b p(t, \theta) d\theta \leq 0$, $1 \leq r(t) \leq 1/p_4$ and that (9) holds.

Then (1) has a bounded nonoscillatory solution.

Proof. Let Λ be the set of all continuous and bounded functions on $[t_0, \infty)$ and the norm be $\|x(t)\| = \sup_{t_0 \leq t < \infty} x(t)$. Set $A = \{x \in \Lambda, M_5 \leq x(t) \leq M_6, t \geq t_0\}$, where M_5, M_6 are two positive constants such that $M_5/p_4 < \alpha < (1 + p_4)M_6$.

From (9), one can choose $t_1 \geq t_0 + \mu$ and sufficiently large $t \geq t_1$ such that

$$\begin{aligned} & \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\beta_1 \int_c^d q_1(s, \tau) d\tau + |h(s)| \right] ds \\ & \leq \Phi((1+p_4)M_6) - \Phi(\alpha), \\ & \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\beta_2 \int_e^f q_2(s, \sigma) d\sigma + |h(s)| \right] ds \end{aligned} \quad (28)$$

$$\leq \Phi(\alpha) - \Phi\left(-\frac{M_5}{p_4}\right), \quad (29)$$

$$\begin{aligned} & \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\int_c^d q_1(s, \tau) d\tau + \int_e^f q_2(s, \sigma) d\sigma \right] ds \\ & \leq \frac{1+p_4}{KL}. \end{aligned} \quad (30)$$

and define an operator T on A as follows:

$$\begin{aligned} & (Tx)(t) \\ & = \begin{cases} \frac{1}{r(t)} \left\{ -\int_a^b p(t, \theta) x(t-\theta) d\theta + \Phi^{-1} \left[\Phi(\alpha) + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\int_c^d q_1(s, \tau) g_1(x(s-\tau)) d\tau - \int_e^f q_2(s, \sigma) g_2(x(s-\sigma)) d\sigma - h(s) \right] ds \right] \right\}, & t \geq t_1, \\ (Tx)(t_1), & t_0 \leq t \leq t_1. \end{cases} \end{aligned} \quad (31)$$

It is easy to see that T is continuous, for $t \geq t_1$, $x \in A$. By using (28), we have

$$\begin{aligned} (Tx)(t) & \leq \frac{1}{r(t)} \left\{ -\int_a^b p(t, \theta) x(t-\theta) d\theta + \Phi^{-1} \left[\Phi(\alpha) \right. \right. \\ & \quad \left. \left. + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\int_c^d q_1(s, \tau) g_1(x(s-\tau)) d\tau - h(s) \right] ds \right] \right\} \\ & \leq -p_4 M_6 + \Phi^{-1} \left[\Phi(\alpha) \right. \\ & \quad \left. + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\beta_1 \int_c^d q_1(s, \tau) d\tau + |h(s)| \right] ds \right] \leq M_6, \end{aligned} \quad (32)$$

and, taking (29) into account, we have

$$(Tx)(t) \geq \frac{1}{r(t)} \left\{ -\int_a^b p(t, \theta) x(t-\theta) d\theta + \Phi^{-1} \left[\Phi(\alpha) \right. \right.$$

$$\begin{aligned} & \left. - \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\int_e^f q_2(s, \sigma) g_2(x(s-\sigma)) d\sigma + h(s) \right] ds \right\} \\ & \geq -p_4 \left\{ \Phi^{-1} \left[\Phi(\alpha) \right. \right. \\ & \quad \left. \left. - \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\beta_2 \int_e^f q_2(s, \sigma) d\sigma + |h(s)| \right] ds \right] \right\} \\ & \geq -p_4 \left\{ \Phi^{-1} \left[\Phi(\alpha) - \Phi(\alpha) + \Phi\left(-\frac{M_5}{p_4}\right) \right] \right\} = M_5. \end{aligned} \quad (33)$$

