Research Article

Bifurcation and Nodal Solutions for the Half-Linear Problems with Nonasymptotic Nonlinearity at 0 and ∞

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We study the existence of nodal solutions for the following problem: $-x'' = \alpha x^+ + \beta x^- + ra(t)f(x)$, $0 < t < 1$, $x(0) = x(1) = 0$, where $r \neq 0$ is a parameter, $a(t) \in C([0, 1], (0, \infty))$ with $a(t) \neq 0$ on any subinterval of $[0, 1]$, $x^+ = \max\{x, 0\}$, $x^- = -\min\{x, 0\}$, and $\alpha, \beta \in C[0, 1]$; $f \in C(\mathbb{R}, \mathbb{R})$, $sf(s) > 0$ for $s \neq 0$, and $f_0, f_\infty \notin (0, \infty)$, where $f_0 = \lim_{|s| \rightarrow 0} f(s)/s$ and $f_\infty = \lim_{|s| \rightarrow +\infty} f(s)/s$. We use bifurcation techniques to prove our main results.

1. Introduction

By applying the bifurcation techniques of Rabinowitz [1, 2], Im et al. [3] studied the existence of positive solutions of the problems and some authors [4–7] studied the existence of nodal solutions of the problems. Recently, Dai and Ma [8] established the unilateral global bifurcation theory for the problems by applying the bifurcation techniques of Dancer [9]. Later, Dai [10] also considered the existence of nodal solutions for the problems with nonasymptotic nonlinearity at 0 or ∞ by applying the unilateral global bifurcation theory of [8]. For the abstract unilateral global bifurcation theory, we refer the reader to [1, 2, 8, 9, 11–13] and the references therein. On the other hand, problems involving nondifferentiable nonlinearity have also been investigated by applying bifurcation techniques; see [13–17]. In particular, Berestycki [14] established the Rabinowitz-type global interval bifurcation result and Ma and Dai [13] established the unilateral global interval bifurcation theorem. Meanwhile, half-linear or half-quasilinear boundary value problems have attracted the attention of many specialists in different equations because of their interesting applications; see [13, 14, 16–18]. Among them, Berestycki [14] (or see [13]) established the spectrum for the following half-linear eigenvalue problem:

$$\begin{aligned} -x'' &= \lambda a(t)x + \alpha x^+ + \beta x^-, \quad 0 < t < 1, \\ x(0) &= x(1) = 0, \end{aligned} \quad (1)$$

where $x^+ = \max\{x, 0\}$, $x^- = -\min\{x, 0\}$, $\alpha, \beta \in C[0, 1]$, and a satisfies the following condition:

(A1) $a \in C([0, 1], (0, \infty))$ with $a(t) \neq 0$ on any subinterval of $[0, 1]$.

Let $Y = C[0, 1]$ with the norm $\|x\|_\infty = \max_{t \in [0, 1]} |x(t)|$. Let $E = \{x \in C^1[0, 1] \mid x(0) = x(1) = 0\}$ with the norm $\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty\}$.

We define the linear operator $L : D(L) \subset E \rightarrow Y$

$$Lx = -x'', \quad x \in D(L) \quad (2)$$

with $D(L) = \{x \in C^2[0, 1] \mid x(0) = x(1) = 0\}$. Then L is a closed operator and $L^{-1} : Y \rightarrow E$ is completely continuous.

Let $E = \mathbb{R} \times E$ under the product topology. Let S_k^+ denote the set of function in E which have exactly $k-1$ interior nodal (i.e., nondegenerate) zeros in $(0, 1)$ and are positive near $t = 0$, set $S_k^- = -S_k^+$, and $S_k = S_k^+ \cup S_k^-$. They are disjoint and open in E . Finally, let $\Phi_k^\pm = \mathbb{R} \times S_k^\pm$ and $\Phi_k = \mathbb{R} \times S_k$.

By Theorem 1 of [14], Berestycki [14] obtained the following result.

Lemma 1 (see [14, Theorem 2]). *Let (A1) hold. There exist two sequences of simple half eigenvalues for (1)*

$$\begin{aligned} \lim_{k \rightarrow \infty} \lambda_k^+ &= +\infty, \quad \lambda_1^+ < \lambda_2^+ < \cdots < \lambda_k^+ < \cdots, \\ \lim_{k \rightarrow \infty} \lambda_k^- &= +\infty, \quad \lambda_1^- < \lambda_2^- < \cdots < \lambda_k^- < \cdots. \end{aligned} \quad (3)$$

The corresponding half-lines solutions are in $\{\lambda_k^+\} \times S_k^+$ and $\{\lambda_k^-\} \times S_k^-$. Further, aside from these solutions and the trivial ones, there are no other solutions of (1).

Furthermore, following Lemma 1, Ma and Dai [13] (or see [14]) considered the existence of nodal solutions for the following half-linear eigenvalue problem:

$$\begin{aligned} -x'' &= \alpha x^+ + \beta x^- + ra(t)f(x), \quad 0 < t < 1, \\ x(0) &= x(1) = 0, \end{aligned} \quad (4)$$

where $r \neq 0$ is a parameter and $f \in C(\mathbb{R}, \mathbb{R})$ satisfies the following assumptions:

(A2) $sf(s) > 0$ for $s \neq 0$.

(A3) There exist $f_0, f_\infty \in (0, +\infty)$ such that

$$\begin{aligned} f_0 &= \lim_{|s| \rightarrow 0} \frac{f(s)}{s}, \\ f_\infty &= \lim_{|s| \rightarrow +\infty} \frac{f(s)}{s}. \end{aligned} \quad (5)$$

Firstly, we show that the existence and uniqueness theorem is valid for (4).

Lemma 2 (see [13, Lemma 2.2]). *If (λ, x) is a nontrivial solution of (4) under assumptions (A1)–(A3) and x has a double zero, then $x \equiv 0$.*

Under assumptions (A1)–(A3), they obtained the following result.

Theorem 3 (see [13, Theorem 4.1]). *Let (A1), (A2), and (A3) hold. For $v \in \{+, -\}$, either $\lambda^v/f_\infty < r < \lambda^v/f_0$ or $\lambda^v/f_0 < r < \lambda^v/f_\infty$. Then problem (4) has a solution x_k^v such that x_k^v has exactly $k - 1$ zeros in $(0, 1)$ and vx_k^v is positive near 0.*

Of course, the natural question is that of what would happen if $f_0 \notin (0, +\infty)$ or $f_\infty \notin (0, +\infty)$. Obviously, the previous results cannot deal with this case. The purpose of this work is to establish several results similar to those of [13]. The main methods used in this work are global bifurcation techniques and the approximation of connected components. Moreover, we consider the cases of $f_0, f_\infty \notin (0, \infty)$, while the authors of [13, 16, 17] only studied the cases of $f_0, f_\infty \in (0, \infty)$.

In this paper, we will investigate the existence of nodal solutions for problem (4), where a satisfies condition (A1). Throughout this paper, we assume that f satisfies (A2) and the following assumptions:

(H1) $f_0 \in (0, \infty)$ and $f_\infty = 0$.

