Research Article

Global Exponential Stability of Periodic Solution for Neutral-Type Complex-Valued Neural Networks

Song Guo and Bo Du

Department of Mathematics, Huaiyin Normal University, Huaian, Jiangsu 223300, China

Correspondence should be addressed to Bo Du; dubo7307@163.com

Received 24 April 2016; Accepted 11 August 2016

Academic Editor: Rigoberto Medina

Copyright © 2016 S. Guo and B. Du. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper deals with a class of neutral-type complex-valued neural networks with delays. By means of Mawhin’s continuation theorem, some criteria on existence of periodic solutions are established for the neutral-type complex-valued neural networks. By constructing an appropriate Lyapunov-Krasovskii functional, some sufficient conditions are derived for the global exponential stability of periodic solutions to the neutral-type complex-valued neural networks. Finally, numerical examples are given to show the effectiveness and merits of the present results.

1. Introduction

As it is well known, in a large amount of applications, complex signals often occur and the complex-valued neural networks (CVNNs) are preferable. Therefore, there have been increasing research interests in the dynamical behaviors of complex-valued recurrent neural networks; see [1–8] and references therein. Recently, Gong et al. [9] considered the complex-valued recurrent neural networks with time-varying delays by using the matrix measure method and the Halanay inequality as follows:

$$\dot{u}(t) = -Cu(t) + A f(u(t)) + B f(u(t - \tau(t))),$$

$$u \in \mathbb{C}^n.$$ (1)

Zhang and Yu [10] studied a class of complex-valued Cohen-Grossberg neural networks with time delays

$$z'(t) = -az(t) [h(z(t)) - Af(z(t)) - Bg(z(t - \tau)) - u],$$ (2)

and some stability results were obtained. In [11], Song et al. considered the global exponential stability of complex-valued neural networks with both time-varying delays and impulsive effects as follows:

$$\dot{z}(t) = -Cz(t) + Af(z(t)) + B f(z(t - \tau(t))) + J_k,$$

$$t \neq t_k,$$

$$z(t) = D_k h_k(z(t^-)) + E_k S_k(z(t^- - \tau(t^-))) + T_k,$$

$$t = t_k.$$ (3)

In [12], Wang and Huang obtained the stability criteria for complex-valued bidirectional associative memory (BAM) with time delay by using the Lyapunov function method and mathematical analysis technique. Pan et al. [13] obtained the global exponential stability of a class of CVNNs with time-varying delays by applying conjugate system of CVNNs, fixed point theorem, contraction mapping principle, and a delay differential inequality. In [3], applying delta differential operator, a complex-valued neural network on time scales was discussed. In [11], a class of CVNNs with probabilistic time-varying delays is considered, and several delay-distribution-dependent sufficient conditions to guarantee the global asymptotic and exponential stability were obtained by constructing proper Lyapunov-Krasovskii functional and employing inequality technique.
On the other hand, neutral-type CNNs can be used for describing these complicated dynamic properties of neural cells. Generally, it can be described as

\[
(x_i') (t) = -a_i(t) x_i(t) + \sum_{j=1}^{n} b_{ij} (t) f_j (x_j (t - \tau_{ij} (t))) + \sum_{j=1}^{n} d_{ij} (t) g_j (x_j (t - \delta_{ij} (t))) + I_i(t),
\]

where

\[
A_p z_p (t) = A_p x_p (t) + i A_p y_p (t),
A_p x_p (t) = x_p (t) - c_p x_p (t - r),
A_p y_p (t) = y_p (t) - c_p y_p (t - r),
\]

\[p = 1, 2, \ldots, n, \quad r, c_p \in \mathbb{R}\] with \(|c_p| \neq 1, d_p (t) \geq 0\) is the self-feedback connection weight, and \(a_{pq}, b_{pq}\) are complex-valued connection weight matrices without and with time delays, respectively. \(f_q (z_q), g_q (z_q) : \mathbb{C} \to \mathbb{C}\) is the activation function of the neurons. \(H_p (t) \in \mathbb{C}\) is the external input vector. \(y_p (t) \geq 0\) corresponds to the transmission delays.

