Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2016, Article ID 3067872, 9 pages
http://dx.doi.org/10.1155/2016/3067872

Research Article

Dynamical Analysis of a Computer Virus Propagation Model with Delay and Infectivity in Latent Period

Zizhen Zhang and Dianjie Bi

Anhui University of Finance and Economics, School of Management Science and Engineering, Caoshan Street 962, Bengbu 233030, China

Correspondence should be addressed to Zizhen Zhang; zzzhaida@163.com

Received 25 February 2016; Revised 8 May 2016; Accepted 23 May 2016

Academic Editor: Kousuke Kuto

A delayed SLB computer virus propagation model with infectivity in latent period is proposed in this paper. We establish sufficient conditions for local stability of the positive equilibrium and existence of Hopf bifurcation by analyzing distribution of the roots of the associated characteristic equation and applying the Hopf bifurcation theorem. Furthermore, properties of the Hopf bifurcation are determined by using the normal form theory and the center manifold theorem. Finally, numerical simulations supporting the theoretical analysis are carried out.

1. Introduction

Due to their striking features such as destruction, polymorphism, and unpredictability [1–3], computer viruses, ranging from conventional viruses to network worms and other malicious codes such as Trojans and spyware, have addressed a serious threat to confidentiality, integrity, and availability of computer resources on the Internet. They have also come to be one major threat to our work and daily life.

In recent years, people have paid attention to the necessity of monitoring computer viruses in the Internet and proposed many mathematical models to depict the spread of computer viruses in the Internet based on the classic epidemic models [4–9]. In [4], Li et al. proposed a worm propagation model with removable storage device and studied stability of the model. Mishra and Keshri proposed a SEIRS-V model to describe the dynamics of worm propagation in wireless sensor networks [6]. However, most of the above models assume that only breaking-out computers have infectivity. This is not consistent with reality. Because computer viruses have a feature of latency, an infected computer which is in latency can also infect other computers through file copying or file downloading. Based on this consideration, in [8], Yang et al. proposed the following computer virus model with graded cure rate:

$$\begin{aligned}\frac{dS(t)}{dt} &= \delta - \beta S(t)(L(t)+B(t)) + \gamma_1 L(t) + \gamma_2 B(t) \\ &\quad - \delta S(t), \\ \frac{dL(t)}{dt} &= \beta S(t)(L(t)+B(t)) - \gamma_1 L(t) - \alpha L(t) \\ &\quad - \delta L(t), \\ \frac{dB(t)}{dt} &= \alpha L(t) - \gamma_2 B(t) - \delta B(t),\end{aligned} \tag{1}$$

where $S(t)$ is the percentage of uninfected computers in the Internet, at time t. $L(t)$ and $B(t)$ are the percentages of latent and breaking-out computers in the Internet, at time t, respectively. δ is the connected rate of external computers to the Internet and it is also the disconnected rate of internal computers from the Internet. β is the infected rate of the uninfected computers. α is the breaking-out rate of the latent computers. γ_1 and γ_2 are the cured rates of the latent and breaking-out computers, respectively. Obviously, it is assumed that both seizing and latent computers have infectivity.

However, system (1) ignores time delay due to the period that a computer user uses to cure the latent and the breaking-out computers by using antivirus software or reinstalling system. As is known, time delay can make a dynamical system lose its stability and then lead to the occurrence of a Hopf bifurcation. Hopf bifurcation of dynamical systems with delay has been investigated by many authors [10–16]. In [10], Zhuang and Zhu analyzed the existence of Hopf bifurcation for an improved HIV model with time delay and cure rate by regarding the time lag from infection of cells to the cells becoming actively infected as a bifurcation parameter. In [11], Liu investigated the existence and properties of the Hopf bifurcation in a delayed SEIQRS model for the transmission of malicious objects in computer network. In [14], Bianca et al. studied the bifurcation of a delayed-energy-based model of capital accumulation. Motivated by the work above, we introduce the time delay due to the period that a computer user uses to cure the latent and breaking-out computers into system (1) and get the following delayed computer virus model:

$$\begin{aligned} \frac{dS(t)}{dt} &= \delta - \beta S(t)(L(t)+B(t)) + \gamma_1 L(t-\tau) \\ &\quad + \gamma_2 B(t-\tau) - \delta S(t), \\ \frac{dL(t)}{dt} &= \beta S(t)(L(t)+B(t)) - \gamma_1 L(t) - \alpha L(t) \\ &\quad - \delta L(t), \\ \frac{dB(t)}{dt} &= \alpha L(t) - \gamma_2 B(t) - \delta B(t), \end{aligned} \tag{2}$$

where τ is the delay due to the period that a computer user uses to cure the latent and breaking-out computers by using antivirus software or reinstalling system.

The remainder of this paper is organized as follows. Local stability and existence of Hopf bifurcation for system (2) are analyzed in Section 2. Properties of the Hopf bifurcation are investigated in Section 3 by means of the center manifold theorem and the normal form theory. A numerical example is presented in Section 4 to illustrate the theoretical results. This work is summarized in Section 5.

2. Existence of Hopf Bifurcation

According to the analysis in [8], we can conclude that if the basic reproduction number $R_0 = \beta(\alpha+\gamma_2+\delta)/(\gamma_2+\delta)(\alpha+\gamma_1+\delta) > 1$, then system (2) has a unique positive equilibrium $E_+(S_*, L_*, B_*)$, where

$$\begin{aligned} S_* &= \frac{(\alpha+\gamma_1+\delta)L_*}{\beta(L_*+B_*)}, \\ L_* &= \frac{(\gamma_2+\delta)(1-1/R_0)}{\alpha+\gamma_2+\delta}, \\ B_* &= \frac{\alpha(1-1/R_0)}{\alpha+\gamma_2+\delta}. \end{aligned} \tag{3}$$