(H2) $f_0 \in (0, \infty)$ and $f_\infty = \infty$.

(H3) $f_0 = \infty$ and $f_\infty \in (0, \infty)$.

(H4) $f_0 = \infty$ and $f_\infty = 0$.

(H5) $f_0 = \infty$ and $f_\infty = \infty$.

(H6) $f_0 = 0$ and $f_\infty \in (0, \infty)$.

(H7) $f_0 = 0$ and $f_\infty = 0$.

(H8) $f_0 = 0$ and $f_\infty = \infty$.

The rest of this paper is arranged as follows. In Section 2, we gave some preliminaries. In Section 3, we give the interval for the parameter r which ensure the existence of single or multiple nodal solutions for half-linear problem (4) under assumptions (A2) and (H1)–(H8) for the nonlinearity f .

2. Preliminaries

Let $\mathcal{S}_k^v$ denote the closure of the set of nontrivial solutions of (4) in $\mathbb{R} \times E$, with $\mathcal{S}_k^v$ denoting the subset of with $u \in S_k^+$ and $\mathcal{S}_k = \mathcal{S}_k^+ \cup \mathcal{S}_k^-$.

In order to prove our main results, we need the following Sturm type comparison result.

Lemma 4. *Let $b_2(t) \geq \max\{b_1(t), b_1(t) + \alpha(t) + \beta(t), b_1(t) - \alpha(t) - \beta(t)\}$ for $t \in (0, 1)$ and $b_i(t) \in C(0, 1)$, $i = 1, 2$. Also let u_1, u_2 be solutions of the following differential equations:*

$$u'' + b_i(t)u + \alpha u^+ + \beta u^- = 0, \quad t \in (0, 1), \quad i = 1, 2, \quad (6)$$

respectively. If $(c, d) \subseteq (0, 1)$, and $u_1(c) = u_1(d) = 0$, $u_1(t) \neq 0$ in (c, d) , then either there exists $\tau \in (c, d)$ such that $u_2(\tau) = 0$ or $b_2 = b_1$ or $b_2 = b_1 + \alpha + \beta$ or $b_2 = b_1 - \alpha - \beta$ and $u_2(t) = \mu u_1(t)$ for some constant $\mu \neq 0$.

Proof. We discuss four cases.

Case 1 ($u_1(t) > 0, u_2(t) > 0$ in (c, d)). By the Picone identity [19] (or (0.4) of [20]), we have

$$\begin{aligned} & \int_c^d \left[\frac{u_1}{u_2} (u_2 u_1' - u_1 u_2') \right] dt \\ &= \int_c^d \frac{u_1}{u_2} (u_2 \widehat{L}u_1 - u_1 \widehat{L}u_2) dt \\ &+ \int_c^d \left(u_1' - \frac{u_1}{u_2} u_2' \right)^2 dt, \end{aligned} \quad (7)$$

where $\widehat{L}x = -Lx = x''$, $x \in D(L)$. It follows that

$$\begin{aligned} & \int_c^d \left[\frac{u_1}{u_2} (u_2 u_1' - u_1 u_2') \right] dt \\ &= \int_c^d \left[(b_2 - b_1) u_1^2 + \left(u_1' - \frac{u_1}{u_2} u_2' \right)^2 \right] dt. \end{aligned} \quad (8)$$

The left-hand side of (8) equals

$$\begin{aligned} & - \lim_{t \rightarrow c^+} \frac{u_1 (u_2 u_1' - u_1 u_2')}{u_2} + \lim_{t \rightarrow d^-} \frac{u_1 (u_2 u_1' - u_1 u_2')}{u_2} \\ &= -H_c + H_d. \end{aligned} \quad (9)$$

We prove that $H_c = 0$. If $u_2(c) \neq 0$, then $H_c = 0$. If $u_2(c) = 0$, noting the conclusion of Lemma 2, then $u_2'(c) > 0$. By L'Hospital rule, we have that

$$\begin{aligned} H_c &= \lim_{t \rightarrow c^+} \frac{u_1 (u_2 u_1' - u_1 u_2')}{u_2} \\ &= \lim_{t \rightarrow c^+} \frac{u_1' (u_2 u_1' - u_1 u_2') + u_1 (u_2 u_1' - u_1 u_2')'}{u_2'} = 0. \end{aligned} \quad (10)$$

Similarly, we can obtain that $H_d = 0$. Therefore, the left-hand side of (8) equals zero. Hence, the right-hand side of (8) also equals zero. It follows that there exists a constant $\mu \neq 0$ such that $u_2 = \mu u_1$ and $b_2 = b_1$.

Case 2 ($u_1(t) > 0$, $u_2(t) < 0$ in (c, d)). Similar to (8), we can get

$$\begin{aligned} &\int_c^d \left[\frac{u_1}{u_2} (u_2 u_1' - u_1 u_2') \right] dt \\ &= \int_c^d \left[(b_2 - b_1 - \alpha - \beta) u_1^2 + \left(u_1' - \frac{u_1}{u_2} u_2' \right)^2 \right] dt. \end{aligned} \quad (11)$$

The above argument is still valid for this case.

Case 3 ($u_1(t) < 0$, $u_2(t) < 0$ in (c, d)). Similar to Case 1, we can get the result.

Case 4 ($u_1(t) < 0$, $u_2(t) > 0$ in (c, d)). Similar to (8), we can get

$$\begin{aligned} &\int_c^d \left[\frac{u_1}{u_2} (u_2 u_1' - u_1 u_2') \right]' dt \\ &= \int_c^d \left[(b_2 - b_1 + \alpha + \beta) u_1^2 + \left(u_1' - \frac{u_1}{u_2} u_2' \right)^2 \right] dt. \end{aligned} \quad (12)$$

Similar to the proof of Case 1, we can obtain the result. $\square$

By Lemma 4, we obtain the following result that will be used later.

Lemma 5. Let $I = [a, b]$ be such that $I \subset [0, 1]$ and

$$\text{meas}(I) > 0. \quad (13)$$

Let $g_n(t) \in C([0, 1], \mathbb{R})$ be such that

$$\lim_{n \rightarrow +\infty} g_n(t) = +\infty \quad \text{uniformly on } I. \quad (14)$$

Let $y_n \in E$ be a solution of the equation

$$y_n''(t) + g_n(t) y_n + \alpha y_n^+ + \beta y_n^- = 0, \quad 0 < t < 1. \quad (15)$$

Then y_n must change sign on I as $n \rightarrow +\infty$.