System (7) can be written in vector form as follows:

\[
\frac{dA z(t)}{dt} = -D(t) z(t) + A(t) f(z(t)) + B(t) g(z(t - \gamma(t))) + H(t),
\]

where \(A z(t) = (A_1 z_1(t), A_2 z_2(t), \ldots, A_n z_n(t)) \in \mathbb{C}^n\) is the state vector, \(D(t) = \text{diag} (d_1(t), d_2(t), \ldots, d_n(t))\) is the self-feedback connection weight matrix, and \(A(t) = (a_{pq})_{n \times n} \in \mathbb{C}^{n \times n}\) and \(B(t) = (b_{pq})_{n \times n} \in \mathbb{C}^{n \times n}\) are complex-valued connection weight matrices without and with time delays, respectively. \(f(z) = (f_1(z_1), f_2(z_2), \ldots, f_n(z_n))^\top : \mathbb{C}^n \to \mathbb{C}^n\) is the activation function of the neurons. \(H(t) = (H_1(t), H_2(t), \ldots, H_n(t))^\top \in \mathbb{C}^n\) is the external input vector.

Throughout the paper, we give some notations:

\[
a^-_{pq} = \min_{t \in [0, T]} \{|a_{pq}|\},
a^+_{pq} = \max_{t \in [0, T]} \{|a_{pq}|\},
b^-_{pq} = \min_{t \in [0, T]} \{|b_{pq}|\},
b^+_{pq} = \max_{t \in [0, T]} \{|b_{pq}|\},
c^-_{pq} = \min_{t \in [0, T]} \{|c_{pq}|\},
c^+_{pq} = \max_{t \in [0, T]} \{|c_{pq}|\},
d^-_{pq} = \min_{t \in [0, T]} \{|d_{pq}|\},
d^+_{pq} = \max_{t \in [0, T]} \{|d_{pq}|\},
\]

\[\bar{f} = \frac{1}{T} \int_{0}^{T} f(t) \, dt.\]

Remark 1. The neural network model (7) shows the neutral character by the \(A_p\) operator, which is different from the corresponding results of other papers; see, for example, [23–27].

Remark 2. When \(c_p = 0, p = 1, 2, \ldots, n\), system (7) is changed into non-neutral-type CVNNs which have been extensively studied; see, for example, [2, 5, 9, 12, 13].

Remark 3. In general, when \(|c_p| \geq 1\) in (7), the operator \(A_p\) has no inverse operator. Hence, it is very difficult for obtaining existence and stability results to (7).
We also make the following assumptions:

(H1) \( d_p(t), a_{pq}(t), b_{pq}(t), d_{pq}(t), \) and \( H_p(t) \) are \( T \)-periodic continuous functions.

(H2) There exist nonnegative constants \( M_{q,R}, M_{q,I}, N_{q,R}, \) and \( N_{q,I} \) such that

\[
\begin{align*}
|f^R_q(z)| &\leq M_{q,R}, \\
|f^I_q(z)| &\leq M_{q,I}, \\
|g^R_q(z)| &\leq N_{q,R}, \\
|g^I_q(z)| &\leq N_{q,I},
\end{align*}
\]

\( \forall z \in \mathbb{C}^n, \ q = 1, 2, \ldots, n. \)

The distinctive contributions of this paper are outlined as follows: (1) the neutral-type neural network model (7) shows the neutral character by the \( A_p \) operator, which is different from other papers. Hence, when the neutral delay term is studied as a neutral operator \( A_p \), novel analysis technique is developed since the conventional analysis tool no longer applies. (2) We use a novel method for studying the stability of periodic solutions, which is different from traditional methods. (3) A unified framework is established to handle complex-valued neural networks, neutral terms, and time-varying delays.

By separating the state, the connection weight, the activation function, and the external input into its real and imaginary part, then system (7) can be rewritten as follows

\[
\begin{align*}
\frac{dA_p x_p(t)}{dt} &= -d_p(t) x_p(t) + \sum_{q=1}^{n} d^R_{pq}(t) f^R_q(z_q(t)) \\
&\quad - \sum_{q=1}^{n} a^I_{pq}(t) f^I_q(z_q(t)) \\
&\quad + \sum_{q=1}^{n} b^R_{pq}(t) g^R_q(z_q(t - y_{pq}(t))) \\
&\quad - \sum_{q=1}^{n} b^I_{pq}(t) g^I_q(z_q(t - y_{pq}(t))) \\
&\quad + H^R_p(t) = \Xi_{p,x}(t),
\end{align*}
\]