For the linearized system of system (2) at the positive equilibrium $E_+(S_*, L_*, B_*)$, the corresponding characteristic equation is

$$\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0 + (B_1\lambda + B_0)e^{-\lambda\tau} = 0, \tag{4}$$

where

$$\begin{aligned} A_0 &= a_{11}(a_{23}a_{32} - a_{22}a_{33}) + a_{21}(a_{12}a_{33} - a_{13}a_{32}), \\ A_1 &= a_{11}(a_{22}+a_{33}) + a_{22}a_{33} - a_{12}a_{21} - a_{23}a_{32}, \\ A_2 &= -(a_{11}+a_{22}+a_{33}), \\ B_0 &= a_{21}(a_{33}b_{12} - a_{32}b_{13}), \\ B_1 &= -a_{21}b_{12}, \\ a_{11} &= -\beta(L_*+B_*) - \delta, \\ a_{12} &= -\beta S_*, \\ a_{13} &= -\beta S_*, \\ a_{21} &= \beta(L_*+B_*), \\ a_{22} &= \beta S_* - (\gamma_1+\alpha+\delta)L_*, \\ a_{23} &= \beta S_*, \\ a_{32} &= \alpha, \\ a_{33} &= -(\gamma_2+\delta), \\ b_{12} &= \gamma_1, \\ b_{13} &= \gamma_2. \end{aligned} \tag{5}$$

When $\tau = 0$, (4) becomes

$$\lambda^3 + A_{2*}\lambda^2 + A_{1*}\lambda + A_{0*} = 0, \tag{6}$$

where

$$\begin{aligned} A_{0*} &= B_0 + C_0, \\ A_{1*} &= A_1 + B_1, \\ A_{2*} &= A_2. \end{aligned} \tag{7}$$

Obviously, if condition (H_1), $A_{2*} > 0$, $A_{2*}A_{1*} > A_{0*} > 0$, holds, then system (2) is locally asymptotically stable when $\tau = 0$. Next, we choose the time delay τ as the parameter to consider the local stability of the positive equilibrium $E_+(S_*, L_*, B_*)$ and the Hopf bifurcation of system (2).

For $\tau > 0$, let $\lambda = i\omega$ $(\omega > 0)$ be the root of (4); then

$$\begin{aligned} B_1\omega\cos\tau\omega - B_0\sin\tau\omega &= \omega^3 - A_1\omega, \\ B_1\omega\sin\tau\omega + B_0\cos\tau\omega &= A_2\omega^2 - A_0, \end{aligned} \tag{8}$$

from which we can get the following sixth-degree equation for ω:

$$\omega^6 + c_2\omega^4 + c_1\omega^2 + c_0 = 0, \tag{9}$$

where

$$\begin{aligned} c_2 &= A_2^2 - 2A_1, \\ c_1 &= A_1^2 - 2A_0A_2 - B_1^2, \\ c_0 &= A_0^2 - B_0^2. \end{aligned} \tag{10}$$

Let $\omega^2 = v$; then (9) becomes

$$v^3 + c_2v^2 + c_1v + c_0 = 0. \tag{11}$$

From the analysis above, we know that if the coefficients of system (2) are given, one can easily get the coefficients of (11) by MATLAB software package. Correspondingly, the roots of (11) can be obtained. Thus, we make the following assumption in order to give the main results in this paper.

(H_2) Equation (11) has at least one positive root.

If condition (H_2) holds, then there exists one positive root v_0 of (11) such that (4) has a pair of purely imaginary roots $\pm i\omega_0 = \pm i\sqrt{v_0}$. For ω_0, the corresponding critical value of delay is

$$\tau_0 = \frac{1}{\omega_0}\arccos\frac{B_1\omega_0^4 + (A_2B_0 - A_1B_1)\,\omega_0^2 - A_0B_0}{B_0^2 + B_1^2\omega_0^2}. \tag{12}$$

And then, we will verify the transversality condition. From (4), we can obtain

$$\begin{aligned} \left[\frac{d\lambda}{d\tau}\right]^{-1} &= -\frac{3\lambda^2 + 2A_2\lambda + A_1}{\lambda\left(\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0\right)} \\ &\quad + \frac{B_1}{\lambda\left(B_1\lambda + B_0\right)} - \frac{\tau}{\lambda}. \end{aligned} \tag{13}$$

This gives

$$\operatorname{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_0} = \frac{f'(v_*)}{B_0^2 + B_1^2\omega_0^2}, \tag{14}$$

where $f(v) = v^3 + c_2v^2 + c_1v + c_0$ and $v_* = \omega_0^2$.

It is clear that if condition (H_3) $f'(v_*) \neq 0$ holds, then $\operatorname{Re}[d\lambda/d\tau]^{-1}_{\tau=\tau_0} \neq 0$. According to the Hopf bifurcation theorem in [15], we have the following results for system (2).

Theorem 1. *If conditions (H_1)–(H_3) hold, then the positive equilibrium $E_+(S_*, L_*, B_*)$ of system (2) is locally asymptotically stable for $\tau \in [0, \tau_0)$ and system (2) undergoes a Hopf bifurcation at the positive equilibrium $E_+(S_*, L_*, B_*)$ when $\tau = \tau_0$ and a family of periodic solutions bifurcate from the positive equilibrium $E_+(S_*, L_*, B_*)$ near $\tau = \tau_0$.*

3. Properties of the Hopf Bifurcation

In the previous section, we have obtained the conditions under which a Hopf bifurcation occurs and a family of periodic solutions bifurcate from the positive equilibrium $E_+(S_*, L_*, B_*)$ of system (2) when the delay passes through the critical value τ_0. In this section, we will derive the explicit formulae determining the direction of the Hopf bifurcation and the stability of bifurcating periodic solution of system (2) at $\tau = \tau_0$. The approach employed here is the normal form method and center manifold theorem introduced by Hassard et al. in [17].