Proof. Set $\alpha^0 := \max_{t \in [0, 1]} |\alpha(t)|$ and $\beta^0 := \max_{t \in [0, 1]} |\beta(t)|$. By simple computation, we can show that

$$g_n(t) + \alpha \frac{y_n^+}{y_n} + \beta \frac{y_n^-}{y_n} \geq g_n(t) - \alpha^0 - \beta^0 \quad \forall t \in (0, 1). \quad (16)$$

After taking a subsequence if necessary, we may assume that

$$g_n(t) - \alpha^0 - \beta^0 \geq \lambda_j \quad \forall t \in I \quad (17)$$

as $j \rightarrow +\infty$, where λ_j is j th eigenvalue of the following problem:

$$\begin{aligned} u'' + \lambda u &= 0, \quad 0 < t < 1, \\ u(0) &= u(1) = 0. \end{aligned} \quad (18)$$

Set $s = (b - a)t + a$, $u(s) = v((b - a)s + a)$. By some simple computations, we can show

$$\begin{aligned} (b - a)^2 v'' + \lambda v &= 0, \quad t \in (a, b), \\ v(a) &= v(b) = 0. \end{aligned} \quad (19)$$

Let ψ_j be the corresponding eigenfunction of λ_j . Since $\lambda_j \rightarrow \infty$ as $j \rightarrow \infty$, Lemma 2.1 of [4] implies that ψ_j must change sign on I . Note that the conclusion of Lemma 4 also is valid if $\alpha = \beta = 0$. Using these facts and Lemma 4, we can obtain the desired result. $\square$

Definition 6 (see [21]). Let X be a Banach space and $\{C_n \mid n = 1, 2, \dots\}$ be a family of subsets of X . Then the superior limit $\mathbb{D}$ of $\{C_n\}$ is defined by

$$\begin{aligned} \mathbb{D} &:= \limsup_{n \rightarrow \infty} C_n = \{x \in X \mid \exists \{n_i\} \subset \mathbb{N}, x_{n_i} \\ &\in C_{n_i}, \text{ such that } x_{n_i} \rightarrow x\}. \end{aligned} \quad (20)$$

Lemma 7 (see [22]). Let X be a Banach space and let $\{C_n \mid n = 1, 2, \dots\}$ be a family of closed connected subsets of X . Assume that

- (i) there exist $z_n \in C_n$, $n = 1, 2, \dots$, and $z^* \in X$, such that $z_n \rightarrow z^*$;
- (ii) $r_n = \sup\{\|x\| \mid x \in C_n\} = \infty$;
- (iii) for all $R > 0$, $(\bigcup_{n=1}^{\infty} C_n) \cap B_R$ is a relative compact set of X , where

$$B_R = \{x \in X \mid \|x\| \leq R\}. \quad (21)$$

Then there exists an unbounded component C in $\mathbb{D}$ and $z^* \in C$.

3. Nodal Solutions for Half-Linear Eigenvalue Problems

In this section, we will study the existence of nodal solutions for problem (4) with $f_0 \notin (0, +\infty)$ or $f_\infty \notin (0, +\infty)$.

We start this section by studying the following eigenvalue problem:

$$\begin{aligned} -x'' &= \alpha x^+ + \beta x^- + \lambda a(t) f(x), \quad 0 < t < 1, \\ x(0) &= x(1) = 0, \end{aligned} \quad (22)$$

where $\lambda > 0$ is a parameter. Firstly, under the conditions (A1)–(A3), let $\zeta \in C(\mathbb{R}, \mathbb{R})$ be such that

$$f(x) = f_0 x + \zeta(x) \quad (23)$$

with $\lim_{|x| \rightarrow 0} (\zeta(x)/x) = 0$. Let us consider

$$-x'' = \lambda r a(t) f_0 x + \alpha x^+ + \beta x^- + \lambda r a(t) \zeta(x), \quad 0 < t < 1, \quad (24)$$

$$x(0) = x(1) = 0,$$

as a bifurcation problem from the trivial solution $x \equiv 0$. By using Theorem 3 of [13] or Ma and Dai [13, Lemma 4.1] obtained the following Lemma.

Lemma 8 (see [13, Lemma 4.1]). *For $v \in \{+, -\}$, $(\lambda_k^v/rf_0, 0)$ is a bifurcation point for problem (24). Moreover, there exists an unbounded continuum $\mathcal{D}_k^v$ of solutions of problem (24), such that $\mathcal{D}_k^v \subset (\Phi_k^v \cup \{(\lambda_k^v/rf_0, 0)\})$.*

Remark 9. Any solution of (22) of the form $(1, x)$ yields a solution x of (4). In order to prove our main results, one will only show that $\mathcal{D}_k^v$ crosses the hyperplane $\{1\} \times E$ in $\mathbb{R} \times E$.

Clearly, (A2) implies $f(0) = 0$. Hence, $x = 0$ is always the solution of (4). Applying Lemma 8 (or Lemma 4.1 of [13]), we will establish the existence of nodal solutions of (4) as follows.

Theorem 10. *Let (A1), (A2), and (H1) hold. For $v \in \{+, -\}$, assume that one of the following conditions holds:*

- (i) $r \in (\lambda_k^v/f_0, +\infty)$ for $\lambda_k^v > 0$.
- (ii) $r \in (-\infty, \lambda_k^-/f_0) \cup (\lambda_k^+/f_0, +\infty)$ for $v\lambda_k^v > 0$.
- (iii) $r \in (-\infty, \lambda_k^+/f_0) \cup (\lambda_k^-/f_0, +\infty)$ for $v\lambda_k^v < 0$.
- (iv) $r \in (-\infty, \lambda_k^v/f_0)$ for $\lambda_k^v < 0$.

Then problem (4) possesses two solutions x_k^+ and x_k^- such that x_k^+ has exactly $k - 1$ zeros in $(0, 1)$ and is positive near 0 and x_k^- has exactly $k - 1$ zeros in $(0, 1)$ and is negative near 0.

Proof. We only prove the case of (i) since the proofs of the cases for (ii), (iii), and (iv) can be given similarly.

In view of the proof to prove [13, Theorem 4.1], we only need to show that $\mathcal{D}_k^v$ joins $(\lambda_k^v/rf_0, 0)$ to (∞, ∞) . To do this, it is enough to prove that $[\lambda_k^v/rf_0, +\infty) \subset \text{Proj}_R \mathcal{D}_k^v$.

Assume on the contrary that $\sup\{\lambda \mid (\lambda, u) \in \mathcal{D}_k^v\} < +\infty$; then there exists a sequence $(\mu_n, x_n) \in \mathcal{D}_k^v$ such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|x_n\| &= +\infty, \\ \mu_n &\leq c_0, \end{aligned} \quad (25)$$

for some positive constant c_0 not depending on n .

By (H1), let $\bar{f}(x) = \max_{0 \leq |s| \leq x} |f(s)|$, then $\bar{f}$ is nondecreasing and

$$\lim_{x \rightarrow +\infty} \frac{\bar{f}(x)}{x} = 0. \quad (26)$$

We consider the equation

$$\begin{aligned} -x_n'' &= \alpha x_n^+ + \beta x_n^- + \mu_n r a(t) f(x_n), \quad t \in (0, 1), \\ x_n(0) &= x_n(1) = 0. \end{aligned} \quad (27)$$

Let $y_n = x_n/\|x_n\|$, and y_n should be the solutions of problem

$$\begin{aligned} -y_n'' &= \alpha y_n^+ + \beta y_n^- + \mu_n r a(t) \frac{f(x_n)}{\|x_n\|}, \quad t \in (0, 1), \\ y_n(0) &= y_n(1) = 0. \end{aligned} \quad (28)$$

Since y_n is bounded in $C^2[0, 1]$, choosing a subsequence and relabelling if necessary, we have that $y_n \rightarrow y$ for some $y \in E$ and $\|y\| = 1$.