\[
\begin{align*}
\frac{dA_p y_p(t)}{dt} &= -d_p(t) y_p(t) + \sum_{q=1}^{n} a^R_{pq}(t) f^R_q(z_q(t)) \\
&\quad + \sum_{q=1}^{n} d^I_{pq}(t) f^I_q(z_q(t)) \\
&\quad + \sum_{q=1}^{n} b^R_{pq}(t) g^I_q(z_q(t - y_{pq}(t))) \\
&\quad + \sum_{q=1}^{n} b^I_{pq}(t) g^R_q(z_q(t - y_{pq}(t))) \\
&\quad + H^I_p(t) = \Xi_{n+p,y}(t),
\end{align*}
\]

where \( x_p(t) = Re(z_p(t)), y_p(t) = Im(z_p(t)), a^R_{pq}(t) = Re(a_{pq}(t)), \)
\( a^I_{pq}(t) = Im(a_{pq}(t)), b^R_{pq}(t) = Re(b_{pq}(t)), b^I_{pq}(t) = Im(b_{pq}(t)), \)
\( H^R_p(t) = Re(H_p(t)), H^I_p(t) = Im(H_p(t)), f^R_q(z_q(t)) = Re(f_q(z_q(t))), \)
\( f^I_q(z_q(t)) = Im(f_q(z_q(t))), g^R_q(z_q(t - y_{pq}(t))) = Re(g_q(z_q(t - y_{pq}(t)))), \)
\( g^I_q(z_q(t - y_{pq}(t))) = Im(g_q(z_q(t - y_{pq}(t)))), \)

\[
\begin{align*}
&\sum_{q=1}^{n} b^I_{pq}(t) g^R_q(z_q(t - y_{pq}(t))) + \sum_{q=1}^{n} b^R_{pq}(t) g^I_q(z_q(t - y_{pq}(t)))
\end{align*}
\]

\( (12) \)

2. Preliminaries

In this section, we state some useful definitions and lemmas.

**Lemma 4** (see [28, 29]). Define operator \( A_0 \) on \( C_T \):

\[
A : C_T \rightarrow C_T,
\]

\[
[Ax](t) = x(t) - cx(t - \tau), \quad \forall t \in \mathbb{R},
\]

where \( C_T = \{ x : x \in C(\mathbb{R}, \mathbb{R}), x(t) \equiv x(t) \} \) and \( c \) is a constant. When \( |c| \neq 1 \), then \( A \) has a unique continuous bounded inverse \( A^{-1} \) satisfying

\[
A^{-1}f(t) = \left\{ \begin{array}{ll}
\sum_{j=0}^{\lfloor \frac{t}{\tau} \rfloor} c^j f(t - j\tau), & \text{if } |c| < 1, \forall f \in C_T, \\
\sum_{j=1}^{\lfloor \frac{t}{\tau} \rfloor} c^{j-1} f(t + j\tau), & \text{if } |c| > 1, \forall f \in C_T,
\end{array} \right.
\]

and \( A^{-1} \) has the following inequality properties:

\[(a) \quad ||A^{-1}|| \leq \frac{1}{|1 - |c||}; \]
\[(b) \quad \int_0^T ||A^{-1}f||^2 dt \leq \left( \frac{1}{|1 - |c||} \right) \int_0^T |f(t)|^2 dt, \forall f \in C_T; \]
\[(c) \quad \int_0^T ||A^{-1}f||^2 dt \leq \left( \frac{1}{|1 - |c||} \right) \int_0^T |f(t)|^2 dt, \forall f \in C_T. \]

**Lemma 5** (see [30]). Suppose that \( X \) and \( Y \) are two Banach spaces, and \( L : D(L) \subset X \rightarrow Y \) is a Fredholm operator with index zero. Furthermore, \( \Omega \subset X \) is an open bounded set and \( N : \overline{\Omega} \rightarrow Y \) is \( L \)-compact on \( \overline{\Omega} \). If all the following conditions hold,

\[(1) \quad Lx \neq \lambda Nx, \quad \forall x \in \partial \Omega \cap D(L), \quad \forall \lambda \in (0, 1), \]
\[(2) \quad Nx \notin Im L, \quad \forall x \in \partial \Omega \cap Ker L, \]
\[(3) \quad deg[JQN, \Omega \cap Ker L, 0] \neq 0, \]

where \( J : Im Q \rightarrow Ker L \) is an isomorphism, then equation \( Lx = Nx \) has a solution on \( \overline{\Omega} \cap D(L) \).

