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Research Article

On the Discrete-Time $Geo/G/1$ Queue with Vacations in Random Environment

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A discrete-time $Geo/G/1$ queue with vacations in random environment is analyzed. Using the method of supplementary variable, we give the probability generating function (PGF) of the stationary queue length distribution at arbitrary epoch. The PGF of the stationary sojourn time distribution is also derived. And we present the various performance measures such as mean number of customers in the system, mean length of the type-i cycle, and mean time that the system resides in phase 0. In addition, we show that the $M/G/1$ queue with vacations in random environment can be approximated by its discrete-time counterpart. Finally, we present some special cases of the model and numerical examples.

1. Introduction

During the last three decades queueing systems with vacations have been largely studied; see, for example, the surveys (Doshi [1]) and the monographs (Takagi [2] and Tian and Zhang [3]). Queueing systems with vacations are characterized by the feature that each time a busy period ends and the system becomes empty, the server starts a vacation of random length of time. These queueing models have been applied in many fields such as production systems and computer and communication systems. Doshi [1] gives a large number of examples. See also Takagi [2].

Due to the applicability in the field of computer and communication systems in which time is slotted, discrete-time queueing models have been widely used over the past several years. For a detailed discussion and applications of discrete-time queues, see, for example, Hunter [4], Bruneel and Kim [5], and Woodward [6]. The first work on discrete-time queues is due to Meisling [7]. Since then, parallel to the continuous-time vacation queues, discrete-time queues have been intensively studied by many researchers. The early results of discrete-time queues can be found in the book of Hunter [4]. Subsequently, Takagi [8] provided a detailed analysis on $Geo/G/1$ queues with a variety of vacation policies. The work about a $Geo/G/1$ queue with multiple adaptive vacations was considered by Zhang and Tian [9]. Alfa [10] discussed a model with nonexhaustive service in which both vacation time and service time follow phase type distributions. Recently, Atencia and Moreno [11] studied a discrete-time $Geo/G/1$ retrial queue, where the server is subject to starting failures. Li and Tian [12] considered a $GI/Geo/1$ queue with working vacations and vacation interruption. Using the matrix-analytic method, Tao et al. [13] analyzed the $GI/Geo/1$ queue with Bernoulli-schedule-controlled vacation and vacation interruption. Vijaya Laxmi et al. [14] analyzed a discrete-time working vacation queue with balking.

In most of the literature on queueing theory, both arrival rate and service rate are homogeneous. But, in real life, the parameters in the queueing models may not be constant; they may change with changes of the environment. Yechiali and Naor [15] study a two-level modification of the $M/M/1$ queueing model, where the rate of arrival and the service capacity are subject to Poisson alternations. Their work is considered to be the first systematic work on queueing system in random environment. The $M/G/1$ queue in a two-phase random environment was studied by Neuts [16], Boxma and Kurkova [17], Huang and Lee [18], and so on. Recently, Cordeiro and Kharoufeh [19] discussed an unreliable $M/M/1$

retrial queue whose arrival, service, failure, repair, and retrial rates are all modulated by an exogenous random environment. B. Kim and J. Kim [20] considered a single server queue with Markov modulated service rates and impatient customers. An $M/M/1$ queue in random environment with disasters was investigated by Paz and Yechiali [21] and Jiang et al. [22] extended their work to $M/G/1$ queue.

To the best of our knowledge, there has been no research on the discrete-time queueing system with vacations in random environment. However, in many practical applications like production, these queueing systems may be utilized. For example, in production system, if there is no workload to be processed, the server has a vacation period. When it returns from vacation, if there are one or more customers in the system, it continues to serve customers until the system becomes empty. But the new service rate may change with the changes of the customers' arrival rate, environmental conditions, and operator experience. This motivated us to study the $Geo/G/1$ queue with vacations in random environment in this paper.

The rest of the paper is organized as follows. Section 2 is devoted to the model description. Section 3 obtains the probability generating function of the steady-state queue size distribution at an arbitrary epoch. Section 4 presents the various performance measures such as mean number of customers in the system, mean length of the type-i cycle, and PGF of the steady-state sojourn time distribution. Section 5 gives the relationship between the discrete-time system and its continuous-time counterpart. Some special cases are presented in Section 6. Numerical results are presented in Section 7 followed by conclusions in Section 8.

2. Model Description

We consider a discrete-time vacation queue operating in random environment where the time axis is segmented into slots of equal length. It is assumed that the time axis is marked by $0, 1, \ldots, t, \ldots$ and all queueing activities occur only at the slot boundaries. We consider late arrival system (LAS) policy in our queueing system. That is, we assume that the arrivals occur in (t^-, t), the departures occur in (t, t^+), and the vacation completion occurs just at the instant t.

When the system operates in phase i, $i = 1, 2, \ldots, n$, customers arrive according to geometrical process with rate λ_i, where λ_i is the probability that a customer arrives in a slot, and the service times are independent and identically distributed according to a general distribution $\{s_{i,k}\}_{k=1}^{\infty}$ with probability generating function $S_i(z) = \sum_{k=1}^{\infty} s_{i,k} z^k$ and mean $1/\mu_i$. Customers are served according to the first-come, first-served (FCFS) discipline. Each time a busy period ends and the system becomes empty, the server leaves for a vacation of random length V, which is geometrically distributed with parameter θ, causing the system to move to vacation phase 0, where θ is the probability that vacation ends in a slot. In vacation phase 0, the arrivals occur according to a geometrical process of rate λ_0. When the server returns, if it finds no customer waiting, it goes on another vacation. Otherwise, the system moves from the vacation phase 0 to some operative phase i with probability q_i, $i = 1, 2, \ldots, n$, where $q_i \geq 0$ and $\sum_{i=1}^{n} q_i = 1$. The interarrival times, the individual service, and vacation durations are assumed to be mutually independent. Throughout the rest of the paper, we denote $\overline{p} = 1 - p$.

3. Steady-State Queue Size Distribution

In operative phase i, $i = 1, 2, \ldots, n$, it is easily seen that the system acts as the classical $Geo/G/1$ queue with geometrical arrival rate λ_i and service rate μ_i. So as long as $\rho_i = \lambda_i/\mu_i < 1$, $i = 1, 2, \ldots, n$, this system that we consider is stable. Assume that the condition for stability of the system is fulfilled. Next, we analyze the steady-state queue size distribution.