Furthermore, from (26) and the fact that $\bar{f}$ is nondecreasing, we have that

$$\lim_{n \rightarrow \infty} \frac{f(x_n)}{\|x_n\|} = 0, \quad (29)$$

since

$$\begin{aligned} \frac{f(x_n)}{\|x_n\|} &\leq \frac{\bar{f}(\|x_n\|)}{\|x_n\|} \leq \frac{\bar{f}(\|x_n\|_\infty)}{\|x_n\|} \leq \frac{\bar{f}(\|x_n\|)}{\|x_n\|} \rightarrow 0, \\ n &\rightarrow +\infty. \end{aligned} \quad (30)$$

By (29) and the compactness of L^{-1} , we obtain that

$$\begin{aligned} -y'' &= \alpha y^+ + \beta y^-, \quad t \in (0, 1), \\ y(0) &= y(1) = 0. \end{aligned} \quad (31)$$

By $y(0) = y(1)$, there exists $\xi \in (0, 1)$ such that $y'(\xi) = 0$. Without loss of generality, it follows that

$$|y(t)| = \left| \int_0^t \left[\int_\xi^s (-\alpha y^+ - \beta y^-) d\tau \right] ds \right| \leq M \int_0^t |y| ds, \quad t \in (0, 1), \quad (32)$$

$$M = \max\{\alpha^0, \beta^0\}.$$

By the Gronwall-Bellman inequality [3, Lemma 2.1], we get $y(t) \equiv 0$, $\forall t \in [0, 1]$.

This contradicts $\|y(t)\| = 1$. $\square$

Theorem 11. *Let (A1), (A2), and (H2) hold. For $v \in \{+, -\}$, assume that one of the following conditions holds:*

- (i) $r \in (0, \lambda_k^v/f_0)$ for $\lambda_k^v > 0$.
- (ii) $r \in (0, \lambda_k^+/f_0) \cup (\lambda_k^-/f_0, 0)$ for $v\lambda_k^v > 0$.
- (iii) $r \in (0, \lambda_k^-/f_0) \cup (\lambda_k^+/f_0, 0)$ for $v\lambda_k^v < 0$.
- (iv) $r \in (\lambda_k^v/f_0, 0)$ for $\lambda_k^v < 0$.

Then problem (4) possesses two solutions x_k^+ and x_k^- such that x_k^+ has exactly $k - 1$ zeros in $(0, 1)$ and is positive near 0 and x_k^- has exactly $k - 1$ zeros in $(0, 1)$ and is negative near 0.

Proof. We will only prove the case of (i) since the proofs of the cases for (ii), (iii), and (iv) are completely analogous.

Inspired by the idea of [23] or see [10], we define the cut-off function of f as follows:

$$f^{[n]}(s) := \begin{cases} f(s), & s \in [-n, n], \\ \frac{2n^2 - f(n)}{n}(s - n) + f(n), & s \in (n, 2n), \\ \frac{2n^2 + f(-n)}{n}(s + n) + f(-n), & s \in (-2n, -n), \\ ns, & s \in (-\infty, -2n] \cup [2n, +\infty). \end{cases} \quad (33)$$

We consider the following problem:

$$\begin{aligned} -x'' &= \alpha x^+ + \beta x^- + \lambda r a(t) f^{[n]}(x), \quad 0 < t < 1, \\ x(0) &= x(1) = 0. \end{aligned} \quad (34)$$

Clearly, we can see that $\lim_{n \rightarrow +\infty} f^{[n]}(s) = f(s)$, $(f^{[n]})_0 = f_0$, and $(f^{[n]})_\infty = n$.

Similar to the proof of [13, Theorem 4.1], by Lemma 8 and Remark 9, there exists an unbounded continuum $\mathcal{D}_k^{v[n]}$ of solutions of problem (34) emanating from $(\lambda_k^v/rf_0, 0)$, such that $\mathcal{D}_k^{v[n]} \subset (\Phi_k^v \cup \{(\lambda_k^v/rf_0, 0)\})$, and $\mathcal{D}_k^{v[n]}$ joins $(\lambda_k^v/rf_0, 0)$ to $(\lambda_k^v/rn, \infty)$.

Taking $z_n = (\lambda_k^v/rn, \infty)$ and $z^* = (0, \infty)$, we have that $z_n \rightarrow z^*$.

So condition (i) in Lemma 7 is satisfied with $z^* = (0, \infty)$. Obviously

$$r_n = \sup \{ \lambda + \|x\| \mid (\lambda, x) \in \mathcal{D}_k^{v[n]} \} = \infty, \quad (35)$$

and, accordingly (ii) in Lemma 7 holds. (iii) in Lemma 7 can be deduced directly from the Arzela-Ascoli Theorem and the definition of $f^{[n]}$.

Therefore, by Lemma 7, $\limsup_{n \rightarrow \infty} \mathcal{D}_k^{v[n]}$ contains an unbounded connected component $\mathcal{D}_k^v$ with $(0, \infty) \in \mathcal{D}_k^v$.

From $\lim_{n \rightarrow +\infty} f^{[n]}(s) = f(s)$, (34) can be converted to equivalent equation (22). Since $\mathcal{D}_k^{v[n]} \subset \Phi_k^v$, we conclude $\mathcal{D}_k^v \subset \Phi_k^v$. Moreover, $\mathcal{D}_k^v \subset \mathcal{S}_k^v$ by (4).

In the following, we show that $(\lambda_k^v/rf_0, 0) \in \mathcal{D}_k^v$.

Let $(\mu_n, x_n) \in \mathcal{D}_k^v \setminus \{(\lambda_k^v/rf_0, 0)\}$ satisfy $\lim_{n \rightarrow \infty} \|x_n\| = 0$. Let $\zeta \in C(\mathbb{R}, \mathbb{R})$ be such that

$$f(s) = f_0 s + \zeta(s). \quad (36)$$

With $\lim_{|s| \rightarrow 0} (\zeta(s)/s) = 0$, let $\bar{\zeta}(x) = \max_{0 \leq |s| \leq x} |\zeta(s)|$, then $\bar{\zeta}$ is nondecreasing and

$$\lim_{x \rightarrow \infty} \frac{\bar{\zeta}(x)}{x} = 0. \quad (37)$$

We divide the equation

$$\begin{aligned} -x_n'' &= \mu_n r a(t) f_0 x_n + \alpha x_n^+ + \beta x_n^- + \lambda r a(t) \zeta(x_n), \\ 0 &< t < 1, \end{aligned} \quad (38)$$

$$x_n(0) = x_n(1) = 0.$$

Let $y_n = x_n/\|x_n\|$, and y_n should be the solutions of problem

$$\begin{aligned} -y_n'' &= \mu_n r a(t) f_0 y_n + \alpha y_n^+ + \beta y_n^- + \mu_n r a(t) \frac{\zeta(x_n)}{\|x_n\|}, \\ t &\in (0, 1), \end{aligned} \quad (39)$$

$$y_n(0) = y_n(1) = 0.$$

Since y_n is bounded in $C^2[0, 1]$, choosing a subsequence and relabelling if necessary, we have that $y_n \rightarrow y$ for some $y \in E$.