**Definition 6.** The periodic solution of (7) \( z_p^* = (z_1^*, z_2^*, \ldots, z_n^*)^T \) is globally exponentially stable if there exist constants \( \alpha > 0 \) and \( \beta \geq 1 \) such that

\[
||z_p - z_p^*|| \leq \beta e^{-\alpha t} ||z_p(0) - z_p^*(0)||. \]

\( (15) \)
3. Existence of Periodic Solution

In order to investigate the periodicity of system (7) by means of Mawhin’s continuation theorem, we need to introduce some function spaces. Let

\[ C_T = \{ \phi : \mathbb{R} \to \mathbb{R} \mid \phi(t) \text{ is continuous and } T\text{-periodic} \} \]

\[ C_T^1 = \{ \phi \in C_T \mid \phi' \in C_T \} \]

\[ I_N = \{ 1, 2, \ldots, n \} \]

\[ U = \{ u(t) = (x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)^\top \mid x_p, y_p \in C_T, \ p \in I_N \} \]

with the norm

\[ \|u\|_U = \left\| (x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)^\top \right\|_U \]

\[ = \sum_{p=1}^n \sup_{t \in [0,T]} |x_p(t)| + \sum_{p=1}^n \sup_{t \in [0,T]} |y_p(t)|. \]  

Theorem 7. Assume that conditions (H_1) and (H_2) hold. Then system (7) has at least one \( T \)-periodic solution, if \( T^2|a_p(t)|^2/[1 - c_p] > 1 \).

Proof. From (12), let

\[ L : D(L) \subset U \to U, \]

\[ Lu = (Au)'(t), \]

\[ N : U \to U, \]

\[ Nu = \left( \begin{array}{c} \Xi_{p,x}(t) \\ \Xi_{p,y}(t) \end{array} \right), \]

\[ u \in U. \]

By Hale’s terminology [31], a solution of system (7) is \( u = (x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)^\top \in C(\mathbb{R}, \mathbb{R}^{2n}) \) such that \( Au \in C^1(\mathbb{R}, \mathbb{R}^{2n}) \) and the equalities in (7) are satisfied on \( \mathbb{R} \). Nevertheless, it is easy to see that \( (Au)'(t) = Au(t) \). So a \( T \)-periodic solution of the system (7) must be from \( C^1(\mathbb{R}, \mathbb{R}^{2n}) \). According to Lemma 5, we can easily obtain that

\[ \operatorname{Im}L = \left\{ u \in U \mid \int_0^T Au(s) \, ds = \mathbf{0} \right\}, \]

\[ \operatorname{Ker}L = \left\{ u \in U \mid u \equiv c \in \mathbb{R}^{2n} \right\}. \]

Obviously, \( \operatorname{Im}L \) is a closed set in \( C_T \) and \( \dim \operatorname{Ker}L = \operatorname{con} \dim \operatorname{Im}L = 2n \). So \( L \) is a Fredholm operator with index zero. Define continuous projectors \( P, Q \):

\[ P : C_T \to \operatorname{Ker}L, \]

\[ (Pu)(t) = \frac{1}{T} \int_0^T u(s) \, ds, \]

\[ Q : C_T \to \frac{C_T}{\operatorname{Im}L}, \]

\[ Qv = \frac{1}{T} \int_0^T v(s) \, ds. \]

Let

\[ L_p = L|_{D(L) \cap \operatorname{Ker}P} : D(L) \cap \operatorname{Ker}P \to \operatorname{Im}L, \]

and then

\[ L_p^{-1} = K_p : \operatorname{Im}L \to D(L) \cap \operatorname{Ker}P. \]

Since \( \operatorname{Im}L \subset C_T \) and \( D(L) \cap \operatorname{Ker}P \subset C_T^1 \), \( K_p \) is an embedding operator. Hence \( K_p \) is a complete operator in \( \operatorname{Im}L \). By the definitions of \( Q \) and \( N \), it is known that \( QN(\Omega) \) is bounded on \( \Omega \). Hence nonlinear operator \( N \) is \( L \)-compact on \( \Omega \). We complete the proof by three steps.