At time t^+, the system can be described by the process $\{(L(t^+), J(t^+), S^+(t^+)),\ t = 0, 1, 2, \ldots\}$, where $L(t^+)$ denotes the number of customers in the system and $J(t^+)$ represents the phase in which the system operates. If $L(t^+) \geq 1$ and $J(t^+) = i$, $i = 1, 2, \ldots, n$, $S^+(t^+) = S_i^+(t^+)$ represents the remaining service time of the customer currently being served during phase i. Then $\{(L(t^+), J(t^+), S^+(t^+)),\ t = 0, 1, 2, \ldots\}$ is a Markov process with the state space expressed as

$$\begin{aligned} S = \{&((m, 0),\ m \geq 0) \\ &\cup ((m, i, k),\ m \geq 1,\ k \geq 1,\ i = 1, 2, \ldots, n)\}. \end{aligned} \tag{1}$$

Since we are interested in the stationary behavior of the system, define

$$\begin{aligned} P_{m,0} &= \lim_{t\to\infty} P\{L(t^+) = m,\ J(t^+) = 0\}, \quad m \geq 0, \\ P_{m,i}(k) &= \lim_{t\to\infty} P\{L(t^+) = m,\ J(t^+) = i,\ S_i^+(t^+) = k\}, \\ &\qquad m \geq 1,\ k \geq 1,\ i = 1, 2, \ldots, n. \end{aligned} \tag{2}$$

The balance equations for the stationary distribution of the system are

$$P_{0,0} = \overline{\lambda}_0 P_{0,0} + \sum_{i=1}^{n} \overline{\lambda}_i P_{1,i}(1), \tag{3}$$

$$P_{m,0} = \overline{\lambda}_0 \overline{\theta} P_{m,0} + \lambda_0 \overline{\theta} P_{m-1,0}, \quad m \geq 1, \tag{4}$$

$$\begin{aligned} P_{m,i}(k) = {} & \lambda_i (1 - \delta_{m,1}) P_{m-1,i}(k+1) + \overline{\lambda}_i P_{m,i}(k+1) \\ & + \lambda_i s_{i,k} P_{m,i}(1) + \overline{\lambda}_i s_{i,k} P_{m+1,i}(1) \\ & + \theta q_i s_{i,k} \left(\overline{\lambda}_0 P_{m,0} + \lambda_0 P_{m-1,0}\right), \\ & \qquad k \geq 1,\ m \geq 1,\ i = 1, 2, \ldots, n, \end{aligned} \tag{5}$$

where $\delta_{m,1}$ is Kronecker's delta.

To solve the above equations (3)–(5) let us define the following PGFs:

$$P_0(z) = \sum_{m=0}^{\infty} P_{m,0} z^m,$$
$$P_i(k,z) = \sum_{m=1}^{\infty} P_{m,i}(k) z^m, \quad k \geq 1, \ i = 1,2,\ldots,n, \tag{6}$$
$$P_i^*(x,z) = \sum_{k=1}^{\infty} P_i(k,z) x^k = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} P_{m,i}(k) z^m x^k.$$

From (4), we obtain

$$P_{m,0} = \frac{\lambda_0\overline{\theta}}{1-\overline{\lambda}_0\overline{\theta}} P_{m-1,0} = \left(\frac{\lambda_0\overline{\theta}}{1-\overline{\lambda}_0\overline{\theta}}\right)^m P_{0,0}, \tag{7}$$

and then, we have

$$\begin{aligned} P_0(z) &= \sum_{m=0}^{\infty}\left(\frac{\lambda_0\overline{\theta}}{1-\overline{\lambda}_0\overline{\theta}}\right)^m z^m P_{0,0} \\ &= \frac{1-\overline{\lambda}_0\overline{\theta}}{1-\overline{\lambda}_0\overline{\theta}-\lambda_0\overline{\theta}z} P_{0,0}. \end{aligned} \tag{8}$$

Multiplying both sizes of (5) by z^m and then taking summation over all possible values of $m \geq 1$, we get

$$\begin{aligned} P_i(k,z) &= \left(\lambda_i z + \overline{\lambda}_i\right) P_i(k+1,z) \\ &\quad + \left(\lambda_i z + \overline{\lambda}_i\right) z^{-1} s_{i,k} P_i(1,z) \\ &\quad + \theta q_i s_{i,k}\left(\overline{\lambda}_0 + \lambda_0 z\right) P_0(z) - \overline{\lambda}_i s_{i,k} P_{1,i}(1) \\ &\quad - \theta q_i s_{i,k} \overline{\lambda}_0 P_{0,0}. \end{aligned} \tag{9}$$

Again multiplying both sides of (9) by x^k and then taking summation over all possible values of $k \geq 1$, we get

$$\begin{aligned} &\left(1-\alpha_i(z)x^{-1}\right) P_i^*(x,z) \\ &\quad = z^{-1}\alpha_i(z)\left(S_i(x)-z\right) P_i(1,z) - \overline{\lambda}_i S_i(x) P_{1,i}(1) \\ &\qquad - \theta q_i \overline{\lambda}_0 S_i(x) P_{0,0} + \theta q_i\left(\overline{\lambda}_0 + \lambda_0 z\right) S_i(x) P_0(z), \end{aligned} \tag{10}$$

where $\alpha_i(z) = \lambda_i z + \overline{\lambda}_i$.

Setting $x = \alpha_i(z)$ in (10), we get

$$\begin{aligned} &\alpha_i(z)\left[S_i(\alpha_i(z)) - z\right] P_i(1,z) = z S_i(\alpha_i(z)) \\ &\quad \cdot \left[\overline{\lambda}_i P_{1,i}(1) + \theta q_i \overline{\lambda}_0 P_{0,0} - \theta q_i\left(\overline{\lambda}_0 + \lambda_0 z\right) P_0(z)\right]. \end{aligned} \tag{11}$$

Setting $z = 1$ in (11), we get

$$P_{1,i}(1) = \frac{\theta q_i\left[P_0(1) - \overline{\lambda}_0 P_{0,0}\right]}{\overline{\lambda}_i}. \tag{12}$$

Substituting (12) into (11), we have

$$\begin{aligned} &P_i(1,z) \\ &\quad = \frac{z S_i(\alpha_i(z))\left[\theta q_i P_0(1) - \theta q_i\left(\overline{\lambda}_0 + \lambda_0 z\right) P_0(z)\right]}{\alpha_i(z)\left[S_i(\alpha_i(z)) - z\right]}. \end{aligned} \tag{13}$$

Substituting (12) and (13) into (10), we obtain

$$\begin{aligned} &P_i^*(x,z) \\ &\quad = \frac{x z \theta q_i\left[P_0(1) - \left(\overline{\lambda}_0 + \lambda_0 z\right) P_0(z)\right]\left[S_i(x) - S_i(\alpha_i(z))\right]}{\left[x - \alpha_i(z)\right]\left[S_i(\alpha_i(z)) - z\right]}. \end{aligned} \tag{14}$$