Furthermore, from (37) and the fact that $\bar{\zeta}$ is nondecreasing, we have that

$$\lim_{n \rightarrow \infty} \frac{\zeta(x_n)}{\|x_n\|} = 0, \quad (40)$$

since

$$\begin{aligned} \frac{\zeta(x_n)}{\|x_n\|} &\leq \frac{\bar{\zeta}(\|x_n\|)}{\|x_n\|} \leq \frac{\bar{\zeta}(\|x_n\|_\infty)}{\|x_n\|} \leq \frac{\bar{\zeta}(\|x_n\|)}{\|x_n\|} \rightarrow 0, \\ n &\rightarrow +\infty. \end{aligned} \quad (41)$$

By (40) and the compactness of L^{-1} , we obtain that

$$Ly = \mu r a(t) f_0 y + \alpha y^+ + \beta y^-. \quad (42)$$

It is clear that $\|y\| = 1$ and $y \in \overline{\mathcal{D}_k^v} \subset \mathcal{D}_k^v$ since $\mathcal{D}_k^v$ is closed in $\mathbb{R} \times E$. Moreover, by Lemma 1, $\mu r f_0 = \lambda_k^v$, so that

$$\mu = \frac{\lambda_k^v}{r f_0}. \quad (43)$$

Thus $\mathcal{D}_k^v$ joins $(\lambda_k^v/rf_0, 0)$ to $(0, \infty)$. $\square$

Theorem 12. Let (A1), (A2), and (H3) hold. For $v \in \{+, -\}$, assume that one of the following conditions holds:

- (i) $r \in (0, \lambda_k^v/f_\infty)$ for $\lambda_k^v > 0$.
- (ii) $r \in (0, \lambda_k^+/f_\infty) \cup (\lambda_k^-/f_\infty, 0)$ for $v\lambda_k^v > 0$.
- (iii) $r \in (0, \lambda_k^-/f_\infty) \cup (\lambda_k^+/f_\infty, 0)$ for $v\lambda_k^v < 0$.
- (iv) $r \in (\lambda_k^v/f_\infty, 0)$ for $\lambda_k^v < 0$.

Then problem (4) possesses two solutions x_k^+ and x_k^- such that x_k^+ has exactly $k-1$ zeros in $(0, 1)$ and is positive near 0 and x_k^- has exactly $k-1$ zeros in $(0, 1)$ and is negative near 0.

Proof. Define

$$f^{[n]}(s) := \begin{cases} ns, & s \in \left[-\frac{1}{n}, \frac{1}{n}\right], \\ \left[f\left(\frac{2}{n}\right) - 1\right](ns - 2) + f\left(\frac{2}{n}\right), & s \in \left(\frac{1}{n}, \frac{2}{n}\right), \\ -\left[f\left(-\frac{2}{n}\right) + 1\right](ns + 2) + f\left(-\frac{2}{n}\right), & s \in \left(-\frac{2}{n}, -\frac{1}{n}\right), \\ f(s), & s \in \left(-\infty, -\frac{2}{n}\right] \cup \left[\frac{2}{n}, +\infty\right). \end{cases} \quad (44)$$

We consider the following problem:

$$\begin{aligned} -x'' &= \alpha x^+ + \beta x^- + \lambda r a(t) f^{[n]}(x), \quad 0 < t < 1, \\ x(0) &= x(1) = 0. \end{aligned} \quad (45)$$

Clearly, we can see that $\lim_{n \rightarrow +\infty} f^{[n]}(s) = f(s)$, $(f^{[n]})_0 = n$, and $(f^{[n]})_\infty = f_\infty$.

Similar to the proof of [13, Theorem 4.1], by Lemma 8 and Remark 9, there exists an unbounded continuum $\mathcal{D}_k^{v[n]}$ of solutions of problem (45) emanating from $(\lambda_k^v/rn, 0)$, such that $\mathcal{D}_k^{v[n]} \subset (\Phi_k^v \cup \{(\lambda_k^v/rn, 0)\})$, and $\mathcal{D}_k^{v[n]}$ joins $(\lambda_k^v/rn, 0)$ to $(\lambda_k^v/rf_\infty, \infty)$.

Taking $z_n = (\lambda_k^v/rn, 0)$ and $z^* = (0, 0)$, we have that $z_n \rightarrow z^*$.

So condition (i) in Lemma 7 is satisfied with $z^* = (0, 0)$.

Obviously

$$r_n = \sup \{ \lambda + \|x\| \mid (\lambda, x) \in \mathcal{D}_k^{v[n]} \} = \infty, \quad (46)$$

and accordingly (ii) in Lemma 7 holds. (iii) in Lemma 7 can be deduced directly from the Arzela-Ascoli Theorem and the definition of $f^{[n]}$.

Therefore, by Lemma 7, $\limsup_{n \rightarrow \infty} \mathcal{D}_k^{v[n]}$ contains an unbounded connected component $\mathcal{D}_k^v$ with $(0, 0) \in \mathcal{D}_k^v$.

Similar to Theorem 11, we obtain $\mathcal{D}_k^v \subset \mathcal{S}_k^v$.

Similar to the method of the proof of Case 2 of [13, Theorem 4.1], we can obtain that $(\lambda_k^v/rf_\infty, \infty) \in \mathcal{D}_k^v$. $\square$

Theorem 13. Let (A1), (A2), and (H4) hold. For $v \in \{+, -\}$, assume that one of the following conditions holds:

- (i) $r \in (0, +\infty)$ for $\lambda_k^v > 0$.
- (ii) $r \in (0, +\infty) \cup (-\infty, 0)$ for $v\lambda_k^v > 0$ or $v\lambda_k^v < 0$.
- (iii) $r \in (-\infty, 0)$ for $\lambda_k^v < 0$.

Then problem (4) possesses two solutions x_k^+ and x_k^- such that x_k^+ has exactly $k - 1$ zeros in $(0, 1)$ and is positive near 0 and x_k^- has exactly $k - 1$ zeros in $(0, 1)$ and is negative near 0.