Step 1. Let \( \Omega_1 = \{ u \in D(L) \subset C_T : Lu = \lambda Nu, \ \lambda \in (0,1) \} \).

We show that \( \Omega_1 \) is a bounded set. If \( u \in \Omega_1 \), then \( Lu = \lambda Nu \); that is, for \( p = 1, 2, \ldots, n \),

\[ \frac{dA_p x_p(t)}{dt} = \lambda \left[ -d_p(t) x_p(t) + \sum_{q=1}^n a_{pq}^R(t) f_q^R(z_q(t)) - \sum_{q=1}^n a_{pq}^I(t) f_q^I(z_q(t)) \right] \]

\[ + \sum_{q=1}^n b_{pq}^R(t) g_q(t) - \sum_{q=1}^n b_{pq}^I(t) g_q(t) \right], \]

\[ \frac{dA_p y_p(t)}{dt} = \lambda \left[ -d_p(t) y_p(t) + \sum_{q=1}^n a_{pq}^R(t) f_q^R(z_q(t)) - \sum_{q=1}^n a_{pq}^I(t) f_q^I(z_q(t)) \right] \]

\[ + \sum_{q=1}^n b_{pq}^R(t) g_q(t) - \sum_{q=1}^n b_{pq}^I(t) g_q(t) \right] \]

\[ + \sum_{q=1}^n b_{pq}^I(t) g_q(t) \right]. \]

There exists \( t_i \in [0,T] \) such that \( A_p x_p(t_i) = [A_p x_p(t)]^\top \).

Hence \( (A_p x_p)'(t_i) = 0 \). In view of (23), this implies that

\[ d_p(t_i) x_p(t_i) = \sum_{q=1}^n a_{pq}^R(t_i) f_q^R(z_q(t_i)) \]

\[ - \sum_{q=1}^n a_{pq}^I(t_i) f_q^I(z_q(t_i)) \]

\[ + \sum_{q=1}^n b_{pq}^R(t_i) g_q(t_i) - \sum_{q=1}^n b_{pq}^I(t_i) g_q(t_i) \right]. \]
By (25) and assumption (H$_1$), for $p = 1, 2, \ldots, n$, we have

$$\left| x_p(t_i) \right| = \left| \sum_{q=1}^{n} a_{pq}^R(t_i) g^R_q(z_q(t_i)) \right| + \sum_{q=1}^{n} h^R_{pq}(t_i) \geq \sum_{q=1}^{n} \left[ a_{pq}^R(t_i) \frac{y_{pq}}{d_p(t_i)} \right] + \sum_{q=1}^{n} \left[ h^R_{pq}(t_i) \right] \geq \sum_{q=1}^{n} \left[ \frac{a_{pq}^R(t_i)}{d_p(t_i)} \right] M_{q,R} + \sum_{q=1}^{n} \left[ h^R_{pq}(t_i) \right] K_{q,I}$$

where $K_{q,I} = \max \left\{ b_{pq}^R(z_q(t)), 1 \right\}$.

Then by (26) there exists a positive constant $h_p$, such that

$$\left| x_p(t_i) \right| \leq h_p, \quad p = 1, 2, \ldots, n$$

where $h_p$ is the $p$th component of vector $h$. Clearly, $h_p, \quad p = 1, 2, \ldots, n$, are independent of $\lambda$. From (27) and Lemma 4 we have

$$\left| x_p(t) \right| \leq h_p + \int_{0}^{T} \left| x_p'(s) \right| ds$$

Thus

$$\left| x_p(t) \right| \leq h_p + \frac{1}{1 - c_p} \int_{0}^{T} \left| A_p x_p'(s) \right| ds.$$
If system (37) has a solution \( u^* = (x^*_1, \ldots, x^*_n, y^*_1, \ldots, y^*_n)^\top \), then similar to the above argument, we have \( \| u^* \| < M_0 \). Thus, (36) holds.