These show that $TA \subset A$. Since A is bounded, close, convex subset of Λ , in order to apply the contraction principle, we have to show that T is a contraction mapping on A . For $\forall x_1, x_2 \in A$, and $t \geq t_1$,

$$\begin{aligned} |(Tx_1)(t) - (Tx_2)(t)| & \leq \frac{1}{r(t)} \left\{ \int_a^b p(t, \theta) |x_1(t-\theta) - x_2(t-\theta)| d\theta + \Phi^{-1} \left[\Phi(\alpha) \right. \right. \\ & \quad \left. \left. + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left| \int_c^d q_1(s, \tau) g_1(x_1(s-\tau)) d\tau - \int_e^f q_2(s, \sigma) g_2(x_1(s-\sigma)) d\sigma - h(s) \right| ds \right] - \Phi^{-1} \left[\Phi(\alpha) \right. \right. \\ & \quad \left. \left. + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left| \int_c^d q_1(s, \tau) g_1(x_2(s-\tau)) d\tau - \int_e^f q_2(s, \sigma) g_2(x_2(s-\sigma)) d\sigma - h(s) \right| ds \right] \right\} \leq \frac{1}{r(t)} \left\{ -p_4 |x_1 - x_2| \right. \\ & \quad \left. + K \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\int_c^d q_1(s, \tau) |g_1(x_1(s-\tau)) - g_1(x_2(s-\tau))| d\tau \right. \right. \\ & \quad \left. \left. + \int_e^f q_2(s, \sigma) |g_2(x_1(s-\sigma)) - g_2(x_2(s-\sigma))| d\sigma \right] ds \right\}. \end{aligned} \quad (34)$$

Or, using (30),

$$\begin{aligned} |(Tx_1)(t) - (Tx_2)(t)| & \leq |x_1 - x_2| \left(-p_4 \right. \\ & \quad \left. + KL \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\int_c^d q_1(s, \tau) d\tau \right. \right. \end{aligned}$$

$$\left. \left. + \int_e^f q_2(s, \sigma) d\sigma \right] ds \right) < \rho |x_1 - x_2|. \quad (35)$$

This implies with the sup norm that

$$\|Tx_1 - Tx_2\| < \rho \|x_1 - x_2\|, \quad (36)$$

where $\rho = 1$, which shows that T is a contraction mapping on A and therefore there exists a unique solution, obviously a bounded positive solution of (1) $x \in A$, such that $Tx = x$. The proof is complete. $\square$

Theorem 4. Assume that $-\infty < 2p_5 < p_6 \leq \int_a^b p(t, \theta) d\theta \leq p_5 < -1$, $-2p_5 < r(t) < -2p_6$ and that (9) holds.

Then (1) has a bounded nonoscillatory solution.

Proof. Let Λ be the set of all continuous and bounded functions on $[t_0, \infty)$ and the norm be $\|x(t)\| = \sup_{t_0 \leq t < \infty} x(t)$. Set $A = \{x \in \Lambda, M_7 \leq x(t) \leq M_8, t \geq t_0\}$, where M_7, M_8 are two positive constants such that $-2p_6 M_7 < \alpha < (-2p_5 + p_6)M_8$. From (9), one can choose $t_1 \geq t_0 + \mu$ and sufficiently large $t \geq t_1$ such that

$$\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\beta_1 \int_c^d q_1(s, \tau) d\tau + |h(s)| \right] ds \leq \Phi(-2p_5 M_8 + p_6 M_8) - \Phi(\alpha), \quad (37)$$

$$\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\beta_2 \int_e^f q_2(s, \sigma) d\sigma + |h(s)| \right] ds \leq \Phi(\alpha) - \Phi(-2p_6 M_7), \quad (38)$$

$$\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\int_c^d q_1(s, \tau) d\tau + \int_e^f q_2(s, \sigma) d\sigma \right] ds \leq \frac{p_6 - 2p_5}{KL}, \quad (39)$$

and define an operator T on A as follows:

$$(Tx)(t) = \begin{cases} \frac{1}{r(t)} \left\{ -\int_a^b p(t, \theta) x(t-\theta) d\theta + \Phi^{-1} \left[\Phi(\alpha) + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\int_c^d q_1(s, \tau) g_1(x(s-\tau)) d\tau - \int_e^f q_2(s, \sigma) g_2(x(s-\sigma)) d\sigma - h(s) \right] ds \right] \right\}, & t \geq t_1, \\ (Tx)(t_1), & t_0 \leq t \leq t_1. \end{cases} \quad (40)$$