Proof. Define

$$f^{[n]}(s) := \begin{cases} \frac{1}{n}s, & s \in (-\infty, -2n] \cup [2n, +\infty), \\ \frac{2 + f(-n)}{n}(s + n) + f(-n), & s \in (-2n, -n), \\ \frac{2 - f(n)}{n}(s - n) + f(n), & s \in (n, 2n), \\ f(s), & s \in \left[-n, -\frac{2}{n}\right] \cup \left[\frac{2}{n}, n\right], \\ -\left[f\left(-\frac{2}{n}\right) + 1\right](ns + 2) + f\left(-\frac{2}{n}\right), & s \in \left(-\frac{2}{n}, -\frac{1}{n}\right), \\ \left[f\left(\frac{2}{n}\right) - 1\right](ns - 2) + f\left(\frac{2}{n}\right), & s \in \left(\frac{1}{n}, \frac{2}{n}\right), \\ ns, & s \in \left[-\frac{1}{n}, \frac{1}{n}\right]. \end{cases} \quad (47)$$

We consider the following problem:

$$\begin{aligned} -x'' &= \alpha x^+ + \beta x^- + \lambda r a(t) f^{[n]}(x), \quad 0 < t < 1, \\ x(0) &= x(1) = 0. \end{aligned} \quad (48)$$

Clearly, we can see that $\lim_{n \rightarrow +\infty} f^{[n]}(s) = f(s)$, $(f^{[n]})_0 = n$, and $(f^{[n]})_\infty = 1/n$.

Applying a similar method used in the proof of Theorem 12, we obtain an unbounded connected component $\mathcal{D}_k^v \subset \mathcal{S}_k^v$ with $(0, 0) \in \mathcal{D}_k^v$.

Similar to the method of the proof of Theorem 10, we can obtain that $(\infty, \infty) \in \mathcal{D}_k^\nu$. It follows that the result is obtained. $\square$

Theorem 14. Let (A1), (A2), and (H5) hold. For $\nu \in \{+, -\}$, assume that one of the following conditions holds:

- (i) There exists $\lambda_{\nu k}^\nu > 0$ for $\lambda_k^\nu > 0$, such that $r \in (\lambda_{\nu k}^\nu, +\infty)$.
- (ii) There exists $\nu\lambda_{\nu k}^\nu > 0$ for $\nu\lambda_k^\nu > 0$, such that $r \in (-\infty, \lambda_{-k}^-) \cup (\lambda_{+k}^+, +\infty)$.

(iii) There exists $\nu\lambda_{\nu k}^\nu < 0$ for $\nu\lambda_k^\nu < 0$, such that $r \in (-\infty, \lambda_{+k}^+) \cup (\lambda_{-k}^-, +\infty)$.

(iv) There exists $\lambda_{\nu k}^\nu < 0$ for $\lambda_k^\nu < 0$, such that $r \in (-\infty, \lambda_{\nu k}^\nu)$.

Then problem (4) possesses two solutions x_k^+ and x_k^- such that x_k^+ has exactly $k-1$ zeros in $(0, 1)$ and is positive near 0 and x_k^- has exactly $k-1$ zeros in $(0, 1)$ and is negative near 0.

Proof. Define

$$f^{[n]}(s) := \begin{cases} ns, & s \in (-\infty, -2n] \cup [2n, +\infty), \\ \frac{2n^2 + f(-n)}{n}(s+n) + f(-n), & s \in (-2n, -n), \\ \frac{2n^2 - f(n)}{n}(s-n) + f(n), & s \in (n, 2n), \\ f(s), & s \in \left[-n, -\frac{2}{n}\right] \cup \left[\frac{2}{n}, n\right], \\ -\left[f\left(-\frac{2}{n}\right) + 1\right](ns+2) + f\left(-\frac{2}{n}\right), & s \in \left(-\frac{2}{n}, -\frac{1}{n}\right), \\ \left[f\left(\frac{2}{n}\right) - 1\right](ns-2) + f\left(\frac{2}{n}\right), & s \in \left(\frac{1}{n}, \frac{2}{n}\right), \\ ns, & s \in \left[-\frac{1}{n}, \frac{1}{n}\right]. \end{cases} \quad (49)$$

We consider the following problem:

$$\begin{aligned} -x'' &= \alpha x^+ + \beta x^- + \lambda r a(t) f^{[n]}(x), \quad 0 < t < 1, \\ x(0) &= x(1) = 0. \end{aligned} \quad (50)$$

Clearly, we can see that $\lim_{n \rightarrow +\infty} f^{[n]}(s) = f(s)$, $(f^{[n]})_0 = n$, and $(f^{[n]})_\infty = n$.

Applying a similar method used in the proof of Theorem 12, we obtain an unbounded connected component $\mathcal{D}_k^\nu \subset \mathbb{R} \times \Sigma_k^\nu$ with $(0, 0) \in \mathcal{D}_k^\nu$.

In the following, we can show that $(0, \infty) \in \mathcal{D}_k^\nu$.

Assume that there exists a sequence $(\mu_n, x_n) \in \mathcal{D}_k^\nu$ such that

$$\begin{aligned} \lim_{n \rightarrow \infty} (\|x_n\| + |\mu_n|) &= +\infty, \\ -x_n'' &= \alpha x_n^+ + \beta x_n^- + \mu_n r a(t) f(x_n), \\ 0 < t < 1, \end{aligned} \quad (51)$$

$$x_n(0) = x_n(1) = 0.$$

We divide the proof into two steps.

Step 1. We show that $\lim_{n \rightarrow \infty} \|x_n\| = +\infty$.

If $\{x_n\}$ is bounded, then there exists a constant M_1 not depending on n such that $\|x_n\| \leq M_1$. It follows that

$$\lim_{n \rightarrow \infty} |\mu_n| = +\infty. \quad (52)$$

Let

$$\begin{aligned} 0 &= \tau(0, n) < \tau(1, n) < \tau(2, n) < \dots < \tau(k-1, n) \\ &< \tau(k, n) = 1 \end{aligned} \quad (53)$$

denote the simple zeros of x_n in $(0, 1)$. Then, after choosing a subsequence and relabelling if necessary,

$$\lim_{n \rightarrow +\infty} \tau(l, n) = \tau(l, \infty), \quad l \in \{0, 1, \dots, k\}. \quad (54)$$

We claim that there exists $l_0 \in \{0, 1, \dots, k\}$ such that

$$\tau(l_0, \infty) < \tau(l_0 + 1, \infty). \quad (55)$$

Otherwise, we have

$$\begin{aligned} 1 &= \sum_{l=0}^{k-1} (\tau(l+1, n) - \tau(l, n)) \\ &\longrightarrow \sum_{l=0}^{k-1} (\tau(l+1, \infty) - \tau(l, \infty)) = 0. \end{aligned} \quad (56)$$

This is a contradiction.

