Research Article

Positive Solutions of a Diffusive Predator-Prey System including Disease for Prey and Equipped with Dirichlet Boundary Condition

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We study a three-dimensional system of a diffusive predator-prey model including disease spread for prey and with Dirichlet boundary condition and Michaelis-Menten functional response. By semigroup method, we are able to achieve existence of a global solution of this system. Extinction of this system is established by spectral method. By using bifurcation theory and fixed point index theory, we obtain existence and nonexistence of inhomogeneous positive solutions of this system in steady state.

1. Introduction

In recent decade, predator-prey systems are significant in mathematical biology and applications [1–3]. These systems are usually described by ordinary differential equation (ODEs). Besides, epidemic models have been deeply studied by differential equations in [4, 5], based on the pioneer work of the classical SIR model by Kermack and McKendrick. However, epidemic disease can spread within a species except for interactions among species. For this case, some ecoepidemiological models were introduced in [6–8] to study how disease affects the dynamic of predator-prey systems. Since we live in the real world, spatial factor plays an important role in populations dynamic. Moreover, space could induce some significant phenomena different from the corresponding ODE systems. At present, a lot of literature has showed that predator-prey systems with spatial diffusion can produce a variety of pattern formation by Turing bifurcation [9–11] and references wherein. In this paper, we are interested

in the following model having Holling-type II functional response with disease in the prey:

$$\begin{aligned} u_t - \Delta u &= u(a - u - v) - \beta uv, & (x, t) \in \Omega \times \mathbb{R}^+, \\ v_t - \Delta v &= \beta uv - bv - \frac{cwv}{mw + v}, & (x, t) \in \Omega \times \mathbb{R}^+, \\ w_t - \Delta w &= -dw + \frac{ewv}{mw + v}, & (x, t) \in \Omega \times \mathbb{R}^+. \end{aligned} \quad (1a)$$

There are two species in the model: prey ($u + v$) and predator (w). In (1a), u is susceptible prey, and v denotes infected prey. Only susceptible prey u is capable of reproducing and its reproducing rate is a . However, the infected prey does possess the capability of reproducing and still contributes to population growth of susceptible prey u according to the logistic growth law. The disease is transmitted only within prey in the form of βuv , and $\beta \in (0, 1)$ is called the transmission coefficient. The disease is not genetically inherited. The infected populations do not recover or become immune. Infected

individuals are less active and can be caught more easily by predator resulting in increasing survival of susceptible prey. Catching rate of predators is a ratio-dependent Michaelis-Menten functional response function; that is, $cwv/(mw + v)$. Infected prey has a death rate $b > 0$ and $d > 0$ is a death rate of predators. Prey and predators are inhabited in a bounded region $\Omega \subseteq \mathbb{R}^N$ with the smooth boundary $\partial\Omega$, where N is a spatial dimension number. Δ is the Laplace operator and represents diffusion of individuals from a high density region to low density one. Zhang et al. [12] have studied (1a) under the homogeneous Neumann boundary condition and obtain permanence and stability.

However, few author, as far as we know, investigated mathematical property of (1a) equipped with Dirichlet boundary condition, such as existence of global time-dependent positive solution and inhomogeneous positive solutions in steady state. To this end, in this paper we make the first attempt to fill this gap and study existence of a global positive solution (1a) with homogeneous Dirichlet boundary conditions:

$$u(x, t) = v(x, t) = w(x, t) = 0, \quad (x, t) \in \partial\Omega \times \mathbb{R}^+, \quad (1b)$$

which imply that predator and prey die out on the boundary $\partial\Omega$ and initial values

$$\begin{aligned} u(x, 0) &= u_0(x) \geq 0, \\ v(x, 0) &= v_0(x) \geq 0, \\ w(x, 0) &= w_0(x) \geq 0, \end{aligned} \quad (1c)$$

$$x \in \Omega.$$

Besides, we first attempt to investigate existence and nonexistence of inhomogeneous positive solutions of (1a), (1b), and (1c) in steady state as follows:

$$\begin{aligned} -\Delta u &= u(a - u - v) - \beta uv, \quad x \in \Omega, \\ -\Delta v &= \beta uv - bv - \frac{cwv}{mw + v}, \quad x \in \Omega, \\ -\Delta w &= -dw + \frac{ewv}{mw + v}, \quad x \in \Omega, \\ u = v = w &= 0, \quad x \in \partial\Omega. \end{aligned} \quad (2)$$

The remainder of the paper is organized as follows: in Section 2, some necessary notations and theoretical results are introduced; in Section 3, we examine existence of a global positive solution to (1a), (1b), and (1c); in Section 4 we discuss extinction of system (1a), (1b), and (1c); existence and nonexistence of inhomogeneous positive solutions are discussed by bifurcation theory and degree theory in Section 5.

2. Preliminary

In this section, some notations and basic well-known results are introduced. For any $q \in C(\bar{\Omega})$, the linear eigenvalue problem [13, 14],

$$\begin{aligned} -\Delta u + q(x)u &= \lambda u, \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega, \end{aligned} \quad (3)$$

has an infinite sequence of eigenvalues. Let $\lambda_1(q)$ be the principal eigenvalue of (9) which is simple and its eigenfunction does not change sign in Ω . We denote $\lambda_1(0)$ by λ_1 for simplicity and the eigenfunction corresponding to λ_1 is denoted by $\varphi_1(x)$. $q_1(x) \leq q_2(x)$ with q_1 being not identically equal to q_2 implies $\lambda_1(q_1) \leq \lambda_1(q_2)$.

Now, consider the nonlinear boundary value problem

$$\begin{aligned} -\Delta u &= u(a - u), \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega. \end{aligned} \quad (4)$$

If $a > \lambda_1$ then (4) has a unique positive solution (cf. [15, 16]). We denote this unique positive solution by u_a and $u_a < a$.

Next, the fixed point index theory, which is used later, is introduced. Let E be a real Banach space. W is called a wedge if W is a closed convex set and $\beta W \subset W$ for all $\beta \geq 0$. For $y \in W$, we define

$$\begin{aligned} W_y &= \{x \in E : \exists r = r(x) > 0, \text{ s.t. } y + rx \in W\}, \\ S_y &= \{x \in \bar{W}_y : -x \in \bar{W}_y\}. \end{aligned} \quad (5)$$

We always assume that $E = \bar{W} - \bar{W}$. Let $T : W_y \rightarrow W_y$ be a compact linear operator on E . We say that T has property α on $\bar{W}_y$ if there exists $t \in (0, 1)$ and $w \in \bar{W}_y \setminus S_y$ such that $(1 - tT)w \in S_y$. Let $A : W \rightarrow W$ be a compact operator with a fixed point $y \in W$ and A is Fréchet differentiable at y . Let $L = A'(y)$ be the Fréchet derivative of A at y . Then L maps $\bar{W}_y$ into itself. We denote by $\text{index}_W(A, y)$ the fixed point index of A at y relative to W .