Let $\tau = \tau_0 + \mu$, $u_1 = S(\tau t)$, $u_2 = L(\tau t)$, $u_3 = B(\tau t)$, $\mu \in R$, $L_\mu : C \to R^2$, and $F : R \times C \to R^3$, so that system (2) is transformed into a functional differential equation in $C = C([-1, 0], R^3)$ as

$$\dot{u}(t) = L_\mu(u_t) + F(\mu, u_t), \tag{15}$$

where

$$\begin{aligned} L_\mu\varphi &= (\tau_0 + \mu)\left[A_*\varphi(0) + B_*\varphi(-1)\right], \\ F_1(\varphi, \mu) &= -\beta(\tau_0 + \mu)\left[\varphi_1(0)\varphi_2(0) + \varphi_1(0)\varphi_3(0)\right], \\ F_2(\varphi, \mu) &= \beta(\tau_0 + \mu)\left[\varphi_1(0)\varphi_2(0) + \varphi_1(0)\varphi_3(0)\right], \\ F_3(\varphi, \mu) &= 0 \end{aligned} \tag{16}$$

with

$$A_* = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{22} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}, \qquad B_* = \begin{bmatrix} 0 & b_{12} & b_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{17}$$

Then, L_μ is a bounded linear operator in $C = C([-1, 0], R^3)$. By the Riesz representation theorem, there exists a function $\eta(\theta, \mu)$ of bounded variation for $\theta \in [-1, 0]$ such that

$$L_\mu\phi = \int_{-1}^{0} d\eta(\theta, \mu)\,\phi(\theta), \quad \phi \in C\left([-1, 0], R^3\right). \tag{18}$$

In fact, we choose

$$\eta(\theta, \mu) = (\tau_0 + \mu)\left(A_*\delta(\theta) + B_*\delta(\theta + 1)\right), \tag{19}$$

where $\delta(\theta)$ is the Dirac delta function.

For $\phi \in C([-1, 0], R^3)$, we define

$$\begin{aligned} A(\mu)\phi &= \begin{cases} \dfrac{d\phi(\theta)}{d\theta}, & -1 \le \theta < 0, \\ \displaystyle\int_{-1}^{0} d\eta(\theta, \mu)\,\phi(\theta), & \theta = 0, \end{cases} \\ R(\mu)\phi &= \begin{cases} 0, & -1 \le \theta < 0, \\ F(\mu, \phi), & \theta = 0. \end{cases} \end{aligned} \tag{20}$$

Then system (15) is equivalent to the following operator equation:

$$\dot{u}(t) = A(\mu)u_t + R(\mu)u_t. \tag{21}$$

Next, we define the adjoint operator A^* of A,

$$A^*(\varphi)=\begin{cases}-\dfrac{d\varphi(s)}{ds}, & 0<s\le 1,\\ \displaystyle\int_{-1}^{0} d\eta^T(s,0)\,\varphi(-s), & s=0,\end{cases} \tag{22}$$

and a bilinear inner product,

$$\begin{aligned}\langle\varphi(s),\phi(\theta)\rangle &= \overline{\varphi}(0)\,\phi(0)\\ &\quad-\int_{\theta=-1}^{0}\int_{\xi=0}^{\theta}\overline{\varphi}(\xi-\theta)\,d\eta(\theta)\,\phi(\xi)\,d\xi,\end{aligned} \tag{23}$$

where $\eta(\theta)=\eta(\theta,0)$.

By the discussion in the previous section, we know that $\pm i\omega_0\tau_0$ are common eigenvalues of $A(0)$ and $A^*(0)$. We need to compute the eigenvectors of $A(0)$ and $A^*(0)$ corresponding to $+i\omega_0\tau_0$ and $-i\omega_0\tau_0$, respectively. Let $q(\theta)=(1,q_2,q_3)^T e^{i\omega_0\tau_0\theta}$ be the eigenvector of $A(0)$ corresponding to $+i\omega_0\tau_0$ and let $q^*(\theta)=D(1,q_2^*,q_3^*)e^{i\omega_0\tau_0 s}$ be the eigenvector of $A^*(0)$ corresponding to $-i\omega_0\tau_0$. Then we have

$$\begin{aligned}A(0)\,q(\theta)&=i\omega_0\tau_0 q(\theta),\\ A^*(0)\,q^{*T}(\theta)&=-i\omega_0\tau_0 q^{*T}(\theta).\end{aligned} \tag{24}$$

From the definitions of $A(0)$ and $A^*(0)$, we have

$$\begin{aligned}A(0)\,q(\theta)&=\frac{dq(\theta)}{d\theta},\\ A^*(0)\,q^{*T}(s)&=-\frac{dq^{*T}(s)}{ds}.\end{aligned} \tag{25}$$

Thus,

$$\begin{aligned}q(\theta)&=q(0)\,e^{i\omega_0\tau_0\theta},\\ q^*(s)&=q(0)\,e^{i\omega_0\tau_0 s}.\end{aligned} \tag{26}$$

In addition,

$$\begin{aligned}\int_{-1}^{0} d\eta(\theta)\,q(\theta)&=A_*q(0)+B_*q(-1)=A(0)\,q(0)\\ &=i\omega_0\tau_0 q(0).\end{aligned} \tag{27}$$

Thus, we can get

$$\begin{aligned}q_2&=\frac{i\omega_0-a_{11}-\left(a_{13}+b_{13}e^{-i\omega_0\tau_0}\right)q_3}{a_{12}+b_{12}e^{-i\omega_0\tau_0}},\\ q_3&=\frac{a_{32}\left(i\omega_0-a_{11}\right)}{\Delta_1+\Delta_2},\end{aligned} \tag{28}$$

where

$$\begin{aligned}\Delta_1&=a_{32}\left(a_{13}+b_{13}e^{-i\omega_0\tau_0}\right),\\ \Delta_2&=\left(i\omega_0-a_{33}\right)\left(a_{12}+b_{12}e^{-i\omega_0\tau_0}\right).\end{aligned} \tag{29}$$