Let $[c, d] \subset (\tau(l_0, \infty), \tau(l_0 + 1, \infty))$, with $c < d$. For all n sufficiently large, we have $[c, d] \subset (\tau(l_0, n), \tau(l_0 + 1, n))$. So $x_n(t)$ does not change its sign in $[c, d]$. Without loss of generality, we assume that $x_n(t) > 0$, $t \in [c, d]$. Moreover, we have $\max_{t \in [c, d]} |x_n(t)| \leq M_1$. Combining this with the fact

$$\frac{f(x_n)}{x_n(t)} \geq \inf \left\{ \frac{f(s)}{s} \mid 0 < s \leq M_1 \right\} > 0, \quad t \in [c, d] \quad (57)$$

and with the relation

$$-x_n'' = \alpha x_n^+ + \beta x_n^- + \mu_n r a(t) \frac{f(x_n)}{x_n(t)} x_n(t), \quad t \in [c, d]. \quad (58)$$

Furthermore, we obtain that $\lim_{n \rightarrow +\infty} \mu_n r a(t) (f(x_n(t))/x_n(t)) = \pm\infty$ for any $t \in [c, d]$. Let φ_1 be the corresponding eigenfunction of λ_1^+ of (1). If $\lim_{n \rightarrow +\infty} \mu_n r a(t) (f(x_n(t))/x_n(t)) = -\infty$ for any $t \in [c, d]$, applying Lemma 4 to y_n and φ_1 on $[c, d]$, we can get that φ_1 must change its sign in $[c, d]$ for n large enough. While, this is impossible. So we have that $\lim_{n \rightarrow +\infty} \mu_n r a(t) (f(x_n(t))/x_n(t)) = +\infty$ for any $t \in [c, d]$. Applying Lemma 5 to $x_n(t)$ on $[c, d]$, we can get that $x_n(t)$ must change its sign in $[c, d]$ for n large enough. This is a contradiction.

Step 2. We show that $\lim_{n \rightarrow \infty} |\mu_n| = 0$.

Assume on the contrary that $|\mu_n| > c_0$. We consider the problem

$$-x_n'' = \alpha x_n(t) + \mu_n r a(t) \frac{f(x_n)}{x_n(t)} x_n(t), \quad t \in [c, d]. \quad (59)$$

$$f^{[n]}(s) := \begin{cases} \frac{1}{n}s, & s \in \left[-\frac{1}{n}, \frac{1}{n}\right], \\ \left[f\left(\frac{2}{n}\right) - \frac{1}{n^2}\right](ns - 2) + f\left(\frac{2}{n}\right), & s \in \left(\frac{1}{n}, \frac{2}{n}\right), \\ -\left[f\left(-\frac{2}{n}\right) + \frac{1}{n^2}\right](ns + 2) + f\left(-\frac{2}{n}\right), & s \in \left(-\frac{2}{n}, -\frac{1}{n}\right), \\ f(s), & s \in \left(-\infty, -\frac{2}{n}\right] \cup \left[\frac{2}{n}, +\infty\right). \end{cases} \quad (63)$$

We consider the following problem:

$$\begin{aligned} -x'' &= \alpha x^+ + \beta x^- + \lambda r a(t) f^{[n]}(x), \quad 0 < t < 1, \\ x(0) &= x(1) = 0. \end{aligned} \quad (64)$$

Clearly, we can see that $\lim_{n \rightarrow +\infty} f^{[n]}(s) = f(s)$, $(f^{[n]})_0 = 1/n$, and $(f^{[n]})_\infty = f_\infty$.

By Lemma 2.1 of [7], we have

$$\lim_{n \rightarrow \infty} \min \{|x_n(t)| \mid t \in [c, d]\} = \infty. \quad (60)$$

We have from (60) that we may assume, after taking a subsequence if necessary, that

$$\begin{aligned} \lim_{n \rightarrow \infty} \min \{x_n(t) \mid t \in [c, d]\} &= +\infty \quad (x_n(t) > 0) \text{ or} \\ \lim_{n \rightarrow \infty} \max \{x_n(t) \mid t \in [c, d]\} &= -\infty \quad (x_n(t) < 0). \end{aligned} \quad (61)$$

From (A1) and (61) and the condition $f_\infty = \infty$, we have that

$$\lim_{n \rightarrow +\infty} \mu_n r a(t) \frac{f(x_n(t))}{x_n(t)} = \pm\infty, \quad (62)$$

uniformly for $t \in [c, d]$.

Similar to the proof of Step 1, we can get a contradiction. $\square$

Theorem 15. Let (A1), (A2), and (H6) hold. For $\nu \in \{+, -\}$, assume that one of the following conditions holds:

- (i) $r \in (\lambda_k^\nu / f_\infty, +\infty)$ for $\lambda_k^\nu > 0$.
- (ii) $r \in (\lambda_k^+ / f_\infty, +\infty) \cup (-\infty, \lambda_k^- / f_\infty)$ for $\nu \lambda_k^\nu > 0$.
- (iii) $r \in (\lambda_k^- / f_\infty, +\infty) \cup (-\infty, \lambda_k^+ / f_\infty)$ for $\nu \lambda_k^\nu < 0$.
- (iv) $r \in (-\infty, \lambda_k^\nu / f_\infty)$ for $\lambda_k^\nu < 0$.

Then problem (4) possesses two solutions x_k^+ and x_k^- such that x_k^+ has exactly $k - 1$ zeros in $(0, 1)$ and is positive near 0 and x_k^- has exactly $k - 1$ zeros in $(0, 1)$ and is negative near 0.

Proof. Define

Similar to the proof of [13, Theorem 4.1], by Lemma 8 and Remark 9, there exists an unbounded continuum $\mathcal{D}_k^{[n]}$ of solutions of problem (64) emanating from $(n\lambda^\nu/r, 0)$, such that $\mathcal{D}_k^{[n]} \subset (\Phi_k^\nu \cup \{(n\lambda^\nu/r, 0)\})$, and $\mathcal{D}_k^{[n]}$ joins $(n\lambda^\nu/r, 0)$ to $(\lambda^\nu/r f_\infty, \infty)$.

Taking $z_n = (n\lambda^\nu/r, 0)$ and $z^* = (\infty, 0)$, we have that $z_n \rightarrow z^*$.

So condition (i) in Lemma 7 is satisfied with $z^* = (\infty, 0)$.

Obviously

$$r_n = \sup \{\lambda + \|x\| \mid (\lambda, x) \in \mathcal{D}_k^{[n]}\} = \infty, \quad (65)$$

and accordingly (ii) in Lemma 7 holds. (iii) in Lemma 7 can be deduced directly from the Arzela-Ascoli Theorem and the definition of $f^{[n]}$.

Therefore, by Lemma 7, $\limsup_{n \rightarrow \infty} \mathcal{D}_k^{[n]}$ contains an unbounded connected component $\mathcal{D}_k^\nu$ with $(\infty, 0) \in \mathcal{D}_k^\nu$.

Similar to Theorem 11, we obtain $\mathcal{D}_k^\nu \subset \mathcal{S}_k^\nu$.