Proposition 1 (see [16–18]). *Assume that $I - L$ is invertible on $\bar{W}_y$. Then*

- (i) *if L has property α on $\bar{W}_y$, then $\text{index}_W(A, y) = 0$;*
- (ii) *if L does not have property α on $\bar{W}_y$, then $\text{index}_W(A, y) = (-1)^\sigma$, where σ is the sum of multiplicities of all eigenvalues of L which is greater than 1.*

For a linear operator L , we denote by $r(L)$ the spectral radius of L .

Proposition 2 (see [16, 18]). *Let $q(x) \in C(\bar{\Omega})$ and M be a positive constant such that $M - q(x) > 0$ on $\bar{\Omega}$. Let $\lambda_1(q)$ be the first eigenvalue of the problem*

$$\begin{aligned} -\Delta u + q(x)u &= \lambda u, \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega. \end{aligned} \quad (6)$$

We have the following conclusions:

- (a) $\lambda_1(q) < 0 \Rightarrow r[(M - \Delta)^{-1}(M - q(x))] > 1$,
- (b) $\lambda_1(q) > 0 \Rightarrow r[(M - \Delta)^{-1}(M - q(x))] < 1$,
- (c) $\lambda_1(q) = 0 \Rightarrow r[(M - \Delta)^{-1}(M - q(x))] = 1$.

Now, we introduce the degree calculations, which was introduced by Dancer and Du in [19]. Assume that E_1 and E_2

are ordered Banach spaces with positive cones W_1 and W_2 , respectively. Let $E = E_1 \oplus E_2$ and $W = W_1 \oplus W_2$. Then E is an ordered Banach space with positive cone W . Let D be an open set in W containing 0 and $F_i : \bar{D} \rightarrow W_i$ be completely continuous operators, $i = 1, 2$. Denote by (u, v) a general element in W with $u \in W_1$ and $v \in W_2$. Let $F : \bar{D} \rightarrow W$ be defined by

$$F(u, v) = (F_1(u, v), F_2(u, v)). \quad (7)$$

Define $W_2(\epsilon) = \{v \in W_2 : \|v\|_{E_2} < \epsilon\}$.

Theorem 3 (see [15]). *Suppose that $U \subset W_1 \cap D$ is relatively open and bounded and*

$$\begin{aligned} F_1(u, 0) &\neq u \quad \text{for } u \in \partial U; \\ F_2(u, 0) &= 0 \quad \text{for } u \in \bar{U}. \end{aligned} \quad (8)$$

Suppose that $F_2 : D \rightarrow W_2$ extends to a continuously differentiable mapping of a neighborhood of D into E_2 , $W_2 - W_2$ is dense in E_2 , and $\Phi = \{u \in U : u = F_1(u, 0)\}$. Then the following are true:

- (i) $\deg_W(I - F, U \times W_2(\epsilon), 0) = 0$ for $\epsilon > 0$ small, if, for any $u \in \Phi$, the spectral radius $r(F'_2(u, 0)|_{W_2}) > 1$ and 1 is not an eigenvalue of $F'_2(u, 0)|_{W_2}$ corresponding to a positive eigenvector;
- (ii) $\deg_W(I - F, U \times W_2(\epsilon), 0) = \deg_W(I - F_1|_{W_1}, U, 0)$ for $\epsilon > 0$ small, if, for any $u \in \Phi$, the spectral radius $r(F'_2(u, 0)|_{W_2}) < 1$.

3. Existence of a Global Positive Solution of (1a), (1b), and (1c)

For convenience, we introduce some notations. For $x, y \in \mathbb{R}$, we denote $x \wedge y := \min\{x, y\}$, $x \vee y := \max\{x, y\}$, $x_+ := x \vee 0$, and $x_- := (-x) \vee 0$ and extend these notations to real-valued functions. If $(V, \geq)$ is partially ordered vector space, we denote its positive cone by $V_+ := \{v \in V : v \geq 0\}$. For $p \in [1, +\infty)$, $p > N/2$, let $X = \{L^p(\Omega)\}^3$. Assume $z = (u, v, w) \in X$ and its norm is defined as $\|z\| = \|u\|_p + \|v\|_p + \|w\|_p$.

Hence, system (1a), (1b), and (1c) can be rewritten as an abstract differential equation:

$$\frac{dz}{dt} = Az + H(z), \quad t \in \mathbb{R}^+, \quad (9)$$

$$z(0) = z_0, \quad t = 0,$$

where $z = (u(x, t), v(x, t), w(x, t))^T$ and $z_0 = (u_0, v_0, w_0)^T$,

$$A = \begin{pmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{pmatrix} \quad (10)$$

with

$$\begin{aligned} D(A) &= \{(u, v, w) \in \{C^2(\bar{\Omega})\}^3, u|_{\partial\Omega} = v|_{\partial\Omega} = w|_{\partial\Omega} \\ &= 0\}, \\ H(z) &= \begin{pmatrix} u(a - u - v) - \beta uv \\ \beta uv - bv - \frac{cwv}{mw + v} \\ -dw + \frac{ewv}{mw + v} \end{pmatrix}. \end{aligned} \quad (11)$$

Next, the existence of the local solution for (9) will be proved. First, we give out the following lemma.

Lemma 4. *For every $z_0 \in X_+$, Cauchy problem (9) has a unique maximal local solution*

$$\begin{aligned} z &\in C([0, T_{\max}); X) \cap C^1((0, T_{\max}); X) \\ &\cap C((0, T_{\max}); D(A_p)), \end{aligned} \quad (12)$$

which satisfies the following Duhamel formula for $t \in [0, T_{\max})$:

$$z(t) = e^{tA_p} z_0 + \int_0^t e^{(t-s)A_p} H[z(s)] ds, \quad (13)$$

where A_p is a operator and the closure of A in X . Moreover, if $T_{\max} < \infty$ then $\limsup_{t \rightarrow T_{\max}^-} \|z(t)\| = \infty$.