Setting $x = 1$ in (14), we get

$$\begin{aligned} &P_i^*(1,z) \\ &\quad = \frac{z \theta q_i\left[P_0(1) - \left(\overline{\lambda}_0 + \lambda_0 z\right) P_0(z)\right]\left[1 - S_i(\alpha_i(z))\right]}{\left[1 - \alpha_i(z)\right]\left[S_i(\alpha_i(z)) - z\right]}. \end{aligned} \tag{15}$$

Then from (8) and (15), we get the distribution of the number of customers in the system having PGF

$$\begin{aligned} P(z) &= P_0(z) + \sum_{i=1}^{n} P_i^*(1,z) = P_0(z) \\ &\quad + \sum_{i=1}^{n} \frac{z\theta q_i\left[P_0(1) - \left(\overline{\lambda}_0 + \lambda_0 z\right) P_0(z)\right]\left[1 - S_i(\alpha_i(z))\right]}{\left[1 - \alpha_i(z)\right]\left[S_i(\alpha_i(z)) - z\right]} \\ &= \left\{\frac{1-\overline{\lambda}_0\overline{\theta}}{1-\overline{\lambda}_0\overline{\theta}-\lambda_0\overline{\theta}z} + \frac{\left(1-\overline{\lambda}_0\overline{\theta}\right)\lambda_0(1-z)z}{1-\overline{\lambda}_0\overline{\theta}-\lambda_0\overline{\theta}z}\right. \\ &\quad \left. \cdot \sum_{i=1}^{n} \frac{q_i\left[1 - S_i(\alpha_i(z))\right]}{\left[1 - \alpha_i(z)\right]\left[S_i(\alpha_i(z)) - z\right]}\right\} P_{0,0}. \end{aligned} \tag{16}$$

Putting $\mathscr{D}(z) = [1-\alpha_i(z)][S_i(\alpha_i(z))-z]$, $\mathscr{N}(z) = z[P_0(1)-(\overline{\lambda}_0 + \lambda_0 z)P_0(z)][1 - S_i(\alpha_i(z))]$. Let $z \to 1$ in (15); we obtain by L'Hospital's Rule

$$P_i^*(1,1) = \lim_{z\to 1} P_i^*(1,z) = \frac{\theta q_i \mathscr{N}''(1)}{\mathscr{D}''(1)}, \tag{17}$$

where

$$\begin{aligned} \mathscr{D}''(1) &= \frac{2\lambda_i(\mu_i - \lambda_i)}{\mu_i}, \\ \mathscr{N}''(1) &= \frac{2\lambda_i\lambda_0\left(1-\overline{\lambda}_0\overline{\theta}\right)}{\mu_i\theta^2} P_{0,0}. \end{aligned} \tag{18}$$

Substituting (18) into (17), we get

$$P_i^*(1,1) = \frac{q_i\lambda_0\left(1-\overline{\lambda}_0\overline{\theta}\right)}{\theta(\mu_i - \lambda_i)} P_{0,0}. \tag{19}$$

From normalizing condition that can be written as

$$P_0(1) + \sum_{i=1}^{n} P_i^*(1,1) = 1, \tag{20}$$

we get on simplification

$$P_{0,0} = \frac{1}{\lambda_0 \beta}, \tag{21}$$

where

$$\beta = \frac{1-\overline{\lambda}_0\overline{\theta}}{\lambda_0\theta} + \sum_{i=1}^{n} \frac{q_i\left(1-\overline{\lambda}_0\overline{\theta}\right)}{\theta\left(\mu_i-\lambda_i\right)}. \tag{22}$$

Now we summarize the results of this section in the following theorem.

Theorem 1. *If the stationary condition $\rho_i = \lambda_i/\mu_i < 1$, $i = 1, 2, \ldots, n$, holds, the PGF of the distribution of the number of customers in the system is given by*

$$P(z) = \left\{ \frac{1-\overline{\lambda}_0\overline{\theta}}{1-\overline{\lambda}_0\overline{\theta}-\lambda_0\overline{\theta}z} + \frac{\left(1-\overline{\lambda}_0\overline{\theta}\right)\lambda_0(1-z)z}{1-\overline{\lambda}_0\overline{\theta}-\lambda_0\overline{\theta}z} \cdot \sum_{i=1}^{n} \frac{q_i\left[1-S_i\left(\alpha_i(z)\right)\right]}{\left[1-\alpha_i(z)\right]\left[S_i\left(\alpha_i(z)\right)-z\right]} \right\} \frac{1}{\lambda_0\beta}, \tag{23}$$

where β is given by (22).

4. Performance Measures

In this section, some important performance measures of the system will be provided.

4.1. Mean Number of Customers in the System. The mean number of customers in the system (L) is obtained as

$$\begin{aligned} L &= \left.\frac{dP(z)}{dz}\right|_{z=1} = \left.\frac{dP_0(z)}{dz}\right|_{z=1} + \sum_{i=1}^{n} \left.\frac{dP_i^*(1,z)}{dz}\right|_{z=1} \\ &= \frac{\left(1-\overline{\lambda}_0\overline{\theta}\right)\overline{\theta}}{\theta^2\beta} + \frac{\left(1-\overline{\lambda}_0\overline{\theta}\right)}{\beta} \\ &\quad \cdot \sum_{i=1}^{n} \frac{q_i\left[\mathcal{N}_1'''(1)\mathcal{D}_1''(1) - \mathcal{N}_1''(1)\mathcal{D}_1'''(1)\right]}{3\left[\mathcal{D}_1''(1)\right]^2}, \end{aligned} \tag{24}$$

where $\mathcal{D}_1(z) = (1-\overline{\lambda}_0\overline{\theta}-\lambda_0\overline{\theta}z)[1-\alpha_i(z)][S_i(\alpha_i(z))-z]$, $\mathcal{N}_1(z) = z(1-z)[1-S_i(\alpha_i(z))]$.