Similar to the method of the proof of Case 2 of [13, Theorem 4.1], we can obtain that $(\lambda_k^\nu / r f_\infty, \infty) \in \mathcal{D}_k^\nu$. $\square$

Theorem 16. Let (A1), (A2), and (H7) hold. For $\nu \in \{+, -\}$, assume that one of the following conditions holds:

- (i) There exists $\lambda_{\nu k}^\nu > 0$ for $\lambda_k^\nu > 0$, such that $r \in (\lambda_{\nu k}^\nu, +\infty)$.

$$f^{[n]}(s) := \begin{cases} \frac{1}{n}s, & s \in (-\infty, -2n] \cup [2n, +\infty), \\ \frac{2 + f(-n)}{n}(s + n) + f(-n), & s \in (-2n, -n), \\ \frac{2 - f(n)}{n}(s - n) + f(n), & s \in (n, 2n), \\ f(s), & s \in \left[-n, -\frac{2}{n}\right] \cup \left[\frac{2}{n}, n\right], \\ -\left[f\left(-\frac{2}{n}\right) + \frac{1}{n^2}\right](ns + 2) + f\left(-\frac{2}{n}\right), & s \in \left(-\frac{2}{n}, -\frac{1}{n}\right), \\ \left[f\left(\frac{2}{n}\right) - \frac{1}{n^2}\right](ns - 2) + f\left(\frac{2}{n}\right), & s \in \left(\frac{1}{n}, \frac{2}{n}\right), \\ \frac{1}{n}s, & s \in \left[-\frac{1}{n}, \frac{1}{n}\right]. \end{cases} \quad (66)$$

We consider the following problem:

$$\begin{aligned} -x'' &= \alpha x^+ + \beta x^- + \lambda r a(t) f^{[n]}(x), \quad 0 < t < 1, \\ x(0) &= x(1) = 0. \end{aligned} \quad (67)$$

Clearly, we can see that $\lim_{n \rightarrow +\infty} f^{[n]}(s) = f(s)$, $(f^{[n]})_0 = 1/n$, and $(f^{[n]})_\infty = 1/n$.

Applying a similar method used in the proof of Theorem 15, we obtain an unbounded connected component $\mathcal{D}_k^\nu \subset \mathcal{S}_k^\nu$ with $(\infty, 0) \in \mathcal{D}_k^\nu$.

Similar to the proof of Theorem 10, we can show that $(\infty, \infty) \in \mathcal{D}_k^\nu$. $\square$

$$f^{[n]}(s) := \begin{cases} ns, & s \in (-\infty, -2n] \cup [2n, +\infty), \\ \frac{2n^2 + f(-n)}{n}(s + n) + f(-n), & s \in (-2n, -n), \\ \frac{2n^2 - f(n)}{n}(s - n) + f(n), & s \in (n, 2n), \\ f(s), & s \in \left[-n, -\frac{2}{n}\right] \cup \left[\frac{2}{n}, n\right], \\ -\left[f\left(-\frac{2}{n}\right) + \frac{1}{n^2}\right](ns + 2) + f\left(-\frac{2}{n}\right), & s \in \left(-\frac{2}{n}, -\frac{1}{n}\right), \\ \left[f\left(\frac{2}{n}\right) - \frac{1}{n^2}\right](ns - 2) + f\left(\frac{2}{n}\right), & s \in \left(\frac{1}{n}, \frac{2}{n}\right), \\ \frac{1}{n}s, & s \in \left[-\frac{1}{n}, \frac{1}{n}\right]. \end{cases} \quad (68)$$

- (ii) There exists $\nu \lambda_{\nu k}^\nu > 0$ for $\nu \lambda_k^\nu > 0$, such that $r \in (-\infty, \lambda_{\nu k}^\nu) \cup (\lambda_{\nu k}^\nu, +\infty)$.

- (iii) There exists $\nu \lambda_{\nu k}^\nu < 0$ for $\nu \lambda_k^\nu < 0$, such that $r \in (-\infty, \lambda_{\nu k}^\nu) \cup (\lambda_{\nu k}^\nu, +\infty)$.

- (iv) There exists $\lambda_{\nu k}^\nu < 0$ for $\lambda_k^\nu < 0$, such that $r \in (-\infty, \lambda_{\nu k}^\nu)$.

Then problem (4) possesses two solutions x_k^+ and x_k^- such that x_k^+ has exactly $k - 1$ zeros in $(0, 1)$ and is positive near 0 and x_k^- has exactly $k - 1$ zeros in $(0, 1)$ and is negative near 0.

Proof. Define

Theorem 17. Let (A1), (A2), and (H8) hold. For $\nu \in \{+, -\}$, assume that one of the following conditions holds:

- (i) $r \in (0, +\infty)$ for $\lambda_k^\nu > 0$.
(ii) $r \in (0, +\infty) \cup (-\infty, 0)$ for $\nu \lambda_k^\nu > 0$ or $\nu \lambda_k^\nu < 0$.
(iii) $r \in (-\infty, 0)$ for $\lambda_k^\nu < 0$.

Then problem (4) possesses two solutions x_k^+ and x_k^- such that x_k^+ has exactly $k - 1$ zeros in $(0, 1)$ and is positive near 0 and x_k^- has exactly $k - 1$ zeros in $(0, 1)$ and is negative near 0.

Proof. Define

We consider the following problem:

$$\begin{aligned} -x'' &= \alpha x^+ + \beta x^- + \lambda r a(t) f^{[n]}(x), \quad 0 < t < 1, \\ x(0) &= x(1) = 0. \end{aligned} \quad (69)$$

Clearly, we can see that $\lim_{n \rightarrow +\infty} f^{[n]}(s) = f(s)$, $(f^{[n]})_0 = n$, and $(f^{[n]})_\infty = 1/n$.

Applying a similar method used in the proof of Theorem 15, we obtain an unbounded connected component $\mathcal{D}_k^v \subset \mathcal{S}_k^v$ with $(\infty, 0) \in \mathcal{D}_k^v$.

Similar to the proof of Theorem 14, we can show that $(0, \infty) \in \mathcal{D}_k^v$. $\square$

Remark 18. Clearly, if $\alpha = \beta = 0$, [6, 7] studied the problem (4) under conditions (A2) and (H4) or (H8). Hence, the results of Theorems 13 and 17 improve on Theorems 1.1 and 1.2 of [6] or [7].

Remark 19. Note that if $\alpha = \beta \equiv 0$ the result of Theorems 10–17 is equivalent to those of Theorems 2.1–2.8 of [10] (when $p = 2$), respectively. Hence, Theorems 10–17 extend the corresponding results of [10] (when $p = 2$).

Remark 20. The nonlinear term of (4) is not necessarily homogeneous linearizable at the origin and infinity because of the influence of the term $\alpha x^+ + \beta x^-$. Clearly, so the bifurcation results of [1, 2, 8–12] cannot be applied directly to obtain our results.

Remark 21. We consider the cases of $f_0, f_\infty \notin (0, \infty)$, while the authors of [13] only studied the cases of $f_0, f_\infty \in (0, \infty)$. Hence, Theorems 10–17 extend Theorem 4.1 of [13].

Remark 22. Previous cases have been investigated by several authors by applying also different methods. In particular, under the conditions $\alpha = \beta = 0$ and (H4) or (H8), we can see [24–29].

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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