Research Article

Dynamic Complexities in 2-Dimensional Discrete-Time Predator-Prey Systems with Allee Effect in the Prey

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The Allee effect is incorporated into a predator-prey model with linear functional response. Compared with the predator-prey which only takes the crowding effect and predator partially dependent on prey into consideration, it is found that the Allee effect of the prey species would increase the extinction risk of both the prey and predator. Moreover, by using a center manifold theorem and bifurcation theory, it is shown that the model with Allee effect undergoes the flip bifurcation and Hopf bifurcation in the interior of R_+^2 with different Allee effect values. In the two bifurcations, we can come to the conclusion that different Allee effect will have different bifurcation value and the increasing of the Allee effect will increase the value of bifurcation, respectively.

1. Introduction

The predator-prey interaction is a topic of great interest for many ecologists and mathematics. Such interactions have motivated various models of phenotypic coevolution in prey-predator communities. Many researchers have studied the dynamical behavior of the predator-prey system in ecology and contributed to growth of continuous models for large size population [1–13]. Also, a number of researchers have established the discrete-time models which are more appropriate and provided more efficient results as compared to the continuous models for small size population.

In [14], the author has investigated the stability and bifurcation of discrete-time predator-prey system with crowding effect of predator where predator is partially dependent on prey, this can be described by the following system (3). By analyzing, the author got the following conclusion: the systems undergoes flip bifurcation and Hopf bifurcation at fixed points under specific conditions when δ varies in small neighborhood. And numerical simulations of the model display cascade of period-doubling bifurcation in orbits of periods 2, 4, 8, chaotic sets and stable window of period 9 orbit in case of flip bifurcation, and smooth invariant circles in case of Hopf bifurcation. In biology, it means that the

predator coexists with prey at period- n orbits and smooth invariant circle.

Now, on the basis of previous systems, we make an improvement in it. We assume the prey with Allee effect and use center manifold theorem and bifurcation theory to study the system with Allee effect.

The concept of Allee effect originates from observations made by an American ecologist Allee in the 1930s [15]. In the 1930s, Allee found that factors affecting the growth rate of single group include the resources space constraints and density of the initial population density. When the initial density of the population is lower than a certain critical value, population will eventually become extinct because of the restriction of the conditions or other conditions of the population reproduction, which is called Allee effect. This effect is probably one of the most exciting concepts in ecology, a concept that has long been neglected but now has attracted renewed interest. The most usual continuous growth function considering Allee effect is expressed by [16–19]

$$\frac{dx}{dt} = rx \left(1 - \frac{x}{k} \right) (x - m), \quad (1)$$

where x denotes the population density; k denotes the carrying capacity; r denotes the intrinsic per capita growth

rate of the population. In (1), when $m > 0$, it is the strong Allee effect. when $-k < m \leq 0$, it is the weak Allee effect. Weak Allee effect growth function always remains positive, while in the strong Allee effect, there is a threshold population level m below which the growth is negative and the species will become extinct.

Other mathematical forms have been proposed to model the Allee effect [16, 20–23] and most of them are topologically equivalent [24]. For instance, in [25, 26] the following equation is proposed:

$$\frac{dx}{dt} = r \left(1 - \frac{x}{k} - \frac{n}{x+b} \right), \quad (2)$$

which has been called additive Allee effect; (2) allows the modelling of multiple Allee effect [20]. Here, n and b are Allee effect parameter.

In our paper, we consider the following system:

$$\begin{aligned} \dot{x}(t) &= ax \left(1 - \frac{x}{k} \right) - \frac{bxy}{x+l}, \\ \dot{y}(t) &= cy + \frac{mbxy}{x+l} - dy^2, \end{aligned} \quad (3)$$

where all the parameters are positive. $x(t)$ and $y(t)$ represent the densities of prey and predator population, respectively; a , b denote the intrinsic growth rate of prey and predator, respectively; c denotes the growth rate of predator due to alternative resources. Again, d denotes competition among individuals of predator species due to overcrowding; l denotes the half saturation constant; m denotes the conversion rate for predator; and k denotes carrying capacity of the prey in a particular habitat.

In this prey-predator system, due to the impact of their own breeding process or because of the prey itself, the prey population will have the Allee effect. By observing some signs, single species of plant and animal which are influenced by Allee effect are very wide. Many facts point out the fact that the prey of the growth rate with Allee effect will fall to very low but remain positive. So, Allee effect provides an important basis for the management of endangered species. Without the Allee effect, the evolution of the system is a sublimite situation, the time to reach the p point of the system is less than that of the system without Allee system which is not conducive to the evolution of biological, under the influence of the Allee effect, and the evolution of the system will reach a state of equilibrium. In our paper, it is shown that Allee effect of the prey species has a significant effect on the dynamics of the predator-prey system with linear functional response. Many studies have demonstrated the potential importance of Allee effect for dynamics of small population, range expansion, community composition, and biological invasion. It is widely accepted that Allee effect may increase the extinction risk of low density populations.