Step 3. Let \( \Omega = \{ u \in U : \| u \| < M_0 \} \), and then \( \Omega_1 \cup \Omega_1 \subseteq \Omega \). We will show that condition (3) in Lemma 5 holds. Take the homotopy

\[
H(u, \mu) = \begin{pmatrix}
-\tilde{d}_p x_p + (1 - \mu) \sum_{q=1}^n \left[ \tilde{a}_{pq} f_q^R(z_q) - \tilde{a}_{pq} f_q^I(z_q) \right] + \tilde{H}_p^R \& 0 \\
-\tilde{d}_p y_p + (1 - \mu) \sum_{q=1}^n \left[ \tilde{a}_{pq} f_q^R(z_q) - \tilde{a}_{pq} f_q^I(z_q) \right] + \tilde{H}_p^I
\end{pmatrix},
\]

where \( \mu \in [0, 1] \). We will show that if \( u \in \partial \Omega \cap \text{Ker} L_0 \), and \( u \) is a constant vector in \( \mathbb{R}^{2n} \) with \( \| u \|_U = M_0 \), then \( H(u, \mu) \neq 0 \). Otherwise, if \( u \in \mathbb{R}^{2n} \) with \( \| u \|_U = M_0 \) satisfying \( H(u, \mu) = 0 \),

\[
-\tilde{d}_p x_p + (1 - \mu) \sum_{q=1}^n \left[ \tilde{a}_{pq} f_q^R(z_q) - \tilde{a}_{pq} f_q^I(z_q) \right] + \tilde{H}_p^R = 0,
\]

\[
-\tilde{d}_p y_p + (1 - \mu) \sum_{q=1}^n \left[ \tilde{a}_{pq} f_q^R(z_q) - \tilde{a}_{pq} f_q^I(z_q) \right] + \tilde{H}_p^I = 0,
\]

which is a contradiction to \( \| u \|_U < M_0 \). And then by the degree theory,

\[
\deg \{ QN, \Omega \cap \text{Ker} L_0 \} = \deg \{ H(\cdot, 0), \Omega \cap \text{Ker} L_0 \} = \deg \{ -I, \Omega \cap \text{Ker} L_0 \} \neq 0.
\]

Applying Lemma 5, we reach the conclusion. \( \square \)

Remark 8. Since the condition \( |c_p| \neq 1, p = 1, 2, \ldots, n \) ensures that operator \( A_p \) has inverse operator which is important for obtaining the existence results of periodic solutions for (7). In critical case, Serra [32] studied a class of neutral differential equations

\[
x'(t) \pm x'(t - \tau) = f(t, x(t)), \quad x: 2\pi\text{-periodic}
\]

and obtained some existence results of periodic solutions. In a very recent paper, Junc and Lombard [33] investigated the following neutral differential equation:

\[
y'(t) \pm y'(t - 1) + f(y(t)) + g(y(t - 1)) = s(t), \quad t > 0
\]

\[
y(t) = y_0(t) \in H^1((-1, 0], \mathbb{R}), \quad -1 < t < 0.
\]

Based on the energy method, the authors obtained new results about asymptotic stability of constant and periodic solutions. Schmitt in [34] has shown how to reduce the case \( |a| > 1 \) to \( |a| < 1 \). However, when \( |c_p| = 1, p = 1, 2, \ldots, n \) (in this case, difference \( A_p \) has no inverse operator), there are no existence results of periodic solutions for system (7). We hope that the authors are interested in doing further research on this issue.

Remark 9. In this paper, we use Mawhin’s continuous theorem and some mathematical analysis technique for obtaining existence results. We can also use other methods (e.g., some fixed point theorem) to discuss the existence of \( T \)-periodic solution for system (7).

4. Global Exponential Stability of Periodic Solution

In this section, we establish some results for the uniqueness and exponential stability of the \( T \)-periodic solution of system (7).