Proof. Since the operator A_p is the closure of A in X , it generates an analytic, condensed, strong continuous operator semigroup e^{tA_p} . Moreover, we observe that $H : D(A_p) \rightarrow X$ is locally Lipschitz on a bounded set [20]. Via Theorem 7.2.1 in [21], we complete this proof. $\square$

Lemma 5. *For initial-boundary problem (1a), (1b), and (1c), the components of its solution $u(x, t)$, $v(x, t)$, $w(x, t)$ are nonnegative.*

Proof. To prove $u(x, t), v(x, t), w(x, t) \geq 0$ for $t \in [0, T_{\max})$ and $x \in \Omega$, we consider the following system:

$$\frac{\partial u'}{\partial t} = \Delta u' + u'_+ (a - u'_+ - v'_+) - \beta u'_+ v'_+, \quad (14a)$$

$$\frac{\partial v'}{\partial t} = \Delta v' + \beta u'_+ v'_+ - bv'_+ - \frac{cw'_+ v'_+}{mw'_+ + v'_+}, \quad (14b)$$

$$\frac{\partial w'}{\partial t} = \Delta w' - dw'_+ + \frac{ew'_+ v'_+}{mw'_+ + v'_+} \quad (14c)$$

with boundary and initial conditions

$$\begin{aligned} u'|_{\partial\Omega} &= v'|_{\partial\Omega} = w'|_{\partial\Omega} = 0, \\ u'(x, 0) &= u_0, \\ v'(x, 0) &= v_0, \\ w'(x, 0) &= w_0. \end{aligned} \quad (15)$$

After multiplying (14a), (14b), and (14c) by u'_- , v'_- , and w'_- , respectively, and integrating by parts on the domain Ω , we obtain

$$\begin{aligned} -\frac{1}{2} \frac{d}{dt} \int_{\Omega} |u'_-|^2 dx - \int_{\Omega} |\nabla u'_-|^2 dx &= 0, \\ -\frac{1}{2} \frac{d}{dt} \int_{\Omega} |v'_-|^2 dx - \int_{\Omega} |\nabla v'_-|^2 dx &= 0, \\ -\frac{1}{2} \frac{d}{dt} \int_{\Omega} |w'_-|^2 dx - \int_{\Omega} |\nabla w'_-|^2 dx &= 0. \end{aligned} \quad (16)$$

Hence, for $t \in (0, T_{\max})$,

$$\begin{aligned} \|u'_-(\cdot, t)\|_2^2 + \|v'_-(\cdot, t)\|_2^2 &\leq \|u'_-(\cdot, 0)\|_2^2 + \|v'_-(\cdot, 0)\|_2^2 \\ &= 0. \end{aligned} \quad (17)$$

Consequently,

$$\begin{aligned} u'(x, t) &\geq 0, \\ v'(x, t) &\geq 0, \\ w'(x, t) &\geq 0. \end{aligned} \quad (18)$$

Now, we can know that $(u'(x, t), v'(x, t), w'(x, t))$ is a solution of (1a), (1b), and (1c), and according to Lemma 4, we finally get that

$$\begin{aligned} u(x, t) &= u'(x, t) \geq 0, \\ v(x, t) &= v'(x, t) \geq 0, \\ w(x, t) &= w'(x, t), \\ t &\in (0, T_{\max}). \end{aligned} \quad (19)$$

□

Next, we will prove the global existence of the positive solution for (1a), (1b), and (1c). Via Lemmas 4 and 5, we only need to show that the solution of (1a), (1b), and (1c) is uniformly bounded, that is, dissipation.

Theorem 6. *Let $(u(x, t), v(x, t), w(x, t))$ be the solution of (1a), (1b), and (1c) with initial value (1c); then*

$$\begin{aligned} 0 &\leq u(x, t) \leq c_1, \\ 0 &\leq v(x, t) \leq c_2, \\ 0 &\leq w(x, t) \leq c_3 \end{aligned} \quad (20)$$

for $(x, t) \in \Omega \times [0, T_{\max})$, where c_i , $i = 1, 2, 3$ are defined by

$$\begin{aligned} c_1 &:= \max \{a, \|u_0\|_{\infty}\}, \\ c_2 &:= \max \left\{ \frac{\beta(a+b)c_1}{b(1+\beta)}, \frac{\beta\|u_0\|_{\infty}}{1+\beta} + \|v_0\|_{\infty} \right\}, \end{aligned}$$

$$\begin{aligned} c_3 &:= \max \left\{ \frac{(a+d)e\beta c_1}{cd(1+\beta)} \right. \\ &\quad \left. + \frac{ec_2}{c}, \frac{e}{c} \left(\frac{\beta\|u_0\|_{\infty}}{1+\beta} + \|v_0\|_{\infty} \right) + \|w_0\|_{\infty} \right\}. \end{aligned} \quad (21)$$

Proof. In view of Lemma 5, we only prove the upper bound of (1a), (1b), and (1c) for $(x, t) \in \Omega \times [0, T_{\max})$. From (1a), we have

$$\begin{aligned} u_t &\leq \Delta u + u(a-u), \quad \text{in } \Omega \times \mathbb{R}^+, \\ u(x, t) &= 0, \quad \text{on } \partial\Omega \times \mathbb{R}^+, \end{aligned} \quad (22)$$

$$u(x, 0) = u_0, \quad \text{in } \Omega.$$

We get $u(x, t) \leq c_1$, $(x, t) \in \Omega \times (0, T_{\max})$ by maximum principle, where $c_1 = \max\{a, \|u_0\|_{\infty}\}$.