On the other hand,

$$\begin{aligned}\int_{-1}^{0} d\eta(\theta)\,q^*(-s)&=A_1^T q^{*T}(0)+B_1^T q^{*T}(-1)\\ &=A^*(0)\,q^{*T}(0)=-i\omega_0\tau_0 q^{*T}(0).\end{aligned} \tag{30}$$

Thus, we get

$$\begin{aligned}q_2^*&=-\frac{i\omega_0+a_{11}}{a_{21}},\\ q_3^*&=\frac{\left(i\omega_0+a_{11}\right)\left(i\omega_0+a_{22}\right)}{a_{21}a_{32}}+\frac{a_{12}+b_{12}e^{i\omega_0\tau_0}}{a_{32}}.\end{aligned} \tag{31}$$

From (23), we have

$$\begin{aligned}\langle q^*(s),q(\theta)\rangle&=\overline{D}\overline{q}^*(0)\,q(0)\\ &\quad-\int_{-1}^{0}\int_{0}^{\theta}\overline{D}\overline{q}^*(0)\,e^{-i\omega_0\tau_0(\xi-\theta)}d\eta(\theta)\,q(0)\,e^{i\omega_0\tau_0\xi}d\xi\\ &=\overline{D}\Big[1+q_2\overline{q}_2^*+q_3\overline{q}_3^*\\ &\quad-\int_{-1}^{0}\overline{q}^*(0)\,\theta e^{i\omega_0\tau_0\theta}d\eta(\theta)\,q(0)\Big]=\overline{D}\Big[1+q_2\overline{q}_2^*\\ &\quad+q_3\overline{q}_3^*+\overline{q}^*(0)\left(B_*e^{-i\omega_0\tau_0}\right)q(0)\Big]=\overline{D}\Big[1+q_2\overline{q}_2^*\\ &\quad+q_3\overline{q}_3^*+\tau_0e^{-i\omega_0\tau_0}\left(b_{12}q_2+b_{13}q_3\right)\Big].\end{aligned} \tag{32}$$

Thus, we can obtain

$$\overline{D}=\left[1+q_2\overline{q}_2^*+q_3\overline{q}_3^*+\tau_0e^{-i\omega_0\tau_0}\left(b_{12}q_2+b_{13}q_3\right)\right]^{-1}, \tag{33}$$

such that $\langle q^*,q\rangle=1$.

On the other hand, since $\langle\varphi,A\phi\rangle=\langle A^*\varphi,\phi\rangle$, we have

$$\begin{aligned}-i\omega_0\tau_0\langle q^*,\overline{q}\rangle&=\langle q^*,A\overline{q}\rangle=\langle A^*q^*,\overline{q}\rangle\\ &=\langle -i\omega_0\tau_0q^*,\overline{q}\rangle=i\omega_0\tau_0\langle q^*,\overline{q}\rangle.\end{aligned} \tag{34}$$

Obviously, $\langle q^*,\overline{q}\rangle=0$.

In the remainder of this section, we use the same notations as those used by Xu and He [15] and we first compute the coordinates to describe the center manifold C_0 at $\mu=0$. Let x_t be the solution of (21) when $\mu=0$.

Define

$$\begin{aligned}z(t)&=\langle q^*,x_t\rangle,\\ W(t,\theta)&=x_t(\theta)-2\mathrm{Re}\{z(t)\,q(\theta)\},\end{aligned} \tag{35}$$

on the center manifold C_0, and we have

$$W(t,\theta)=W(z(t),\overline{z}(t),\theta), \tag{36}$$

where

$$\begin{aligned}W(z(t),\overline{z}(t),\theta)&=W(z,\overline{z})\\ &=W_{02}\frac{z^2}{2}+W_{11}z\overline{z}+W_{02}\frac{\overline{z}^2}{2}+\cdots,\end{aligned} \tag{37}$$

and z and $\overline{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\overline{q}^*$. Note that W is real if x_t is real; we only deal with real solutions. It is easy to see that

$$\begin{aligned}\dot{z}(t) &= \langle q^*, \dot{x}_t\rangle = \langle q^*, A(0)x_t\rangle + \langle q^*, R(0)x_t\rangle \\ &= \langle A^*(0)q^*, x_t\rangle + \langle q^*, R(0)x_t\rangle \\ &= \langle A^*(0)q^*, x_t\rangle + \overline{q}^*(0)R(0)x_t \\ &\quad - \int_{-1}^{0}\int_{0}^{\theta}\overline{q}^*(\xi-\theta)\,d\eta(\theta)A(0)R(0)x_t(\xi)\,d\xi \\ &= i\omega_0\tau_0 z(t) + \overline{q}^*(0)f(0, x_t(\theta)) \\ &\coloneqq i\omega_0\tau_0 z(t) + \overline{q}^*(0)f_0(z(t), \overline{z}(t)).\end{aligned} \tag{38}$$

We rewrite this equation as

$$\dot{z}(t) = i\omega_0\tau_0 z(t) + g(z, \overline{z}), \tag{39}$$

where

$$g(z,\overline{z}) = g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots. \tag{40}$$

So, we can get

$$\begin{aligned}g(z,\overline{z}) &= \overline{q}^*(0)f_0(z,\overline{z}) \\ &= \overline{D}(1, \overline{q}_2^*, \overline{q}_3^*)(f_1(0,x_t), f_2(0,x_t), 0)^T,\end{aligned} \tag{41}$$

where

$$\begin{aligned}f_1(0,x_t) &= -\beta\tau_0[\varphi_1(0)\varphi_2(0) + \varphi_1(0)\varphi_3(0)], \\ f_2(0,x_t) &= \beta\tau_0[\varphi_1(0)\varphi_2(0) + \varphi_1(0)\varphi_3(0)].\end{aligned} \tag{42}$$