After some calculations we get

$$\begin{aligned} \mathcal{D}_1''(1) &= 2\theta\lambda_i\left(1-\frac{\lambda_i}{\mu_i}\right), \\ \mathcal{D}_1'''(1) &= 6\lambda_0\overline{\theta}\lambda_i\left(\frac{\lambda_i}{\mu_i}-1\right) - 3\theta\lambda_i^3\left(\beta_i^{(2)}-\frac{1}{\mu_i}\right), \\ \mathcal{N}_1''(1) &= 2\frac{\lambda_i}{\mu_i}, \\ \mathcal{N}_1'''(1) &= 6\frac{\lambda_i}{\mu_i} + 3\lambda_i^2\left(\beta_i^{(2)}-\frac{1}{\mu_i}\right), \end{aligned} \tag{25}$$

where $\beta_i^{(2)}$ is the 2nd moment of the service time. Substituting the above results into (24), we have

$$\begin{aligned} L &= \frac{\left(1-\overline{\lambda}_0\overline{\theta}\right)\overline{\theta}}{\theta^2\beta} + \frac{\left(1-\overline{\lambda}_0\overline{\theta}\right)}{\theta^2\beta} \\ &\quad \cdot \sum_{i=1}^{n} \frac{q_i\left[2\left(\mu_i-\lambda_i\right)\left(\theta+\lambda_0\overline{\theta}\right) + \theta\lambda_i\left(\mu_i^2\beta_i^{(2)}-\mu_i\right)\right]}{2\left(\mu_i-\lambda_i\right)^2}. \end{aligned} \tag{26}$$

4.2. Mean Time That the System Resides in Phase 0. We denote A as the interarrival time during a vacation period, which has geometrical distribution with parameter λ_0. Then, the probability that there is no customer arrival during a vacation period is given by

$$\begin{aligned} P(A > V) &= \sum_{k=1}^{\infty} P(V=k)\,P(A>k \mid V=k) \\ &= \frac{\overline{\lambda}_0\theta}{1-\overline{\lambda}_0\overline{\theta}}; \end{aligned} \tag{27}$$

that is to say, the probability that at least one customer arrives during a vacation period is $\lambda_0/(1-\overline{\lambda}_0\overline{\theta})$. We can easily find that the number of vacations in phase 0 is geometrically distributed with parameter $\lambda_0/(1-\overline{\lambda}_0\overline{\theta})$ and the mean time of each vacation is $1/\theta$. Hence, the mean time that the system resides in phase 0, denoted by T_0, is given by

$$T_0 = \sum_{k=1}^{\infty} \left(\frac{\overline{\lambda}_0\theta}{1-\overline{\lambda}_0\overline{\theta}}\right)^{k-1} \cdot \frac{\lambda_0}{1-\overline{\lambda}_0\overline{\theta}} \cdot \frac{k}{\theta} = \frac{1-\overline{\lambda}_0\overline{\theta}}{\lambda_0\theta}, \tag{28}$$

which is just the product of the mean number of vacations in phase 0 and the mean time of each vacation.

4.3. Mean Number of Customers in the System at the End of Phase 0. Let a_m be the probability that m customers arrive during a vacation V, and let b_m represent the probability that m customers arrive during phase 0. Then

$$\begin{aligned} a_m &= \sum_{k=m}^{\infty} \overline{\theta}^{k-1}\theta C_k^m \lambda_0^m \overline{\lambda}_0^{k-m}, \\ b_m &= \sum_{k=1}^{\infty} \left(\frac{\overline{\lambda}_0\theta}{1-\overline{\lambda}_0\overline{\theta}}\right)^{k-1} a_m = \frac{1-\overline{\lambda}_0\overline{\theta}}{\lambda_0} a_m. \end{aligned} \tag{29}$$

Then, the mean number of customers in the system at the end of phase 0, denoted by N_0, is given by

$$N_0 = \sum_{m=1}^{\infty} m \cdot b_m = \frac{1-\overline{\lambda}_0\overline{\theta}}{\lambda_0} \cdot \sum_{m=1}^{\infty} m \cdot a_m = \frac{1-\overline{\lambda}_0\overline{\theta}}{\theta}, \tag{30}$$

which is the product of the mean time that the system resides in phase 0, that is, T_0, and the mean number of customers that arrive to the system per unit time during phase 0, that is, λ_0.

4.4. Mean Time That the System Resides in Phase i. We denote T_i as the mean time that the system resides in phase i. Then

$$T_i = \sum_{m=1}^{\infty} b_m \frac{m}{\mu_i - \lambda_i} = \frac{1 - \overline{\lambda}_0 \overline{\theta}}{\theta(\mu_i - \lambda_i)}, \tag{31}$$

where $1/(\mu_i - \lambda_i)$ is the mean busy period that one customer induces in phase i. Actually, the mean time that the system resides in phase i is the product of the mean number of customers in the system at the end of phase 0 and the mean busy period that one customer induces in phase i.

4.5. Probability That the System Resides in Phase 0. From (8) and (21), the probability that the system resides in phase 0 is given by

$$P_0(1) = \frac{1 - \overline{\lambda}_0 \overline{\theta}}{\lambda_0 \theta \beta}. \tag{32}$$

Intuitively, in the time interval between two consecutive epochs at which phase 0 begins, the probability that the system resides in phase 0 is equal to the proportion of time the system resides in phase 0. Then, we have

$$\begin{aligned} P_0(1) &= \frac{\left(1 - \overline{\lambda}\,\overline{\theta}\right)/\lambda_0\theta}{\left(1 - \overline{\lambda}\,\overline{\theta}\right)/\lambda_0\theta + \sum_{i=1}^{n}\left(q_i\left(1 - \overline{\lambda}\,\overline{\theta}\right)/\theta(\mu_i - \lambda_i)\right)} \\ &= \frac{1 - \overline{\lambda}\,\overline{\theta}}{\lambda_0\theta\beta}, \end{aligned} \tag{33}$$

where $(1 - \overline{\lambda}\,\overline{\theta})/\lambda_0\theta$ is the mean time that the system resides in vacation phase and $\sum_{i=1}^{n}(q_i(1 - \overline{\lambda}\,\overline{\theta})/\theta(\mu_i - \lambda_i))$ is the mean time that the system resides in operative phase.

4.6. Probability That the System Resides in Phase i. From (19) and (21), the probability that the system resides in phase i is given by

$$P_i^*(1,1) = \frac{q_i\left(1 - \overline{\lambda}\,\overline{\theta}\right)}{\theta(\mu_i - \lambda_i)\beta}, \tag{34}$$

which can be intuitively explained in a way similar to the one described above for the probability that the system resides in phase 0.

4.7. Mean Length of the Type-i Cycle. The length of type-i cycle, denoted by C_i, is the length of time from the beginning of the last phase i to the beginning of the next phase i, $i = 0, 1, \ldots, n$. Then, there are $(n + 1)$ types of cycles. From (28) and (31), we can easily get the mean length of the type-0 cycle, denoted by $E(C_0)$, as equal to β. Now, we calculate the mean length of the type-i cycle, denoted by $E(C_i)$, $i = 1, 2, \ldots, n$.