Let $v_1 = \beta u/(1+\beta) + v$. From (1b) we have

$$\begin{aligned} (v_1)_t - \Delta v_1 &= \frac{\beta}{1+\beta} u(a-u) - bv - \frac{cuv}{mw+v} \\ &\leq \frac{a\beta u}{1+\beta} - b \left(v_1 - \frac{\beta u}{1+\beta} \right) \\ &\leq \frac{\beta(a+b)c_1}{1+\beta} - bv_1. \end{aligned} \quad (23)$$

Hence, by maximum principle we have that

$$\begin{aligned} v(x, t) &\leq v_1(x, t) \\ &\leq \max \left\{ \frac{\beta(a+b)c_1}{b(1+\beta)}, \frac{\beta\|u_0\|_{\infty}}{1+\beta} + \|v_0\|_{\infty} \right\} \\ &:= c_2, \quad t \in [0, T). \end{aligned} \quad (24)$$

By using the same method, we consider $w_1 = (e/c)(\beta u/(1+\beta) + v) + w$; then from (1c) we have that

$$\begin{aligned} (w_1)_t - \Delta w_1 &= \frac{e\beta}{c(1+\beta)} u(a-u) - \frac{be}{c} v - dw \\ &\leq \frac{ae\beta}{c(1+\beta)} u \\ &\quad - d \left[w_1 - \frac{e\beta u}{c(1+\beta)} - \frac{ev}{c} \right] \\ &\leq \frac{(a+d)e\beta c_1}{c(1+\beta)} + \frac{dec_2}{c} - dw_1. \end{aligned} \quad (25)$$

Thus, the maximum principle shows that

$$\begin{aligned} w(x, t) &\leq w_1(x, t) \leq \max \left\{ \frac{(a+d)e\beta c_1}{cd(1+\beta)} \right. \\ &\quad \left. + \frac{ec_2}{c}, \frac{e}{c} \left(\frac{\beta\|u_0\|_{\infty}}{1+\beta} + \|v_0\|_{\infty} \right) + \|w_0\|_{\infty} \right\} := c_3. \end{aligned} \quad (26)$$

□

Remark 7. From Theorem 6, we know that the upper bound of the solution of (1a), (1b), and (1c) is independent of the maximal time $T_{\max}$, which implies that the solution of (1a), (1b), and (1c) is a global solution.

By Lemmas 4 and 5 and Theorem 6, we can get the following theorem.

Theorem 8. System (1a), (1b), and (1c) has a unique, nonnegative, and bounded solution $z(x, t) = (u(x, t), v(x, t), w(x, t))$, such that

$$\begin{aligned} z(x, t) &\in C([0, \infty), X) \cap C^1((0, \infty), X) \\ &\cap C((x, \infty), D(A)). \end{aligned} \quad (27)$$

4. Extinction of (1a), (1b), and (1c)

Theorem 9. Let $(u(x, t), v(x, t), w(x, t))$ be a positive solution of (1a), (1b), and (1c), then we have the following:

- (i) if $a \leq \lambda_1$ and $e < d$ then $(u, v, w) \rightarrow (0, 0, 0)$ as $t \rightarrow \infty$;
- (ii) if $a < \lambda_1$ and $-b < \lambda_1(-\beta u_a)$ then $(u, v, w) \rightarrow (u_a, 0, 0)$ as $t \rightarrow \infty$.

Proof. (i) First, we observe that solution (u, v, w) of (1a), (1b), and (1c) satisfies

$$\begin{aligned} u_t - \Delta u &\leq u(a - u), \quad (x, t) \in \Omega \times \mathbb{R}^+, \\ u &= 0, \quad (x, t) \in \partial\Omega \times \mathbb{R}^+. \end{aligned} \quad (28)$$

Let ε be a sufficiently small positive constant such that $\varepsilon \leq b/\beta$. Since $a \leq \lambda_1$, we see that $u(x, t) \rightarrow 0$ uniformly as $t \rightarrow \infty$ by using comparison principle for elliptic problems. Then there exists $T_\varepsilon \geq 0$ such that $u(x, t) \leq \varepsilon$ for all $t \geq T_\varepsilon$. Therefore, we have

$$\begin{aligned} v_t - \Delta v &\leq v(\beta\varepsilon - b), \quad (x, t) \in \Omega \times (T_\varepsilon, \infty), \\ v &= 0, \quad (x, t) \in \partial\Omega \times (T_\varepsilon, \infty), \end{aligned} \quad (29)$$

which concludes that $v(x, t) \rightarrow 0$ uniformly as $t \rightarrow \infty$. Similarly, there exists $T'_\varepsilon > 0$ such that $v(x, t) \leq \varepsilon$ for all $t > T'_\varepsilon$. Further, since $ewv/(mw + v)$ is monotone increasing with respect to the variable v , we have

$$\begin{aligned} w_t - \Delta w &\leq w\left(\frac{e\varepsilon}{mw + \varepsilon} - d\right) \leq w(e - d), \\ (x, t) &\in \Omega \times (T'_\varepsilon, \infty), \end{aligned} \quad (30)$$

$$w = 0, \quad (x, t) \in \partial\Omega \times (T'_\varepsilon, \infty),$$

which concludes that $w(x, t) \rightarrow 0$ as $t \rightarrow \infty$.

(ii) Let ε be a sufficiently small positive constant such that

$$\begin{aligned} \text{(a)} \quad \varepsilon &< \frac{\lambda_1(b - \beta u_a)}{\beta}, \\ \text{(b)} \quad \varepsilon &< \frac{a - \lambda_1}{\beta + 1}. \end{aligned} \quad (31)$$

Since $a > \lambda_1$, we get from (28) and comparison theorem that

$$\limsup_{t \rightarrow \infty} u(x, t) \leq u_a. \quad (32)$$

Then, there exists $T_\varepsilon > 0$ such that $u(x, t) \leq u_a + \varepsilon$ for all $t > T_\varepsilon$. Thus, we have

$$\begin{aligned} v_t - \Delta v &\leq v[\beta(u_a + \varepsilon) - b], \quad (x, t) \in \Omega \times (T_\varepsilon, \infty), \\ v &= 0, \quad (x, t) \in \partial\Omega \times (T_\varepsilon, \infty). \end{aligned} \quad (33)$$

Since ε satisfies (a) and $-b < \lambda_1(-\beta u_a)$, we conclude that $v(x, t) \rightarrow 0$ uniformly as $t \rightarrow \infty$. Similarly, we can see that $w(x, t) \rightarrow 0$ uniformly as $t \rightarrow \infty$.

Besides, there exists $T'_\varepsilon \geq 0$ such that $v(x, t), w(x, t) < \varepsilon$ for all $t > T'_\varepsilon$. Hence, we have

$$\begin{aligned} u_t - \Delta u &\geq u(a - u - (\beta + 1)\varepsilon), \\ (x, t) &\in \Omega \times (T'_\varepsilon, \infty), \end{aligned} \quad (34)$$

$$u = 0, \quad (x, t) \in \partial\Omega \times (T'_\varepsilon, \infty).$$

Since ε satisfies (b), we get that

$$\liminf_{t \rightarrow \infty} u(x, t) \geq u_{(a - (\beta + 1)\varepsilon)}. \quad (35)$$

Thus, from (32), (35), and the arbitrariness of ε , we can conclude that $u(x, t) \rightarrow u_a$ as $t \rightarrow \infty$. We finish this proof. $\square$

5. Existence and Nonexistence of Positive Solutions of (2)

To prove the existence of positive solution of (2) by using fixed point index theory, we need a priori estimate for nonnegative solutions of (1a), (1b), and (1c). So we first give the following theorem.