Since

$$\begin{aligned}x_t &= x(t+\theta) = W(z,\overline{z},\theta) + zq(\theta) + \overline{z}\overline{q}(\theta), \\ q(\theta) &= (1, q_2, q_3)^T e^{i\omega_0\tau_0\theta},\end{aligned} \tag{43}$$

we have

$$\begin{aligned}x_t &= \begin{bmatrix} x_1(t+\theta) \\ x_2(t+\theta) \\ 0 \end{bmatrix} + \begin{bmatrix} W^{(1)}(t+\theta) \\ W^{(2)}(t+\theta) \\ 0 \end{bmatrix} \\ &\quad + z\begin{bmatrix} 1 \\ q_2 \\ q_3 \end{bmatrix} e^{i\omega_0\tau_0\theta} + \overline{z}\begin{bmatrix} 1 \\ \overline{q}_2 \\ \overline{q}_3 \end{bmatrix} e^{-i\omega_0\tau_0\theta},\end{aligned}$$

$$\begin{aligned}\varphi_1(0) &= z + \overline{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\overline{z} + W_{02}^{(1)}\frac{\overline{z}^2}{2} \\ &\quad + \cdots, \\ \varphi_2(0) &= zq_2 + \overline{z}\overline{q}_2 + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\overline{z} \\ &\quad + W_{02}^{(2)}\frac{\overline{z}^2}{2} + \cdots, \\ \varphi_3(0) &= zq_3 + \overline{z}\overline{q}_3 + W_{20}^{(3)}(0)\frac{z^2}{2} + W_{11}^{(3)}(0)z\overline{z} \\ &\quad + W_{02}^{(3)}\frac{\overline{z}^2}{2} + \cdots.\end{aligned} \tag{44}$$

From (40) and (41), we have

$$\begin{aligned}g(z,\overline{z}) &= \overline{D}(1, \overline{q}_2^*, \overline{q}_3^*) \\ &\quad \times \begin{bmatrix} K_{11}z^2 + K_{12}z\overline{z} + K_{13}\overline{z}^2 + K_{14}z^2\overline{z} \\ K_{21}z^2 + K_{22}z\overline{z} + K_{23}\overline{z}^2 + K_{24}z^2\overline{z} \\ 0 \end{bmatrix} \\ &\quad + \cdots,\end{aligned} \tag{45}$$

where

$$\begin{aligned}K_{11} &= -\beta\tau_0(q_2 + q_3), \\ K_{12} &= -\beta\tau_0(q_2 + \overline{q}_2 + q_3 + \overline{q}_3), \\ K_{13} &= -\beta\tau_0(\overline{q}_2 + \overline{q}_3), \\ K_{14} &= -\beta\tau_0\Big(W_{11}^{(1)}(0)q_2 + \frac{1}{2}W_{20}^{(1)}(0)\overline{q}_2 + W_{11}^{(2)}(0) \\ &\quad + \frac{1}{2}W_{20}^{(2)}(0) + W_{11}^{(1)}(0)q_3 + \frac{1}{2}W_{20}^{(1)}(0)\overline{q}_3 \\ &\quad + W_{11}^{(3)}(0) + \frac{1}{2}W_{20}^{(3)}(0)\Big), \\ K_{21} &= \beta\tau_0(q_2 + q_3), \\ K_{22} &= \beta\tau_0(q_2 + \overline{q}_2 + q_3 + \overline{q}_3), \\ K_{23} &= \beta\tau_0(\overline{q}_2 + \overline{q}_3), \\ K_{24} &= \beta\tau_0\Big(W_{11}^{(1)}(0)q_2 + \frac{1}{2}W_{20}^{(1)}(0)\overline{q}_2 + W_{11}^{(2)}(0) \\ &\quad + \frac{1}{2}W_{20}^{(2)}(0) + W_{11}^{(1)}(0)q_3 + \frac{1}{2}W_{20}^{(1)}(0)\overline{q}_3 \\ &\quad + W_{11}^{(3)}(0) + \frac{1}{2}W_{20}^{(3)}(0)\Big).\end{aligned} \tag{46}$$

Thus, we have

$$\begin{aligned}g(z,\overline{z}) &= \overline{D}\big[(K_{11} + \overline{q}_2^*K_{21})z^2 + (K_{12} + \overline{q}_2^*K_{22})z\overline{z} \\ &\quad + (K_{13} + \overline{q}_2^*K_{23})\overline{z}^2 + (K_{14} + \overline{q}_2^*K_{24})z^2\overline{z}\big] + \cdots.\end{aligned} \tag{47}$$

Comparing the coefficients in (47) with those in (40), we can get

$$\begin{aligned} g_{20} &= 2\overline{D}\left(K_{11}+\overline{q}_2^* K_{21}\right), \\ g_{11} &= \overline{D}\left(K_{12}+\overline{q}_2^* K_{22}\right), \\ g_{02} &= 2\overline{D}\left(K_{13}+\overline{q}_2^* K_{23}\right), \\ g_{21} &= 2\overline{D}\left(K_{14}+\overline{q}_2^* K_{24}\right). \end{aligned} \tag{48}$$

In order to get the expression of g_{21}, we need to compute $W_{11}(\theta)$ and $W_{20}(\theta)$. From (21) and (39), we have

$$W' = \begin{cases} AW - 2\operatorname{Re}\left\{\overline{q}^*(0)\, f q(\theta)\right\}, & -1 \le \theta < 0, \\ AW - 2\operatorname{Re}\left\{\overline{q}^*(0)\, f q(\theta)\right\} + f, & \theta = 0. \end{cases} \tag{49}$$

Define (49) as

$$W' = AW + H(z,\overline{z},\theta), \tag{50}$$

where

$$H(z,\overline{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)\frac{z\overline{z}}{2} + H_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots. \tag{51}$$