We can easily find that the times of the system visiting phase 0 during type-i cycle are distributed with geometric distribution with parameter q_i, and the mean length of type-i cycle during which the system visits phase 0 k times is

$$\begin{aligned} &\frac{1 - \overline{\lambda}_0\overline{\theta}}{\lambda_0\theta} + \frac{1 - \overline{\lambda}_0\overline{\theta}}{\theta(\mu_i - \lambda_i)} + (k-1) \\ &\quad \cdot \left(\frac{1 - \overline{\lambda}_0\overline{\theta}}{\lambda_0\theta} + \sum_{j=1, j\neq i}^{n} \frac{q_j\left(1 - \overline{\lambda}_0\overline{\theta}\right)}{\theta(\mu_j - \lambda_j)(1 - q_i)}\right). \end{aligned} \tag{35}$$

Then, the mean length of the type-i cycle is given by

$$\begin{aligned} E(C_i) &= \sum_{k=1}^{\infty}(1 - q_i)^{k-1} q_i \left[\frac{1 - \overline{\lambda}_0\overline{\theta}}{\lambda_0\theta} + \frac{1 - \overline{\lambda}_0\overline{\theta}}{\theta(\mu_i - \lambda_i)}\right. \\ &\quad + (k-1) \\ &\quad \left. \cdot \left(\frac{1 - \overline{\lambda}_0\overline{\theta}}{\lambda_0\theta} + \sum_{j=1, j\neq i}^{n} \frac{q_j\left(1 - \overline{\lambda}_0\overline{\theta}\right)}{\theta(\mu_j - \lambda_j)(1 - q_i)}\right)\right] \\ &= \frac{1 - \overline{\lambda}_0\overline{\theta}}{\lambda_0\theta} + \frac{1 - \overline{\lambda}_0\overline{\theta}}{\theta(\mu_i - \lambda_i)} + \sum_{k=1}^{\infty}(k-1)(1 - q_i)^{k-1} \\ &\quad \cdot q_i \left[\frac{1 - \overline{\lambda}_0\overline{\theta}}{\lambda_0\theta} + \left(\beta - \frac{1 - \overline{\lambda}_0\overline{\theta}}{\lambda_0\theta} - \frac{q_i\left(1 - \overline{\lambda}_0\overline{\theta}\right)}{\theta(\mu_i - \lambda_i)}\right)\right. \\ &\quad \left. \cdot \frac{1}{1 - q_i}\right] = \frac{1 - \overline{\lambda}_0\overline{\theta}}{\lambda_0\theta} + \frac{1 - \overline{\lambda}_0\overline{\theta}}{\theta(\mu_i - \lambda_i)} \\ &\quad + \frac{1 - q_i}{q_i}\left[\frac{1 - \overline{\lambda}_0\overline{\theta}}{\lambda_0\theta}\frac{-q_i}{1 - q_i} + \left(\beta - \frac{q_i\left(1 - \overline{\lambda}_0\overline{\theta}\right)}{\theta(\mu_i - \lambda_i)}\right)\right. \\ &\quad \left. \cdot \frac{1}{1 - q_i}\right] = \frac{1 - \overline{\lambda}_0\overline{\theta}}{\theta(\mu_i - \lambda_i)} + \frac{1}{q_i}\left(\beta - \frac{q_i\left(1 - \overline{\lambda}_0\overline{\theta}\right)}{\theta(\mu_i - \lambda_i)}\right) \\ &= \frac{\beta}{q_i}. \end{aligned} \tag{36}$$

4.8. PGF of the Stationary Sojourn Time Distribution of an Arbitrary Customer. The stationary sojourn times of an arbitrary customer, a customer who arrives in the state $(m, 0)$ and a customer who arrives in the state (m, i, k), $i = 1, 2, \ldots, n$, respectively, are denoted by W, $W_{m,0}$, and $W_{m,i,k}$, with their respective PGFs $W(z)$, $W_{m,0}(z)$, and $W_{m,i,k}(z)$. We also define $W_{m,i}$ and $W_{m,i}(z)$, respectively, as the sum of the $(m + 1)$ customers' service times and its PGF, where $i = 1, 2, \ldots, n$.

When a customer arrives in state $(m, 0)$, $m = 0, 1, 2, \ldots$, his sojourn time until departure is the sum of the remaining vacation time and the $(m + 1)$ customers' service times. Because the vacation time is distributed with a geometrical

distribution, the remaining vacation time is identically distributed with vacation time. Then, we have

$$\begin{aligned} W_{m,0}(z) &= \sum_{i=1}^{n} q_i V(z) W_{m,i}(z) \\ &= \left[\sum_{i=1}^{n} q_i \left(S_i(z)\right)^{m+1}\right] \frac{\theta z}{1-\overline{\theta} z}, \end{aligned} \tag{37}$$

where $V(z)$ is the PGF of the vacation time distribution and $V(z) = \theta z/(1-\overline{\theta} z)$.

When a customer arrives in state (m, i, k), $m \geq 1$, $k \geq 1$, $i = 1, 2, \ldots$, the PGF of the customer's stationary sojourn time until departure is given by

$$W_{m,i,k}(z) = E\left(z^{k+W_{m-1,i}}\right) = z^k \left[S_i(z)\right]^m. \tag{38}$$

Hence, we have the PGF of the stationary sojourn time distribution of an arbitrary customer as follows:

$$\begin{aligned} &W(z) \\ &= \sum_{m=0}^{\infty} P_{m,0} W_{m,0}(z) + \sum_{i=1}^{n}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} P_{m,i}(k) W_{m,i,k}(z) \\ &= \sum_{m=0}^{\infty} \left(\frac{\lambda_0 \overline{\theta}}{1-\overline{\lambda}_0 \overline{\theta}}\right)^m P_{0,0} \left[\sum_{i=1}^{n} q_i \left(S_i(z)\right)^{m+1}\right] \frac{\theta z}{1-\overline{\theta} z} \\ &\quad + \sum_{i=1}^{n}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} P_{m,i}(k) z^k \left[S_i(z)\right]^m \\ &= \frac{\theta z P_{0,0}}{1-\overline{\theta} z} \left[\sum_{i=1}^{n} q_i \sum_{m=0}^{\infty} \left(\frac{\lambda_0 \overline{\theta}}{1-\overline{\lambda}_0 \overline{\theta}}\right)^m \left(S_i(z)\right)^{m+1}\right] \\ &\quad + \sum_{i=1}^{n} P_i^*\left(z, S_i(z)\right) \\ &= \left[\sum_{i=1}^{n} \frac{q_i S_i(z)}{1-\overline{\lambda}_0 \overline{\theta} - \lambda_0 \overline{\theta} S_i(z)}\right] \frac{\left(1-\overline{\lambda}_0 \overline{\theta}\right)\theta z}{\left(1-\overline{\theta} z\right)\lambda_0 \beta} \\ &\quad + \sum_{i=1}^{n} P_i^*\left(z, S_i(z)\right). \end{aligned} \tag{39}$$

5. Relation to the Continuous-Time System

In this section, we analyze the relationship between the discrete-time system and its continuous-time counterpart. The continuous-time system can be approximated by the discrete-time system. If we assume that time is slotted into intervals of constant length Δ, when Δ goes to zero, the approximation tends to the exact value.