Theorem 10. Any nonnegative solution $(u(x), v(x), w(x))$ of (2) has a priori bounds:

$$\begin{aligned} u(x) &\leq a, \\ v(x) &\leq c_4, \\ w(x) &\leq c_5, \end{aligned} \quad (36)$$

where

$$\begin{aligned} c_4 &:= \frac{\beta(a + b)a}{b(1 + \beta)}, \\ c_5 &:= \frac{a(a + d)e\beta}{cd(1 + \beta)} + \frac{ec_4}{c}. \end{aligned} \quad (37)$$

Proof. From the first equation of (2), we have

$$\begin{aligned} -\Delta u &\leq u(a - u), \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega, \end{aligned} \quad (38)$$

and we get $u(x) \leq a$ by maximum principle. Let $v_1 = \beta u / (1 + \beta) + v$ with $v_1|_{\partial\Omega} = 0$. From (1b), we have

$$\begin{aligned} -\Delta v_1 &= \frac{\beta}{1+\beta} u(a-u) - bv - \frac{cuv}{mw+v} \\ &\leq \frac{a\beta u}{1+\beta} - b \left(v_1 - \frac{\beta u}{1+\beta} \right) \leq \frac{\beta(a+b)a}{1+\beta} - bv_1. \end{aligned} \quad (39)$$

Thus, by maximum principle, we get

$$v(x) \leq v_1(x) \leq \frac{\beta(a+b)a}{b(1+\beta)} = c_4. \quad (40)$$

We set $w_1 = (e/c)(\beta u / (1 + \beta) + v) + w$ with $w_1|_{\partial\Omega} = 0$, and then we have

$$\begin{aligned} -\Delta w_1 &= \frac{e\beta}{c(1+\beta)} u(a-u) - \frac{be}{c} v - dw \\ &\leq \frac{ae\beta}{c(1+\beta)} u - d \left[w_1 - \frac{e\beta u}{c(1+\beta)} - \frac{ev}{c} \right] \\ &\leq \frac{a(a+d)e\beta}{c(1+\beta)} + \frac{dec_4}{c} - dw_1. \end{aligned} \quad (41)$$

By using maximum principle again, we obtain

$$w \leq w_1 \leq \frac{a(a+d)e\beta}{cd(1+\beta)} + \frac{ec_4}{c} = c_5. \quad (42)$$

Therefore, we complete this proof. $\square$

Now, we introduce some notations:

$E = \{C_0^1(\bar{\Omega})\}^3$, where $C_0^1(\Omega) = \{\phi \in C^1(\bar{\Omega}) : \phi = 0, \text{ on } \partial\Omega\}$,

$W = \{K\}^3$, where $K = \{\phi \in C^1(\bar{\Omega}) : \phi \geq 0 \text{ in } \Omega \text{ and } \phi = 0 \text{ on } \partial\Omega\}$,

$D = \{(u, v, w) \in W : u < a+1, v < c_4+1, w < c_5+1\}$, according to Theorem 10.

Taking

$$M = \max \left\{ a^2 + ac_4 + \beta ac_4, bc_4 + \frac{c}{m}, dc_5 \right\}, \quad (43)$$

we can define a positive and compact operator $F : E \rightarrow E$ by

$$\begin{aligned} F(u, v, w) &= (-\Delta + M)^{-1} \begin{pmatrix} u(a-u-v) - \beta uv + Mu \\ \beta uv - bv - \frac{cuv}{mw+v} + Mv \\ -dw + \frac{euw}{mw+v} + Mw \end{pmatrix}. \end{aligned} \quad (44)$$

It is obvious that F having a positive fixed point in interior of D implies that (2) has a positive solution.

Next, we give the degree of $I - F$ in D relative to K and the fixed point index of F at the trivial solution $(0, 0, 0)$ of (1a), (1b), and (1c) relative to K .

Lemma 11. Assume that $a > \lambda_1$, then one has

- (i) $\deg_W(1 - F, D) = 1$,
- (ii) $\text{index}_W(F, (0, 0, 0)) = 0$.

Proof. (i) It is easy to see that F has no fixed point on $\partial\Omega$, so $\deg_W(I - F, D)$ is well defined. For $s \in [0, 1]$, we define a positive and compact operator $F_s : E \rightarrow E$ by

$$\begin{aligned} F_s(u, v) &= (-\Delta + M)^{-1} \begin{pmatrix} su(a-u-v) - s\beta uv + Mu \\ s\beta uv - sbv - \frac{scuv}{mw+v} + Mv \\ -dsw + \frac{seuv}{mw+v} + Mw \end{pmatrix}, \end{aligned} \quad (45)$$

and then $F_1 = F$. For each s , a fixed point F_s is a solution of the following problem:

$$\begin{aligned} -\Delta u &= su(a-u-v) - s\beta uv, \quad x \in \Omega, \\ -\Delta v &= s\beta uv - sbv - \frac{scuv}{mw+v}, \quad x \in \Omega, \\ -\Delta w &= -dsw + \frac{seuv}{mw+v}, \quad x \in \Omega, \\ u &= v = 0, \quad x \in \partial\Omega. \end{aligned} \quad (46)$$

Form Theorem 10 F_s has no fixed point $\partial\Omega$ and $\deg_W(1 - F_s, D)$ is well defined. Hence,

$$\begin{aligned} \deg(I - F, D) &= \deg_W(I - F_1, D) \\ &= \deg_W(I - F_0, D). \end{aligned} \quad (47)$$

Note that (46) has only the trivial solution $(0, 0, 0)$ when $s = 0$. Set

$$L = F'_0(0, 0, 0) = (-\Delta + M)^{-1} \begin{pmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & M \end{pmatrix}. \quad (48)$$

Let $L(\xi_1, \xi_2, \xi_3) = (\xi_1, \xi_2, \xi_3)$ for some $(\xi_1, \xi_2, \xi_3) \in \bar{W}_{(0,0,0)} = K \times K \times K$. Obviously, $(\xi_1, \xi_2, \xi_3) = (0, 0, 0)$.