It is easy to see that T is continuous, for $t \geq t_1$, $x \in A$. By using (37), we have

$$\begin{aligned} (Tx)(t) &\leq \frac{1}{r(t)} \left\{ -\int_a^b p(t, \theta) x(t-\theta) d\theta + \Phi^{-1} \left[\Phi(\alpha) \right. \right. \\ &\quad \left. \left. + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\int_c^d q_1(s, \tau) g_1(x(s-\tau)) d\tau - h(s) \right] ds \right] \right\} \\ &\leq \frac{1}{-2p_5} \left\{ -p_6 M_8 + \Phi^{-1} \left[\Phi(\alpha) \right. \right. \\ &\quad \left. \left. + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\beta_1 \int_c^d q_1(s, \tau) d\tau + |h(s)| \right] ds \right] \right\} \leq M_8, \end{aligned} \quad (41)$$

and, taking (38) into account, we have

$$\begin{aligned} (Tx)(t) &\geq \frac{1}{r(t)} \left\{ \Phi^{-1} \left[\Phi(\alpha) \right. \right. \\ &\quad \left. \left. - \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\int_e^f q_2(s, \sigma) g_2(x(s-\sigma)) d\sigma + h(s) \right] ds \right] \right\} \\ &\geq \frac{1}{-2p_6} \left\{ \Phi^{-1} \left[\Phi(\alpha) \right. \right. \\ &\quad \left. \left. - \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\beta_2 \int_e^f q_2(s, \sigma) d\sigma + |h(s)| \right] ds \right] \right\} \geq M_7. \end{aligned} \quad (42)$$

These show that $TA \subset A$. Since A is bounded, close, convex subset of Λ , in order to apply the contraction principle, we have to show that T is a contraction mapping on A . For $\forall x_1, x_2 \in A$, and $t \geq t_1$,

$$\begin{aligned} |(Tx_1)(t) - (Tx_2)(t)| &\leq \frac{1}{r(t)} \left\{ \int_a^b p(t, \theta) |x_1(t-\theta) - x_2(t-\theta)| d\theta + \Phi^{-1} \left[\Phi(\alpha) \right. \right. \\ &\quad \left. \left. + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left| \int_c^d q_1(s, \tau) g_1(x_1(s-\tau)) d\tau - \int_e^f q_2(s, \sigma) g_2(x_1(s-\sigma)) d\sigma - h(s) \right| ds \right] - \Phi^{-1} \left[\Phi(\alpha) \right. \right. \\ &\quad \left. \left. + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left| \int_c^d q_1(s, \tau) g_1(x_2(s-\tau)) d\tau - \int_e^f q_2(s, \sigma) g_2(x_2(s-\sigma)) d\sigma - h(s) \right| ds \right] \right\} \leq \frac{1}{r(t)} \left\{ -p_6 |x_1 - x_2| \right. \\ &\quad \left. + K \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\int_c^d q_1(s, \tau) |g_1(x_1(s-\tau)) - g_1(x_2(s-\tau))| d\tau \right. \right. \\ &\quad \left. \left. + \int_e^f q_2(s, \sigma) |g_2(x_1(s-\sigma)) - g_2(x_2(s-\sigma))| d\sigma \right] ds \right\}. \end{aligned} \quad (43)$$