We consider the continuous-time $M/G/1$ queue with vacations in multiphase random environment. In phase i, $i = 1, 2, \ldots, n$, customers arrive according to a Poisson stream with rate $\widetilde{\lambda}_i$ and service times are independent and identically distributed with general distribution function $B_i(x)$, LST $B_i^*(s)$, and finite means $1/\widetilde{\mu}_i$. In phase 0, customers arrive according to a Poisson stream with rate $\widetilde{\lambda}_0$. Vacation times are independent exponential random variables with parameter $\widetilde{\theta}$. If we assume time is slotted into intervals of constant length Δ, the continuous-time system can be approximated by a discrete-time system for which

$$\begin{aligned} &\lambda_0 = \widetilde{\lambda}_0 \Delta, \\ &\lambda_i = \widetilde{\lambda}_i \Delta, \\ &\mu_i = \widetilde{\mu}_i \Delta, \\ &\qquad\qquad i = 1, 2, \ldots, n, \\ &\theta = \widetilde{\theta} \Delta, \\ &s_{i,k} = \int_{(k-1)\Delta}^{k\Delta} dB_i(x), \quad i = 1, 2, \ldots, n, \ k \geq 1, \end{aligned} \tag{40}$$

where Δ is sufficiently small so that λ_0, λ_i, μ_i, and θ are probabilities.

Using the same technique with Yang and Li [23], it is not difficult to show that $\lim_{\Delta\to 0} P(z)$ is the probability generating function of the number of customers in the $M/G/1$ queue with vacations in multiphase random environment.

From (8), we have

$$\begin{aligned} \lim_{\Delta\to 0} P_0(z) &= \lim_{\Delta\to 0} \frac{1-\overline{\lambda}_0 \overline{\theta}}{1-\overline{\lambda}_0 \overline{\theta} - \lambda_0 \overline{\theta} z} P_{0,0} \\ &= \lim_{\Delta\to 0} \frac{1-\left(1-\widetilde{\lambda}_0 \Delta\right)\left(1-\widetilde{\theta}\Delta\right)}{1-\left(1-\widetilde{\lambda}_0 \Delta\right)\left(1-\widetilde{\theta}\Delta\right) - \widetilde{\lambda}_0 \Delta \left(1-\widetilde{\theta}\Delta\right) z} P_{0,0} \\ &= \lim_{\Delta\to 0} \frac{\widetilde{\lambda}_0 \Delta + \widetilde{\theta}\Delta - \widetilde{\lambda}_0 \widetilde{\theta} \Delta^2}{\widetilde{\lambda}_0 \Delta + \widetilde{\theta}\Delta - \widetilde{\lambda}_0 \widetilde{\theta} \Delta^2 - \widetilde{\lambda}_0 z \Delta + \widetilde{\lambda}_0 \widetilde{\theta} z \Delta^2} P_{0,0} \\ &= \lim_{\Delta\to 0} \frac{\widetilde{\lambda}_0 + \widetilde{\theta} - \widetilde{\lambda}_0 \widetilde{\theta}\Delta}{\widetilde{\lambda}_0 + \widetilde{\theta} - \widetilde{\lambda}_0 \widetilde{\theta}\Delta - \widetilde{\lambda}_0 z + \widetilde{\lambda}_0 \widetilde{\theta} z \Delta} P_{0,0}. \end{aligned} \tag{41}$$

Since

$$\lim_{\Delta\to 0} S_i'(1)\Delta = \lim_{\Delta\to 0} \sum_{k=1}^{\infty} k\Delta s_{i,k} = \lim_{\Delta\to 0} \sum_{k=1}^{\infty} k\Delta \left[B_i(k\Delta) - B_i((k-1)\Delta)\right] = \int_0^{\infty} x dB_i(x) = \frac{1}{\widetilde{\mu}_i},$$

$$\lim_{\Delta\to 0} P_{0,0} = \lim_{\Delta\to 0} \frac{1}{\lambda_0 \beta} = \lim_{\Delta\to 0} \frac{1}{\lambda_0 \left[\left(1-\overline{\lambda}_0 \overline{\theta}\right)/\lambda_0 \theta + \sum_{i=1}^{n} \left(q_i \left(1-\overline{\lambda}_0 \overline{\theta}\right)/\theta\left(\mu_i - \lambda_i\right)\right)\right]}$$

$$
\begin{aligned}
&= \lim_{\Delta\to 0} \frac{1}{\widetilde{\lambda}_0\Delta\left[\left(1-\left(1-\widetilde{\lambda}_0\Delta\right)\left(1-\widetilde{\theta}\Delta\right)\right)/\widetilde{\lambda}_0\Delta\widetilde{\theta}\Delta+\sum_{i=1}^{n}\left(q_i\left(1-\left(1-\widetilde{\lambda}_0\Delta\right)\left(1-\widetilde{\theta}\Delta\right)\right)/\widetilde{\theta}\Delta\left(1/S_i'(1)-\widetilde{\lambda}_i\Delta\right)\right)\right]} \\
&= \lim_{\Delta\to 0} \frac{1}{\left[\left(\widetilde{\lambda}_0+\widetilde{\theta}-\widetilde{\lambda}_0\widetilde{\theta}\Delta\right)/\widetilde{\theta}+\sum_{i=1}^{n}\left(q_i\widetilde{\lambda}_0\left(\widetilde{\lambda}_0+\widetilde{\theta}-\widetilde{\lambda}_0\widetilde{\theta}\Delta\right)/\widetilde{\theta}\left(1/S_i'(1)\Delta-\widetilde{\lambda}_i\right)\right)\right]} \\
&= \lim_{\Delta\to 0} \frac{1}{\left[\left(\widetilde{\lambda}_0+\widetilde{\theta}-\widetilde{\lambda}_0\widetilde{\theta}\Delta\right)/\widetilde{\theta}+\sum_{i=1}^{n}\left(q_i\widetilde{\lambda}_0\left(\widetilde{\lambda}_0+\widetilde{\theta}-\widetilde{\lambda}_0\widetilde{\theta}\Delta\right)/\widetilde{\theta}\left(\widetilde{\mu}_i-\widetilde{\lambda}_i\right)\right)\right]} \\
&= \frac{1}{\left[\left(\widetilde{\lambda}_0+\widetilde{\theta}\right)/\widetilde{\theta}+\sum_{i=1}^{n}\left(q_i\widetilde{\lambda}_0\left(\widetilde{\lambda}_0+\widetilde{\theta}\right)/\widetilde{\theta}\left(\widetilde{\mu}_i-\widetilde{\lambda}_i\right)\right)\right]} = \widetilde{P}_{0,0},
\end{aligned}
\tag{42}
$$

we obtain

$$
\lim_{\Delta\to 0} P_0(z) = \frac{\widetilde{\lambda}_0+\widetilde{\theta}}{\widetilde{\lambda}_0(1-z)+\widetilde{\theta}}\widetilde{P}_{0,0}. \tag{43}
$$