From (37), (39), (50), and (51), we obtain

$$\left(2i\omega_0\tau_0 - A\right)W_{20}(\theta) = H_{20}(\theta), \tag{52}$$

$$AW_{11}(\theta) = -H_{11}(\theta). \tag{53}$$

From (40) and (49), we have

$$\begin{aligned} H(z,\overline{z},\theta) &= -2\operatorname{Re}\left\{\overline{q}^*(0)\, f_0 q(\theta)\right\} \\ &= -2\operatorname{Re}\left\{g(z,\overline{z})\, q(\theta)\right\} \\ &= -g(z,\overline{z})\, q(\theta) - \overline{g}(z,\overline{z})\,\overline{q}(\theta) \\ &= -\left(g_{02}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2}\right)q(\theta) \\ &\quad - \left(\overline{g}_{02}\frac{\overline{z}^2}{2} + \overline{g}_{11}z\overline{z} + \overline{g}_{02}\frac{z^2}{2} + \overline{g}_{21}\frac{z\overline{z}^2}{2}\right)\overline{q}(\theta) \\ &\quad - \cdots. \end{aligned} \tag{54}$$

Comparing the coefficients of (51) and (54), we get

$$H_{20}(\theta) = -g_{20}q(\theta) - \overline{g}_{02}\overline{q}(\theta), \tag{55}$$

$$H_{11}(\theta) = -g_{11}q(\theta) - \overline{g}_{11}\overline{q}(\theta), \tag{56}$$

when $\theta \in [-1,0)$.

From (52) and (55) and the definition of $A(0)$, we have

$$\dot{W}_{20}(\theta) = 2i\omega_0\tau_0 W_{20}(\theta) + g_{20}q(\theta) + \overline{g}_{02}\overline{q}(\theta). \tag{57}$$

According to $q(\theta) = q(0)e^{i\omega_0\tau_0}$, we get

$$\begin{aligned} W_{20}(\theta) &= \frac{ig_{20}}{\omega_0\tau_0}q(0)\,e^{i\omega_0\tau_0} + \frac{i\overline{g}_{02}}{3\omega_0\tau_0}\overline{q}(0)\,e^{-i\omega_0\tau_0} \\ &\quad + E_1 e^{2i\omega_0\tau_0}, \end{aligned} \tag{58}$$

and by a similar method, we can get

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega_0\tau_0}q(0)\,e^{i\omega_0\tau_0} + \frac{i\overline{g}_{11}}{\omega_0\tau_0}\overline{q}(0)\,e^{-i\omega_0\tau_0} + E_2, \tag{59}$$

where E_1 and E_2 are both constant vectors.

In what follows, we will find out E_1 and E_2. It follows from the definition of $A(0)$ and (55) and (56) that

$$\int_{-1}^{0} d\eta(\theta)\, W_{20}(\theta) = 2i\omega_0\tau_0 W_{20}(0) - H_{20}(0), \tag{60}$$

$$\int_{-1}^{0} d\eta(\theta)\, W_{11}(\theta) = -H_{11}(0), \tag{61}$$

where $\eta(\theta) = \eta(0,\theta)$.

From (52) and (53), we get

$$H_{20}(0) = -g_{20}q(\theta) - \overline{g}_{02}\overline{q}(0) + \left(K_{11}, K_{21}, 0\right)^T, \tag{62}$$

$$H_{11}(0) = -g_{11}q(\theta) - \overline{g}_{11}\overline{q}(0) + \left(K_{12}, K_{22}, 0\right)^T. \tag{63}$$

Notice that

$$\begin{aligned} \left(i\omega_0\tau_0 I - \int_{-1}^{0} d\eta(\theta)\, e^{i\omega_0\tau_0}\right) q(0) &= 0, \\ \left(-i\omega_0\tau_0 I - \int_{-1}^{0} d\eta(\theta)\, e^{-i\omega_0\tau_0}\right) \overline{q}(0) &= 0, \end{aligned} \tag{64}$$

and substituting (58) and (62) into (60), we obtain

$$\left(2i\omega_0\tau_0 I - \int_{-1}^{0} d\eta(\theta)\, e^{2i\omega_0\tau_0\theta}\right) E_1 = \begin{pmatrix} K_{11} \\ K_{21} \\ 0 \end{pmatrix}. \tag{65}$$

That is,

$$\left(2i\omega_0\tau_0 I - A_* - B_* e^{-2i\omega_0\tau_0}\right) E_1 = \begin{pmatrix} K_{11} \\ K_{21} \\ 0 \end{pmatrix}. \tag{66}$$

Thus, we have

$$E_1 = \frac{2}{\tau_0}\begin{pmatrix} a'_{11} & a'_{12} & a'_{13} \\ -a_{21} & a'_{22} & -a_{23} \\ 0 & -a_{32} & a'_{33} \end{pmatrix}^{-1} \times \begin{pmatrix} K_{11} \\ K_{21} \\ 0 \end{pmatrix}, \tag{67}$$

where

$$\begin{aligned} a'_{11} &= 2i\omega_0 - a_{11}, \\ a'_{12} &= -a_{12} - b_{12}e^{-2i\omega_0\tau_0}, \\ a'_{13} &= -a_{13} - b_{13}e^{-2i\omega_0\tau_0}, \\ a'_{22} &= 2i\omega_0 - a_{22}, \\ a'_{33} &= 2i\omega_0 - a_{33}. \end{aligned} \tag{68}$$

Similarly, we have

$$\int_{-1}^{0} d\eta(\theta) E_2 = \begin{pmatrix} K_{12} \\ K_{22} \\ 0 \end{pmatrix}. \tag{69}$$

Then,

$$(A_* + B_*) E_2 = \begin{pmatrix} K_{12} \\ K_{22} \\ 0 \end{pmatrix}. \tag{70}$$

Therefore, we can get

$$E_2 = \frac{1}{\tau_0} \begin{pmatrix} a_{11} & b'_{12} & b'_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}^{-1} \times \begin{pmatrix} K_{12} \\ K_{22} \\ 0 \end{pmatrix}, \tag{71}$$

where

$$\begin{aligned} b'_{12} &= a_{12} + b_{12}, \\ b'_{13} &= a_{13} + b_{13}. \end{aligned} \tag{72}$$

Thus, we can compute the following coefficients:

$$\begin{aligned} C_1(0) &= \frac{i}{2\omega_0\tau_0}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{\operatorname{Re}\{C_1(0)\}}{\operatorname{Re}\{\lambda'(\tau_0)\}}, \\ \beta_2 &= 2\operatorname{Re}\{C_1(0)\}, \\ T_2 &= -\frac{\operatorname{Im}\{C_1(0)\} + \mu_2 \operatorname{Im}\{\lambda'(\tau_0)\}}{\omega_0\tau_0}, \end{aligned} \tag{73}$$

which determine the properties of the bifurcating periodic solutions in the center manifold at the critical value τ_0 and we obtain the following results.