Hence, $I - L$ has no nontrivial kernel in $\bar{W}_{(0,0,0)}$. Since $\lambda_1 > 0$, it is easy to see that $r((-\Delta + M)^{-1}(M)) < 1$ by Proposition 2. It implies that L has no property α . Therefore, by Proposition 1, we get the first result of Lemma 11.

(ii) It is obvious that $F(0, 0, 0) = (0, 0, 0)$. Let $L = F'_0(0, 0, 0)$; then

$$L = (-\Delta + M)^{-1} \begin{pmatrix} a+M & 0 & 0 \\ 0 & -b+M & 0 \\ 0 & 0 & -d+M \end{pmatrix}. \quad (49)$$

Assume that $L(\xi_1, \xi_2, \xi_3) = (\xi_1, \xi_2, \xi_3)$ for some $(\xi_1, \xi_2, \xi_3) \in \overline{W}_{(0,0,0)}$, and then

$$\begin{aligned} -\Delta \xi_1 &= a\xi_1, & x \in \Omega, \\ \xi_1 &= 0, & x \in \partial\Omega, \end{aligned} \quad (50)$$

$$\begin{aligned} -\Delta \xi_2 &= -b\xi_2, & x \in \Omega, \\ \xi_2 &= 0, & x \in \partial\Omega, \end{aligned} \quad (51)$$

$$\begin{aligned} -\Delta \xi_3 &= -d\xi_3, & x \in \Omega, \\ \xi_3 &= 0, & x \in \partial\Omega. \end{aligned} \quad (52)$$

Since $b > 0$ and $d > 0$, from (51) and (52), we know that $\xi_2 = \xi_3 = 0$. If $\xi_1 \in K$ with $\xi_1 \neq 0$, then $a = \lambda_1$ from (50), which contradicts with the assumption. Thus $(\xi_1, \xi_2, \xi_3) = (0, 0, 0)$. So $I - L$ has nontrivial kernel in $\overline{W}_{(0,0,0)}$.

For $a > \lambda_1$, by Proposition 2, we see that $r_0 = r((-\Delta + M)^{-1}(a + M)) > 1$, and r_0 is the principle eigenvalue of $(-\Delta + M)^{-1}(a + M)$ with a corresponding eigenfunction $\phi_0 > 0$. Since $S_{(0,0,0)} = \{(0, 0, 0)\}$, we see that $(\phi_0, (0, 0, 0)) \in \overline{W}_{(0,0,0)} \setminus S_{(0,0,0)}$. Set $s \in 1/r_0 \in (0, 1)$, and then $(I - sL)(\phi_0, (0, 0, 0)) = (0, 0, 0) \in S_{(0,0,0)}$. Thus, L has the property α . By Proposition 1 we have $\text{index}_W(F, (0, 0, 0)) = 0$. $\square$

Now, we give out the index of the semitrivial solution $(u_a, 0, 0)$ of (2).

Lemma 12. Assume that $a > \lambda_1$, and then one has

- (i) if $-b > \lambda_1(-\beta u_a)$, then $\text{index}_W(F, (u_a, 0, 0)) = 0$;
- (ii) if $-b < \lambda_1(-\beta u_a)$, then $\text{index}_W(F, (u_a, 0, 0)) = 1$.

Proof. Let $L = F'(u_a, 0, 0)$; then

$$L = (-\Delta + M)^{-1} \cdot \begin{pmatrix} a - 2u_a + M & -(1 + \beta)u_a & 0 \\ 0 & \beta u_a - b + M & 0 \\ 0 & 0 & -d + M \end{pmatrix}. \quad (53)$$

Let $L(\xi_1, \xi_2, \xi_3) = (\xi_1, \xi_2, \xi_3)$ for some $(\xi_1, \xi_2, \xi_3) \in \overline{W}_{(u_a,0,0)} = C_0^1(\overline{\Omega}) \times K \times K$. Then

$$\begin{aligned} -\Delta \xi_1 + (2u_a - a)\xi_1 &= -(1 + \beta)u_a \xi_2, & x \in \Omega, \\ -\Delta \xi_2 + (b - \beta u_a)\xi_2 &= 0, & x \in \Omega, \\ -\Delta \xi_3 + d\xi_3 &= 0, & x \in \Omega, \\ \xi_1 = \xi_2 = \xi_3 &= 0, & x \in \partial\Omega. \end{aligned} \quad (54)$$

Since $\xi_2, \xi_3 \in K$, $-b \neq \lambda_1(-\beta u_a)$ and $d > 0$. So we get that $\xi_2 = \xi_3 = 0$. Hence, from the first equation of (54) we have

$$\begin{aligned} -\Delta \xi_1 + (2u_a - a)\xi_1 &= -(1 + \beta)u_a \xi_2, & x \in \Omega, \\ \xi_1 &= 0, & x \in \partial\Omega. \end{aligned} \quad (55)$$

If $\xi_1 \neq 0$, then $\lambda_1(2u_a - a) = 0$ by Proposition 2. On the other hand, $\lambda_1(2u_a - a) > \lambda_1(u_a - a) = 0$, and we get a contradiction. Hence, $(\xi_1, \xi_2, \xi_3) = (0, 0, 0)$. Then $I - L$ has no nontrivial kernel in $\overline{W}_{(u_a,0,0)}$.

(i) Since $-b > \lambda_1(-\beta u_a)$, by Proposition 1, we have $r_0 = r((-\Delta + M)^{-1}(\beta u_a - b + M)) > 1$ which is an eigenvalue of $(-\Delta + M)^{-1}(\beta u_a - b + M)$ with a corresponding eigenfunction $\phi_0 > 0$. Since $S_{(u_a,0,0)} = C_0^1(\overline{\Omega}) \times \{0\} \times \{0\}$, we have $(0, \phi_0, 0) \in \overline{W}_{(u_a,0,0)} \setminus S_{(u_a,0,0)}$. Set $s \in 1/r_0 \in (0, 1)$, and then

$$\begin{aligned} (I - sL) \begin{pmatrix} 0 \\ \phi_0 \\ 0 \end{pmatrix} &= \begin{pmatrix} 0 \\ \phi_0 \\ 0 \end{pmatrix} - s(-\Delta + M)^{-1} \begin{pmatrix} -(1 + \beta)u_a \phi_0 \\ (\beta u_a - b + M)\phi_0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} s(-\Delta + M)^{-1}(1 + \beta)u_a \phi_0 \\ 0 \\ 0 \end{pmatrix} \in S_{(u_a,0,0)}. \end{aligned} \quad (56)$$

Hence, L has the property α on $\overline{W}_{(u_a,0,0)}$ resulting in $\text{index}_W(F, (u_a, 0, 0)) = 0$ according to Proposition 1.