Since

$$
\begin{aligned}
\lim_{\Delta\to 0} S_i(\alpha_i(z)) &= \lim_{\Delta\to 0} S_i\left(\overline{\lambda}_i+\lambda_i z\right) \\
&= \lim_{\Delta\to 0}\sum_{k=1}^{\infty}\left(1-\widetilde{\lambda}_i\Delta+\widetilde{\lambda}_i\Delta z\right)^k \\
&\quad\cdot\left[B_i(k\Delta)-B_i((k-1)\Delta)\right] \\
&= \lim_{\Delta\to 0}\sum_{k=1}^{\infty}\left\{\left[1-\widetilde{\lambda}_i(1-z)\Delta\right]^{1/\Delta}\right\}^{k\Delta} \\
&\quad\cdot\left[B_i(k\Delta)-B_i((k-1)\Delta)\right] = \int_0^{\infty} e^{-\widetilde{\lambda}_i(1-z)x}dB_i(x) \\
&= B_i^*\left(\widetilde{\lambda}_i(1-z)\right),
\end{aligned}
\tag{44}
$$

we get

$$
\begin{aligned}
\lim_{\Delta\to 0} P_i^*(1,z) &= \lim_{\Delta\to 0}\frac{\left(1-\overline{\lambda}_0\overline{\theta}\right)\lambda_0 q_i z(1-z)\left[1-S_i(\alpha_i(z))\right]}{(1-\alpha_i(z))\left[S_i(\alpha_i(z))-z\right]\left(1-\overline{\lambda}_0\overline{\theta}-\lambda_0\overline{\theta}z\right)}P_{0,0} \\
&= \lim_{\Delta\to 0}\frac{\left(\widetilde{\lambda}_0\Delta+\widetilde{\theta}\Delta-\widetilde{\lambda}_0\widetilde{\theta}\Delta^2\right)\widetilde{\lambda}_0\Delta q_i z(1-z)\left[1-S_i(\alpha_i(z))\right]}{\left(\widetilde{\lambda}_i\Delta-\widetilde{\lambda}_i\Delta z\right)\left[S_i(\alpha_i(z))-z\right]\left(\widetilde{\lambda}_0\Delta+\widetilde{\theta}\Delta-\widetilde{\lambda}_0\widetilde{\theta}\Delta^2-\widetilde{\lambda}_0 z\Delta+\widetilde{\lambda}_0\widetilde{\theta}z\Delta^2\right)}P_{0,0} \\
&= \frac{\left(\widetilde{\lambda}_0+\widetilde{\theta}\right)\widetilde{\lambda}_0 q_i z\left[1-B_i^*\left(\widetilde{\lambda}_i(1-z)\right)\right]}{\widetilde{\lambda}_i\left(\widetilde{\lambda}_0+\widetilde{\theta}-\widetilde{\lambda}_0 z\right)\left[B_i^*\left(\widetilde{\lambda}_i(1-z)\right)-z\right]}\widetilde{P}_{0,0} = \widetilde{P}_i(z).
\end{aligned}
\tag{45}
$$

From (43) and (45), we obtain the PGF of the number of customers in the $M/G/1$ queue with vacations in multiphase random environment as follows:

$$
\begin{aligned}
\widetilde{P}(z) &= \widetilde{P}_0(z)+\sum_{i=1}^{n}\widetilde{P}_i(z) \\
&= \frac{\left(\widetilde{\lambda}_0+\widetilde{\theta}\right)\widetilde{P}_{0,0}}{\widetilde{\lambda}_0(1-z)+\widetilde{\theta}} \\
&\quad+\frac{\left(\widetilde{\lambda}_0+\widetilde{\theta}\right)\widetilde{\lambda}_0 z\widetilde{P}_{0,0}}{\widetilde{\lambda}_0(1-z)+\widetilde{\theta}}\sum_{i=1}^{n}\frac{q_i\left[1-B_i^*\left(\widetilde{\lambda}_i(1-z)\right)\right]}{\widetilde{\lambda}_i\left[B_i^*\left(\widetilde{\lambda}_i(1-z)\right)-z\right]}.
\end{aligned}
\tag{46}
$$

6. Special Cases

6.1. The Geo/G/1 Queue with Vacations. When the system is homogeneous, that is, $\lambda_i = \lambda$ and $\mu_i = \mu$, the model translates into a regular $Geo/G/1$ queue with multiple vacations. From (23), $P(z)$ reduces to

$$
P(z) = \frac{\theta(1-\rho)(1-z)S(\alpha(z))}{\left[1-\overline{\theta}\alpha(z)\right]\left[S(\alpha(z))-z\right]}, \tag{47}
$$

which is the PGF of the number of customers in the regular $Geo/G/1$ queue with multiple vacations, where $S(z)$ is the PGF of the service time distribution, $\alpha(z) = \overline{\lambda} + \lambda z$.

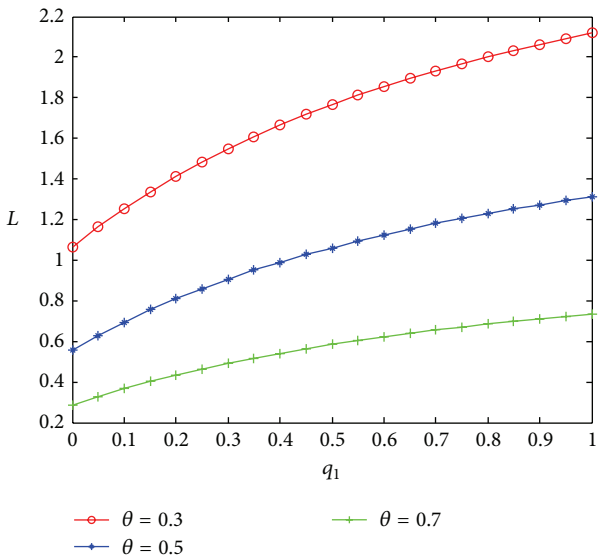

Figure 1: The effect of q_1 on L.