Or, using (39),

$$\begin{aligned} |(Tx_1)(t) - (Tx_2)(t)| &\leq |x_1 - x_2| \frac{1}{-2p_5} \left\{ -p_6 \right. \\ &+ KL \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\int_c^d q_1(s, \tau) d\tau \right. \\ &\left. \left. + \int_e^f q_2(s, \sigma) d\sigma \right] ds \right\} < \rho |x_1 - x_2|. \end{aligned} \quad (44)$$

This implies with the sup norm that

$$\|Tx_1 - Tx_2\| < \rho \|x_1 - x_2\|, \quad (45)$$

where $\rho = 1$, which shows that T is a contraction mapping on A and therefore there exists a unique solution, obviously a bounded positive solution of (1) $x \in A$, such that $Tx = x$. The proof is complete. $\square$

3. Remark

We note that when $n = 1$, $r(t) \equiv 1$, $\Phi(u) = u^y$, $q_2(t, \sigma) = 0$, and $h(t) = 0$, we obtain (8). Thus, this paper great achievement was further obtained compared to that of [7].

4. Example

Example 1. Consider high-order neutral differential equation with distributed deviating arguments:

$$\begin{aligned} &\left[\left(\frac{3}{2 + \sin t} x(t) + \int_{\pi/2}^\pi e^{-t} x(t - \theta) d\theta \right)^3 \right]'' \\ &+ 36 \int_\pi^{3\pi/2} \tau e^{-t} x(t - \tau) d\tau \\ &- 72 \int_{\pi/2}^\pi \sigma e^{-2t} x(t - \sigma) d\sigma = e^{-3t} \left(15 \cos^3 t \right. \\ &- 30 \sin^3 t + 63 \pi \sin^2 t - 18 \pi \cos^2 t \\ &+ \frac{1}{2} \sin 2t (15 \cos t - 27 \sin^2 t + 33 \pi) \\ &+ 6 \pi^2 (\cos t - 7 \sin t) + 9 \pi^3 \Big) + 18 e^{-2t} (8 \pi \cos t \\ &+ \cos 2t - \sin 2t + 4 \sin t - 4 \cos t - 3 \pi^2) \\ &+ 9 e^{-t} (4 \pi \cos t - 6 \pi \sin t + 5 \pi^2 + 2 \pi). \end{aligned} \quad (46)$$

Here, $n = 2$, $r(t) = 3/(2 + \sin t)$, $P(t, \theta) = e^{-t}$, $q_1(t, \tau) = 36 \tau e^{-t}$, $q_2(t, \sigma) = 72 \sigma e^{-2t}$, $\Phi(u) = u^3$, $g_1(u) = g_2(u) = u$, $a = e = \pi/2$, $b = c = f = \pi$, $d = 3\pi/2$. $1 < r(t) = 3/(2 + \sin t) < 3$, $\int_{\pi/2}^\pi e^{-t} d\theta = (\pi/2)e^{-t} < 1$, $\int_{t_0}^\infty \int_{3\pi/2}^\pi 36 \tau s e^{-s} d\tau ds < \infty$, $\int_{t_0}^\infty \int_{\pi/2}^\pi \sigma s e^{-2s} d\sigma ds < \infty$, $\int_{t_0}^\infty e^{-3s} |15 \cos^3 s - 30 \sin^3 s + 63 \pi \sin^2 s - 18 \pi \cos^2 s + (1/2) \sin 2s (15 \cos s - 27 \sin^2 s + 33 \pi) + 6 \pi^2 (\cos s - 7 \sin s) + 9 \pi^3| + 18 e^{-2s} |8 \pi \cos s + \cos 2s - \sin 2s +$

$4 \sin s - 4 \cos s - 3 \pi^2| + 9 e^{-s} |4 \pi \cos s - 6 \pi \sin s + 5 \pi^2 + 2 \pi| ds < \infty$.

Then it is easy to see that all the conditions of Theorem 1 are satisfied. In fact, $x(t) = 2 + \sin t$ is a nonoscillatory solution of (46).

Competing Interests

The authors declare that they have no competing interests.

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