Theorem 10. Under conditions of Theorem 7, assume further that

\[
(H_3') \text{ there exist } L_q > 0 \text{ and } \bar{L}_q > 0 \text{ such that for any } u, v \in C, \text{ one has}
\]

\[
|f_q(u) - f_q(v)| \leq L_q |u - v|, \quad q = 1, 2, \ldots, n;
\]

\[
|g_q(u) - g_q(v)| \leq \bar{L}_q |u - v|, \quad q = 1, 2, \ldots, n;
\]

\[
(H_4') \text{ [ } d_p(t)^- - \sum_{q=1}^n [d_{pq}(t)]^+ \text{] } L_q - \sum_{q=1}^n \max_{t \in [0,T]} (1/1 - y'_{pq}(v_{pq}(t))) \bar{L}_q |b_p(t)| > 0, \quad p = 1, 2, \ldots, n,
\]

where \( y'_{pq}(t) < 1 \) and \( v_{pq}(t) \) is an inverse function of \( t - y_{pq}(t) \). Then system (7) has unique \( T \)-periodic solution which is globally exponentially stable.

Proof. Using conditions of Theorem 7 and condition (H_3'), we can also use other methods (e.g., some fixed point theorem) to discuss the existence of \( T \)-periodic solution for system (7).
(z_1(t), z_2(t), ..., z_n(t))^\top is an arbitrary solution of (7). Then by (7), for \( p = 1, 2, ..., n, \ t > 0 \) we have

\[
\frac{d}{dt} \left[ A_p z_p(t) - A_p z_p^*(t) \right] = -d_p(t) \left[ z_p(t) - z_p^*(t) \right] + \sum_{q=1}^{n} a_{pq}(t) \left[ f_q(z_q(t)) - f_q(z_q^*(t)) \right] + \sum_{q=1}^{n} b_{pq}(t) \cdot g_q(z_q(t - \gamma_{pq}(t))) - g_q(z_q^*(t - \gamma_{pq}(t))) \cdot [g_q(z_q(t - \gamma_{pq}(t))) - g_q(z_q^*(t - \gamma_{pq}(t)))].
\]

In view of condition (H_3), we have

\[
\frac{d^+}{dt} \left[ A_p z_p(t) - A_p z_p^*(t) \right] = - \left[ d_p(t) \right]^- z_p(t) - z_p^*(t) + \sum_{q=1}^{n} \left[ a_{pq}(t) \right]^+ L_q \left[ z_q(t) - z_q^*(t) \right] + \sum_{q=1}^{n} \left[ b_{pq}(t) \right]^+ \left. \tilde{L}_q \left[ b_{pq}(t) \right]^+ \right| z_{p}(s) - z_{p}^*(s) \right| ds,
\]

and by (45) we have

\[
\frac{d^+ V(t)}{dt} \leq \sum_{p=1}^{n} \left[ - \left[ d_p(t) \right]^- z_p(t) - z_p^*(t) \right] + \sum_{q=1}^{n} \left[ a_{pq}(t) \right]^+ L_q \left[ z_q(t) - z_q^*(t) \right] + \sum_{q=1}^{n} \left[ b_{pq}(t) \right]^+ \left. \tilde{L}_q \left[ b_{pq}(t) \right]^+ \right| z_{p}(s) - z_{p}^*(s) \right| ds,
\]

According to condition (H_4) there exists a real number \( \alpha > 0 \) such that

\[
\left[ d_p(t) \right]^- - \sum_{q=1}^{n} \left[ a_{pq}(t) \right]^+ L_q - \sum_{q=1}^{n} \max_{t \in [0, T]} \left( 1 - \gamma'_{pq}(\gamma_{pq}(t)) \right) \tilde{L}_q \left[ b_{pq}(t) \right]^+ \right| > \alpha,
\]

and it follows that

\[
\frac{d^+ V(t)}{dt} \leq -\alpha V(t), \ t > 0.
\]

Then by (49) we have

\[
V(t) \leq e^{-\alpha t} V(0), \ t > 0.
\]

Thus

\[
\sum_{p=1}^{n} \left| z_p(s) - z_p^*(s) \right| \leq e^{-\alpha t} \left| z_p(0) - z_p^*(0) \right|, \ t > 0.
\]

Then system (7) has unique \( T \)-periodic solution \( z_q^*(t) \) which is globally exponentially stable. \( \square \)

Remark 11. It is well known that Lyapunov method has been widely used for studying stability problems. In this paper we construct a novel Lyapunov functional for studying the stability of periodic solutions, which is different from traditional Lyapunov functional method. And the proposed analysis method is also easy to extend to the case of other type neural networks. In the future, we will further study the synchronization problem and/or the Markovian jumping problem of complex-valued neural networks.