(ii) First, we prove that L does not have the property α on $\overline{W}_{(u_a,0,0)}$.

Since $-b < \lambda_1(-\beta u_a)$, we have $r((-\Delta + M)^{-1}(-\beta u_a)) < 1$. On the contrary, we suppose that L has the property α on $\overline{W}_{(u_a,0,0)}$. Then there exist $s \in (0, 1)$ and $(\xi_1, \xi_2, \xi_3) \in \overline{W}_{(u_a,0,0)}$ such that $(I - sL)(\xi_1, \xi_2, \xi_3) \in S_{(u_a,0,0)}$. So

$$(-\Delta + M)^{-1}(\beta u_a - b + M)\xi_2 = \frac{\xi_2}{s}. \quad (57)$$

Then $1/s$ is an eigenvalue of the operator $(-\Delta + M)^{-1}(\beta u_a - b + M)$, which contradicts $r((-\Delta + M)^{-1}(\beta u_a - b + M)) < 1$. So L does not have the property α on $\overline{W}_{(u_a,0,0)}$. By Proposition 2, we have $\text{index}_W(F, (u_a, 0, 0)) = (-1)^\sigma$, where σ is the sum of the multiplicities of all eigenvalues of L . Next, we will prove that $\sigma = 0$.

Assume that $1/\rho > 1$ is an eigenvalue of L with corresponding eigenfunction (ξ_1, ξ_2, ξ_3) , and then $L(\xi_1, \xi_2, \xi_3) = 1/\rho(\xi_1, \xi_2, \xi_3)$; that is,

$$\begin{aligned} -\Delta \xi_1 + M\xi_1 &= \rho((a - 2u_a + M)\xi_1 - (1 + \beta)u_a \xi_2), \\ & x \in \Omega, \\ -\Delta \xi_2 + M\xi_2 &= \rho(\beta u_a - b + M)\xi_2, & x \in \Omega, \\ -\Delta \xi_3 + M\xi_3 &= \rho(-d + M)\xi_3, & x \in \Omega, \end{aligned} \quad (58)$$

$$\xi_1 = \xi_2 = \xi_3 = 0, \quad x \in \partial\Omega.$$

Since $0 < \rho < 1$ and $d > 0$, it follows from the third equation (58) that $\xi_3 = 0$.

We suppose that $\xi_2 \neq 0$, and it follows from the second equation of (58) and Proposition 2 that

$$0 = \lambda_1 (M(1 - \rho) - \rho(\beta u_a - b)) > \lambda_1 (-\beta u_a) + b, \quad (59)$$

which contradicts $-b < \lambda_1(-\beta u_a)$, and so $\xi_2 = 0$.

If $\xi_1 \neq 0$, it follows from the first equation of (58) that

$$\begin{aligned} 0 &= \lambda_1 (M(1 - \rho) - \rho(a - 2u_a)) > \lambda_1 (2u_a - a) \\ &> \lambda_1 (u_a - a) = 0. \end{aligned} \quad (60)$$

Thus, we get the contradiction and this implies that $\xi_1 = 0$. Hence, $(\xi_1, \xi_2, \xi_3) = (0, 0, 0)$, which implies that L has no eigenvalue being greater than 1; that is, $\sigma = 0$, and $\text{index}_W(F, (u_a, 0, 0)) = 1$. We complete the proof. $\square$

Next, we consider the other semitrivial solutions of (2). Let us consider the following three subsystems:

$$\begin{aligned} -\Delta u &= u(a - u - v) - \beta uv, \quad x \in \Omega, \\ -\Delta v &= \beta uv - bv, \quad x \in \Omega, \end{aligned} \quad (61)$$

$$u = v = 0, \quad x \in \partial\Omega.$$

$$\begin{aligned} -\Delta u &= u(a - u), \quad x \in \Omega, \\ -\Delta w &= -dw, \quad x \in \Omega, \end{aligned} \quad (62)$$

$$u = w = 0, \quad x \in \partial\Omega.$$

$$\begin{aligned} -\Delta v &= -bv - \frac{c w v}{m w + v}, \quad x \in \Omega, \\ -\Delta w &= -dw + \frac{e w v}{m w + v}, \quad x \in \Omega, \end{aligned} \quad (63)$$

$$v = w = 0, \quad x \in \partial\Omega.$$

It is obvious that (63) has trivial solution $(v, w) = (0, 0)$ and that (62) only has a trivial solution $(u, w) = (0, 0)$ and a semitrivial solution $(u, w) = (u_a, 0)$ if and only if $a > \lambda_1$. For subsystem (61), a lot of papers [15, 22] and references wherein have discussed systems in this case, so we directly give out some results about (61).

Theorem 13. Equation (61) has a positive solution (u^*, v^*) with $u^* < u_a$ if and only if $a > \lambda_1$ and $-b > \lambda_1(-\beta u_a)$. Moreover, if $a - \beta - 1 > \lambda_1$ and $-b > \lambda_1(-\beta u_{(a-\beta-1)})$, then the positive solution (u^*, v^*) satisfies $u_{(a-\beta-1)} \leq u^*$ and $\bar{v} \leq v^*$, where $\bar{v}$ is the unique positive solution of the following equations:

$$\begin{aligned} -\Delta v &= v(\beta u_{(a-\beta-1)} - b), \quad x \in \Omega, \\ v &= 0, \quad x \in \partial\Omega. \end{aligned} \quad (64)$$

Let $\{(u, v)\}$ be the set of all positive solutions of (61), denoted by Φ , and $\Psi = \{(u, v, 0), (u, v)\}$. It is easy to see that (61) has semitrivial solution $(u^*, v^*, 0) \in \Psi$ from Theorem 13. Next, we give the degree of $(I - F)$ in Ψ relative to W .

Lemma 14. Assume $\Psi \neq \emptyset$, and then one has the following:

- (i) if $-d > \lambda_1 - e$, for any $(u^*, v^*, 0) \in \Psi$, then $\deg_W(I - F, \Psi) = 0$;
- (ii) if $-d < \lambda_1 - e$, for any $(u^*, v^*, 0) \in \Psi$, then $\deg_W(I - F, \Psi) = 1$.