Theorem 2. *For system (2),*

(i) *the sign of μ_2 determines the direction of the Hopf bifurcation: if $\mu_2 > 0$ ($\mu_2 < 0$), then the Hopf bifurcation is supercritical (subcritical);*

(ii) *the sign of β_2 determines the stability of the bifurcating periodic solutions: if $\beta_2 < 0$ ($\beta_2 > 0$), then the bifurcating periodic solutions are stable (unstable);*

(iii) *the sign of T_2 determines the period of the bifurcating periodic solutions: if $T_2 > 0$ ($T_2 < 0$), then the bifurcating periodic solutions increase (decrease).*

4. Numerical Simulation

In this section, we try to present some numerical simulations for system (2) to validate the previous main results. By extracting some values from [8] and considering the conditions for the existence of the Hopf bifurcation, we choose a set of parameters as follows: $\alpha = 0.3$, $\beta = 0.35$, $\gamma_1 = 0.1$, $\gamma_2 = 0.3$, and $\delta = 0.05$. Then, we can get a specific case of system (2).

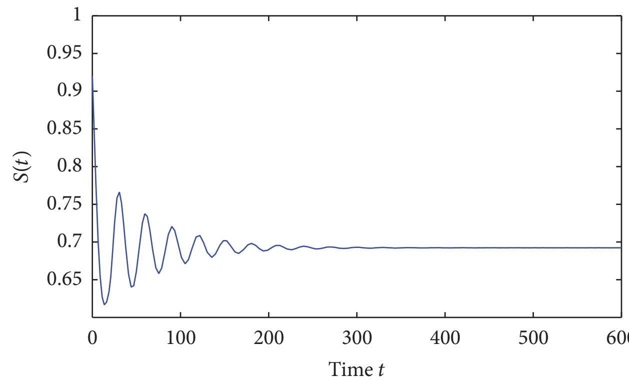

Figure 1: E_+ is locally asymptotically stable when $\tau = 18.3656 < \tau_0$ with initial value "0.92, 0.155, 0.112."

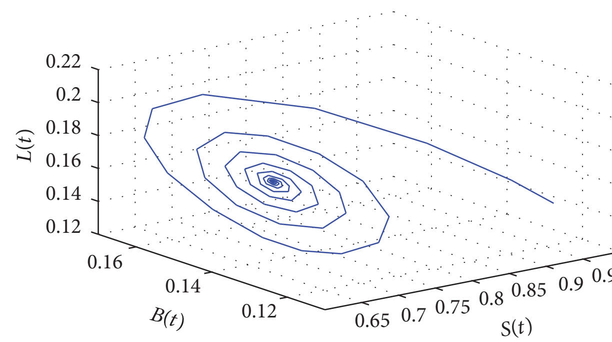

Figure 2: The phase plot of the states S_*, L_*, and B_* when $\tau = 18.3656 < \tau_0$ with initial value "0.92, 0.155, 0.112."

By a simple computation, we get $R_0 = 1.4444 > 1$ and the unique positive equilibrium $E_+(0.6924, 0.1657, 0.1420)$ of the system. Further, from (11) and (12), we obtain $\omega_0 = 0.2766$ and $\tau_0 = 22.9208$. According to Theorem 1, we can conclude that if $\tau \in [0, 6.0897)$, then the positive equilibrium $E_+(0.6924, 0.1657, 0.1420)$ is locally asymptotically stable. As can be seen from Figures 1 and 2, when we choose $\tau = 18.3656 < \tau_0$, the positive equilibrium $E_+(0.6924, 0.1657, 0.1420)$ is locally asymptotically stable. Once the value of the delay τ passes through the critical value τ_0, as is shown in Figures 3 and 4, when we choose $\tau = 32.8965 > \tau_0$, a Hopf bifurcation occurs and a family of periodic solutions bifurcate from the positive equilibrium $E_+(0.6924, 0.1657, 0.1420)$.

In addition, we also find that onset of the Hopf bifurcation can be delayed if the values of the parameters γ_1 and γ_2 increase from the numerical simulation. And this can be realized by means of strengthening the immunization of the new computers connected to the network. Thus, we can conclude that the managers of the network should strengthen the immunization of the new computers connected to the network so as to predict and control the propagation of the computer virus in the network easily.

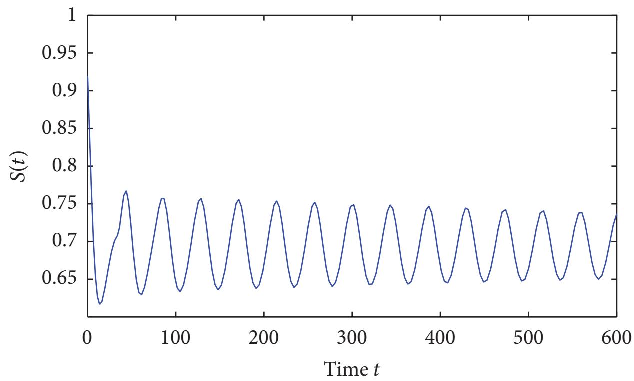

FIGURE 3: E_+ loses stability and a Hopf bifurcation occurs when $\tau = 32.8965 > \tau_0$ with initial value "0.92, 0.155, 0.112."