Remark 12. In studying the stability problems of time-delay systems, various methods have been developed. The important methods are based on the LKF methods [35], combined with other techniques such as the free-weighting matrix approach [36], the descriptor system approach [37], state estimation [38], and the triple-integral terms approach [39]. The results by the above methods are comparatively conservative. The important feature of the above LKF method is that the resulting conditions can check systems with interval delays, which may be unstable for small delays or delay-free cases. For more details, see, for example, [40–42].

5. Example

In order to verify the feasibility of our results, consider the following neutral-type CVNNs:

\[
\frac{dA z(t)}{dt} = -D(t) z(t) + A(t) f(z(t)) + B(t) g(z(t - \gamma(t))) + H(t),
\]

where \( D(t), A(t), f(t), g(t), H(t) \) are matrices and functions, respectively.
where
\[ A_p^r(t) = A_p^x(t) + iA_p^y(t), \]
\[ A_p^x(t) = x_p(t) - \frac{1}{2}x_p(t - \frac{\pi}{2}), \]
\[ A_p^y(t) = y_p(t) - \frac{1}{2}y_p(t - \frac{\pi}{2}), \]
\[ p = 1, 2, \]
\[ D(t) = \begin{pmatrix} 2.1 - 1.6 \sin \pi t & 0 \\ 0 & 2.1 - 1.6 \cos \pi t \end{pmatrix}, \]
\[ A(t) = \begin{pmatrix} \frac{1}{25} - \frac{i}{23} \sin 2\pi t & \frac{1}{25} + \frac{i}{23} \cos 2\pi t \\ \frac{1}{25} + \frac{i}{23} \cos 2\pi t & \frac{1}{25} - \frac{i}{23} \sin 2\pi t \end{pmatrix}, \]
\[ B(t) = \begin{pmatrix} \frac{1}{21} - \frac{i}{23} \sin 2\pi t & \frac{1}{21} - \frac{i}{23} \cos 2\pi t \\ \frac{1}{21} - \frac{i}{23} \cos 2\pi t & \frac{1}{21} - \frac{i}{23} \sin 2\pi t \end{pmatrix}, \]
\[ H(t) = \begin{pmatrix} -2 + 4 \sin 2\pi t \\ -2 + \cos 2\pi t \end{pmatrix}, \]
\[ \gamma(t) = \frac{1}{10} \sin t, \]
\[ f_q(z_q) = g_q(z_q) \]
\[ = \frac{2}{3 + \exp(x_q + y_q)} + i \frac{1 - \exp(-3x_q - y_q)}{2 + \exp(-x_q + y_q)}, \quad q = 1, 2. \]

Obviously, the conditions in Theorem 10 are all satisfied. It follows from Theorem 10 that the periodic solution of system (52) is globally exponentially stable.

The numerical simulations of (52) are shown in Figures 1 and 2. Figure 1 shows the state trajectories of the real part...
and the imaginary part of (52). Figure 2 shows the amplitude curves of neuro states of (52). From the simulation results, it can be seen that the periodic solution of (52) is unique and stable.

**Remark 13.** We greatly want to provide the circuit diagram of system (7). However, system (52) contains neutral-type operators and time-varying delays influence is more complicated. So far, we cannot obtain the circuit diagram of (52). We hope that some authors research this subject in the future.

### 6. Conclusions

In this paper, we have investigated stability problems of periodic solutions for a class of neutral-type complex-valued neural networks with time-varying delays. By utilizing novel Lyapunov-Krasovskii functionals, the sufficient conditions are derived to guarantee global exponential stability for the involved systems. A simulation example has been provided to show the usefulness of the proposed global exponential stability conditions.

We mention here that some finer approaches to deal with time delays would be the delay-slope-dependent method [43] and the delay-fraction approach [44], which could be the further work to reduce the possible conservatism in the dynamical analysis. And another future research topics would be the extension of the present results to more general cases, for example, the case that there exists impulsive influence, and the case that the neural network of neutral-type is a difference system. The results will appear in the near future.

### Competing Interests

The authors declare to have no competing interests in this paper.

### Acknowledgments

The authors acknowledge the funding of NNSF (no. 11571136) of China.

### References