Proof. Let

$$\begin{aligned} E_1 &= \{C_0^1(\bar{\Omega})\}^2, \\ E_2 &= C_0^1(\bar{\Omega}), \\ W_1 &= \{K\}^2, \\ W_2 &= K. \end{aligned} \quad (65)$$

Define

$$\begin{aligned} F_1(u, v, w) &= (-\Delta + M)^{-1} \begin{pmatrix} u(a - u) - \beta uv + Mu \\ \beta uv - bv - \frac{c w v}{m w + v} + Mv \end{pmatrix}, \end{aligned} \quad (66)$$

$$F_2(u, v, w) = (-\Delta + M)^{-1} \left(\frac{e w v}{m w + v} - dw \right).$$

Then $F = (F_1, F_2)$ and we are in the framework to use Theorem 3. We choose a neighborhood $U \subset W_1 \cap D$ of $\Psi \cap W_1$ such that $(u_a, 0) \notin \bar{U}$.

Now, $F_1(u, v, 0) = (u, v)$ with $(u, v) \in \bar{U}$ if and only if $(u, v, 0) \in \Psi$. By the results in [22], we have

$$\begin{aligned} \deg_{W_1}(I - F_1|_{W_1}, U, 0) &= \begin{cases} 1, & \text{if } a > \lambda_1, -b > \lambda_1(-\beta u_a), \\ -1, & \text{if } a < \lambda_1, -b < \lambda_1(-\beta u_a), \\ 0, & \text{if } (a - \lambda_1) \cdot [-b - \lambda_1(-\beta u_a)] < 0. \end{cases} \end{aligned} \quad (67)$$

For any $(u^*, v^*, 0) \in \Psi$, a direct calculation shows that

$$F_2'(u^*, v^*, 0)|_{W_2} w = (-\Delta + M)^{-1} (M - b + e) w. \quad (68)$$

Notice that, by our choice of $M, M - b + e > 0$, so it follows from the maximum principle that $F_2'(u^*, v^*, 0)|_{W_2}$ is u_0 -positive in the sense of [23] with $u_0 = (-\Delta)^{-1}$. Hence, $r(F_2'(u^*, v^*, 0)|_{W_2}) > 0$ and is the only eigenvalue corresponding to a positive eigenvector. By a simple calculation, we can easily show that $r(F_2'(u^*, v^*, 0)|_{W_2}) > 1$ if and only if $-d > \lambda_1 - e$ and $r(F_2'(u^*, v^*, 0)|_{W_2}) < 1$ if and only if $-d < \lambda_1 - e$. Therefore, by Theorem 3 and the discussion above, we have

$$\begin{aligned} \deg_W(I - F, U \times W_2(\varepsilon), 0) &= \begin{cases} 0, & \text{if } -d > \lambda_1 - e, \\ \deg_W(I - F_1|_{W_1}, U, 0), & \text{if } -d < \lambda_1 - e. \end{cases} \end{aligned} \quad (69)$$

Again since the degree does not depend on the particular choice of U and $\varepsilon, \Psi \neq \emptyset$ means $a > \lambda_1$ and $-b > \lambda_1(-\beta u_a)$. Thus, we complete this proof. $\square$

From the discussion above, we can obtain the existence of positive solutions of (2).

Theorem 15. Assume $a > \lambda_1$. Equation (2) has at least one positive solution if

$$\begin{aligned} a &> \lambda_1 + \beta + 1, \\ -b &> \lambda_1 (-\beta u_{(a-\beta-1)}), \\ -d &> \lambda_1 - e, \end{aligned} \quad (70)$$

and then (2) has at least one positive solution.

Proof. Since $a > \lambda_1$, we obtain $\deg_W(I - F, D) = 1$ and $\text{index}_W(F, (0, 0, 0)) = 0$ from Lemma 11. Hence, we have

$$\text{index}_W(F, (u_a, 0, 0)) + \deg_W(I - F, D) = 1. \quad (71)$$

Since $-b > \lambda_1(-\beta u_{(a-\beta-1)}) > \lambda_1(-\beta u_a)$, we have $\text{index}_W(F, (u_a, 0, 0)) = 0$ from Lemma 12. Moreover, Theorem 13 implies $\Psi \neq \emptyset$. Let $(u^*, v^*, 0) \in \Psi$. Since $u^* \geq u_{(a-\beta)}$ and $v^* \geq \bar{v}$ by Theorem 13, we have $-d > \lambda_1 - e$ and thus $\deg_W(I - F, \Psi) = 0$ from Lemma 14. Therefore, $\text{index}_W(F, (u_a, 0, 0)) + \deg_W(I - F, \Psi) = 0$. So (71) holds. We complete this proof. $\square$

Next, we give out the result of nonexistence of positive solutions.

Theorem 16. If any one of the following conditions holds, then (2) has no positive solution:

- (i) $a < \lambda_1$;
- (ii) $a > \lambda_1$ and $-b \leq \lambda_1(-\beta u_a)$.

Proof. Assume that (2) has positive solution (u, v, w) , and then u, v, w satisfy the following three equations, respectively:

$$\begin{aligned} -\Delta u &= u(a - u - v) - \beta uv, \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega, \end{aligned} \quad (72)$$

$$\begin{aligned} -\Delta v &= \beta uv - bv - \frac{cuv}{mw + v}, \quad x \in \Omega, \\ v &= 0, \quad x \in \partial\Omega, \end{aligned} \quad (73)$$

$$\begin{aligned} -\Delta w &= \frac{ewv}{mw + v} - dw, \quad x \in \Omega, \\ w &= 0, \quad x \in \partial\Omega. \end{aligned} \quad (74)$$

(i) Since $u > 0$ in Ω , we get from (72) that

$$a = \lambda_1(u + (\beta + 1)v) > \lambda_1. \quad (75)$$

By Proposition 2 and comparison property of principle eigenvalue, this contradicts $a < \lambda_1$. Then (73) and (74) have $v = 0$ and $w = 0$.

(ii) Since $v > 0$ in Ω , we get from (73) that

$$-b = \lambda_1\left(\frac{cw}{mw + v} - \beta u\right) > \lambda_1(-\beta u_a), \quad (76)$$

by Proposition 2 and comparison property of principle eigenvalue, we get a contradiction $-b \leq \lambda_1(-\beta u_a)$. $\square$

Competing Interests

The authors declare that they have no competing interests.

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