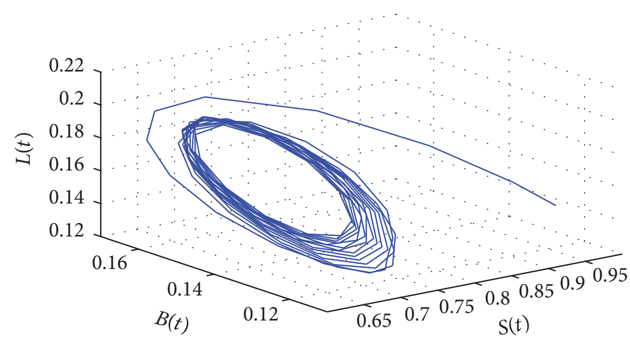

FIGURE 4: The phase plot of the states S_*, L_*, and B_* when $\tau = 32.8965 > \tau_0$ with initial value "0.92, 0.155, 0.112."

5. Conclusion

A new epidemic model of computer virus with time delay has been proposed in this paper. Compared with the model considered in the literature [8], we mainly investigate the effect of the delay due to the period that a computer user uses to cure the latent and breaking-out computers by using antivirus software or reinstalling system on system (2) by analyzing the corresponding characteristic equation and regarding the time delay as a bifurcation parameter.

It is found that when the value of the time delay is below the critical value τ_0 the model is asymptotically stable. In this case, the prevalence of the viruses in the Internet can be easily controlled. However, when the value of the time delay is above τ_0, the model will lose its stability and a Hopf bifurcation occurs, which indicates that computers of the three classes in system (2) may coexist in an oscillatory mode under some conditions. This phenomenon is not welcome in the Internet and it is difficult to control and prevent the viruses in the Internet in this case. Hence, we should control the occurrence of the Hopf bifurcation by combining some bifurcation control strategies and other relative features of virus prevalence, in order to make the propagation of computer viruses possible to be predicted and controlled easily. Specially, with the help of the normal form theory and the center manifold theorem, explicit formulae determining the properties of the Hopf bifurcation have been derived. Finally, numerical examples are given to illustrate the main results obtained in the paper.

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

Acknowledgments

This research was supported by the Natural Science Foundation of Anhui Province (nos. 1608085QF145 and 1608085QF151) and the Natural Science Foundation of the Higher Education Institutions of Anhui Province (no. KJ2015A144).

References

[1] F. Cohen, "Computer viruses: theory and experiments," *Computers & Security*, vol. 6, no. 1, pp. 22–35, 1987.

[2] F. Cohen, *A Short Course on Computer Viruses*, John Wiley & Sons, New York, NY, USA, 2nd edition, 1986.

[3] C. Gan, X. Yang, W. Liu, Q. Zhu, and X. Zhang, "Propagation of computer virus under human intervention: a dynamical model," *Discrete Dynamics in Nature and Society*, vol. 2012, Article ID 106950, 8 pages, 2012.

[4] Q. R. Li, F. Q. Yu, W. Han, and Y. K. Zhang, "A worm propagation model with removable storage devices and its stability study," *ICIC Express Letters*, vol. 8, no. 10, pp. 2719–2724, 2014.

[5] B. K. Mishra and S. K. Pandey, "Effect of anti-virus software on infectious nodes in computer network: a mathematical model," *Physics Letters A*, vol. 376, no. 35, pp. 2389–2393, 2012.

[6] B. K. Mishra and N. Keshri, "Mathematical model on the transmission of worms in wireless sensor network," *Applied Mathematical Modelling*, vol. 37, no. 6, pp. 4103–4111, 2013.

[7] J. R. Piqueira and V. O. Araujo, "A modified epidemiological model for computer viruses," *Applied Mathematics and Computation*, vol. 213, no. 2, pp. 355–360, 2009.

[8] L. X. Yang, X. F. Yang, Q. Y. Zhu, and L. S. Wen, "A computer virus model with graded cure rates," *Nonlinear Analysis: Real World Applications*, vol. 14, no. 1, pp. 414–422, 2013.

[9] F. Wang, Y. Zhang, C. Wang, J. Ma, and S. Moon, "Stability analysis of a SEIQV epidemic model for rapid spreading worms," *Computers and Security*, vol. 29, no. 4, pp. 410–418, 2010.

[10] K. Zhuang and H. Zhu, "Stability and bifurcation analysis for an improved HIV model with time delay and cure rate," *WSEAS Transactions on Mathematics*, vol. 12, no. 8, pp. 860–869, 2013.

[11] J. Liu, "Hopf bifurcation in a delayed SEIQRS model for the transmission of malicious objects in computer network," *Journal of Applied Mathematics*, vol. 2014, Article ID 492198, 8 pages, 2014.

[12] C. Bianca, M. Ferrara, and L. Guerrini, "Qualitative analysis of a retarded mathematical framework with applications to living systems," *Abstract and Applied Analysis*, vol. 2013, Article ID 736058, 7 pages, 2013.

[13] M. Haque, S. Sarwardi, S. P. Preston, and E. Venturino, "Effect of delay in a Lotka–Volterra type predator–prey model with a transmissible disease in the predator species," *Mathematical Biosciences*, vol. 234, no. 1, pp. 47–57, 2011.

[14] C. Bianca, M. Ferrara, and L. Guerrini, "Hopf bifurcations in a delayed-energy-based model of capital accumulation," *Applied Mathematics & Information Sciences*, vol. 7, no. 1, pp. 139–143, 2013.

[15] C. Xu and X. He, "Stability and bifurcation analysis in a class of two-neuron networks with resonant bilinear terms," *Abstract and Applied Analysis*, vol. 2011, Article ID 697630, 21 pages, 2011.

[16] C. Bianca, L. Guerrini, and J. Riposo, "A delayed mathematical model for the acute inflammatory response to infection," *Applied Mathematics and Information Sciences*, vol. 9, no. 6, pp. 2775–2782, 2015.

[17] B. D. Hassard, N. D. Kazarinoff, and Y. H. Wan, *Theory and Applications of Hopf Bifurcation*, Cambridge University Press, Cambridge, UK, 1981.