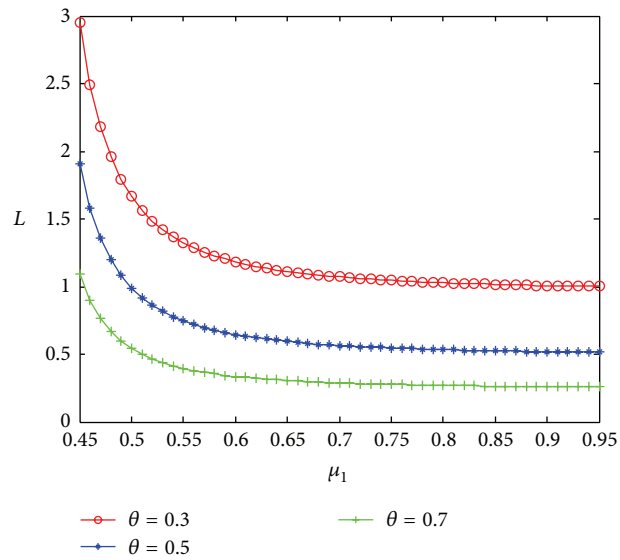

Figure 2: The effect of μ_1 on L.

Expression (47) is in agreement with that in Tian and Zhang [3].

6.2. The Geo/Geo/1 Queue with Vacations in Random Environment. Suppose that the service times in phase i, $i = 1, 2, \ldots, n$, follow the geometric distribution with finite means $1/\mu_i$ in the model; then $S_i(z) = \mu_i z/(1-\overline{\mu}_i z)$ in (23), and (26) yields

$$L = \frac{\left(1-\overline{\lambda}_0\overline{\theta}\right)\overline{\theta}}{\theta^2\beta} + \frac{\left(1-\overline{\lambda}_0\overline{\theta}\right)}{\theta^2\beta}\sum_{i=1}^{n}\frac{q_i\left[\left(\mu_i-\lambda_i\right)\left(\theta+\lambda_0\overline{\theta}\right)+\theta\lambda_i\overline{\mu}_i\right]}{\left(\mu_i-\lambda_i\right)^2}, \tag{48}$$

which is the mean number of customers in the $Geo/Geo/1$ queue with vacations in random environment.

7. Numerical Examples

To observe the effect of the different parameters on the main performance measures, we present some numerical examples in this section. We consider the $Geo/Geo/1$ queue with vacations in random environment, and we assume $n = 2$; that is to say, the system has three phases. We set $\lambda_0 = 0.3$, $\lambda_1 = 0.4$, $\lambda_2 = 0.5$, $\mu_1 = 0.5$, $\mu_2 = 0.8$, $q_1 = 0.4$, $q_2 = 0.6$, and $\theta = 0.3, 0.5, 0.7$, unless they are considered as variables or their values are given in the respective figures.

In Figure 1, we study the influence of q_1 on L for different vacation rate θ. As intuition tells us, the mean number of customers in the system L increases with increasing values of q_1 for any θ. On the other hand, for a fixed q_1, L decreases with the increase of θ. Intuitively, as θ increases, the mean time of vacation decreases; then there are fewer customers in the system. And the behavior of L with respect to μ_1 is displayed in Figure 2. As to be expected, when μ_1 increases the mean number of customers in the system L decreases for any θ. If μ_1 is fixed, L decreases with the increase of θ.

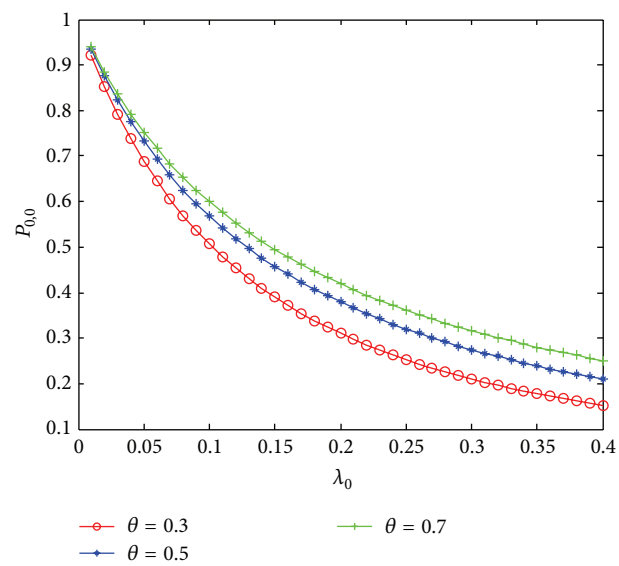

Figure 3: The effect of λ_0 on $P_{0,0}$.

The influence of arrival rate λ_0 on the probability that the system is empty $P_{0,0}$ is illustrated in Figure 3. It is observed that $P_{0,0}$ decreases as λ_0 increases, which also agrees with the intuitive expectations. In addition, for a fixed λ_0, $P_{0,0}$ increases with the increase of θ. Figure 4 describes the influence of μ_1 on the probability that the system is empty $P_{0,0}$. $P_{0,0}$ increases with the increase of μ_1, and, for a fixed μ_1, $P_{0,0}$ increases with the increase of θ, as we expected.

8. Conclusion

In this paper, we consider a $Geo/G/1$ queue with vacations in random environment. For this model, we derive

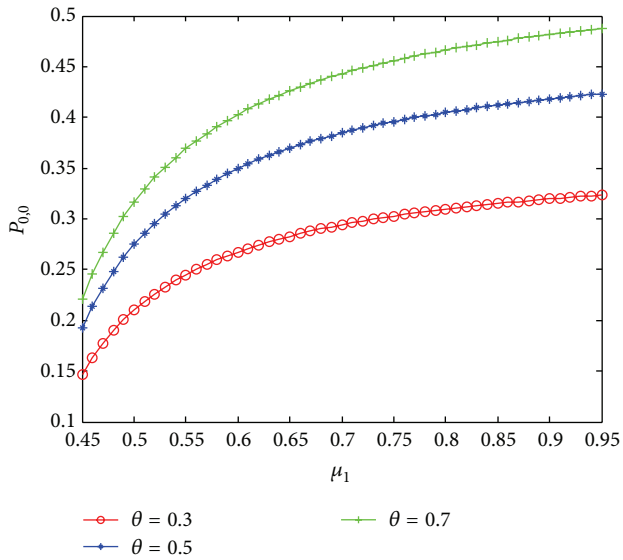

FIGURE 4: The effect of μ_1 on $P_{0,0}$.

the probability generating function of the number of customers in the system. Various system performance characteristics are obtained. And we show that the $M/G/1$ queue with vacations in random environment can be approximated by its discrete-time counterpart. We also perform some numerical examples to demonstrate the effect of various parameters on the performance characteristics.

Competing Interests

The authors declare that they have no competing interests.

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