Because of the difficulties in finding mates when the prey population density becomes low, the Allee effect may occur in the prey species. So, in this paper, we consider the following Allee effect [27]. For example, this might correspond to the density below which it is so difficult to find a mate that

reproduction does not compensate for mortality. Let $f(x)$ be the fertility of a species that has x adults in an isolated patch; then the fertility increases with the population density, which is described by

$$f(x) = \frac{ax}{A_1 + x}, \quad (4)$$

where a is the maximum per capita birth rate of the species, $x/(A_1 + x)$ is the per capita fertilization probability of the species, and A_1 is the Allee effect constant of the species. Here, if $A_1 > 0$, it is strong Allee effect; the fertility of the species will be therefore zero when x is zero and approaches to a when x becomes very large. Then, $f(x)$ increases with x , which depends on the parameter A_1 . Biologically, A_1 is the population density at which a species reaches half its maximum fertility. If $A_1 < 0$, it is the weak Allee effect. When $A_1 = 0$, the fertility is density independent with $f(x) = a$.

Now, we apply forward Euler's scheme to the system of (3); we choose a time step size $\delta > 0$. For any $t \geq 0$, let $x_t = x(t)$, $y_t = y(t)$, $x_{t+1} = x(t + \delta)$, and $y_{t+1} = y(t + \delta)$, since

$$\begin{aligned} \dot{x} &= \lim_{\delta \rightarrow 0} \frac{x_{t+1} - x_t}{\delta}, \\ \dot{y} &= \lim_{\delta \rightarrow 0} \frac{y_{t+1} - y_t}{\delta}. \end{aligned} \quad (5)$$

We can assume that for small enough $\delta > 0$

$$\begin{aligned} \dot{x} &= \frac{x_{t+1} - x_t}{\delta}, \\ \dot{y} &= \frac{y_{t+1} - y_t}{\delta}. \end{aligned} \quad (6)$$

Thus, from system (1), we get the discrete-time system as

$$\begin{aligned} x &\longrightarrow x + \delta \left[ax \left(1 - \frac{x}{k} \right) - \frac{bxy}{x+l} \right], \\ y &\longrightarrow y + \delta \left(cy + \frac{mbxy}{x+l} - dy^2 \right), \end{aligned} \quad (7)$$

where δ is the step size. Shrinking the step size in Euler's method will yield numerical solutions that more accurately approximate the true solution.

So, when the prey species is subject to Allee effect

$$f(x) = \frac{ax}{A_1 + x}, \quad (8)$$

the system of (7) can transform the following system:

$$\begin{aligned} x &\longrightarrow x + \delta x \left[\left(1 - \frac{x}{k} \right) \frac{ax}{A_1 + x} - \frac{by}{x+l} \right], \\ y &\longrightarrow y + \delta y \left(c + \frac{mbx}{x+l} - dy \right), \end{aligned} \quad (9)$$

where A_1 is Allee effect threshold value, if $A_1 > 0$, the Allee effect will be strong, and if $A_1 < 0$, the Allee effect will be weak. Here A_1 is a positive parameter denoting Allee

effect imposed on the prey population. Next, we investigate model (9) by way of mathematical analysis and numerical simulations.

Keeping in view of the literature, the rest of the paper is organized as follows: Section 2 deals with the stability criterion of the system at different fixed points in interior of R_+^2 . In Section 3, we obtain the specific conditions of existence of flip bifurcation and Hopf bifurcation. Numerical simulations are presented to support our analytical findings in Section 4, especially for the period doubling bifurcation and chaotic behavior. Finally a brief conclusion is given in the last section.

2. Stability of Fixed Points

The fixed points of system (9) are $p_0(0, 0)$, $p_1(k, 0)$, $p_2(0, c/d)$, and $p_3(x^*, y^*)$, where x^*, y^* satisfy

$$\begin{aligned} a \left(1 - \frac{x^*}{k} \right) \frac{x^*}{A_1 + x^*} - \frac{by^*}{x^* + l} &= 0, \\ c + \frac{mbx^*}{x^* + l} - dy^* &= 0. \end{aligned} \quad (10)$$

The Jacobian matrix of (9) at the fixed point (x, y) is written as

$$J = \begin{pmatrix} 1 + \delta \left[\frac{ax^2 + 2A_1ax}{(A_1 + x)^2} - \frac{2ax^3 + 3A_1ax^2}{k(A_1 + x)^2} - \frac{bly}{(x + l)^2} \right] & -\frac{\delta bx}{x + l} \\ \frac{\delta mbly}{(x + l)^2} & 1 + \delta \left(c + \frac{mbx}{x + l} - 2dy \right) \end{pmatrix}. \quad (11)$$

The characteristic equation of the Jacobian matrix is given by

$$\lambda^2 + p(x, y)\lambda + q(x, y) = 0, \quad (12)$$

where

$$\begin{aligned} p(x, y) &= -\text{tr } J = -2 - \delta \left[\frac{ax^2 + 2A_1ax}{(A_1 + x)^2} - \frac{2ax^3 + 3A_1ax^2}{k(A_1 + x)^2} - \frac{bly}{(x + l)^2} + c + \frac{mbx}{x + l} - 2dy \right], \\ q(x, y) &= \det J = 1 + \delta \left[\frac{ax^2 + 2A_1ax}{(A_1 + x)^2} - \frac{2ax^3 + 3A_1ax^2}{k(A_1 + x)^2} - \frac{bly}{(x + l)^2} \right] \left[1 + \delta \left(c + \frac{mbx}{x + l} - 2dy \right) \right] + \frac{\delta^2 mb^2 lxy}{(x + l)^3}. \end{aligned} \quad (13)$$

Clearly, we can see the Jacobian matrix of (9) at the point $p_0(0, 0)$ is

$$J = \begin{pmatrix} 1 & 0 \\ 0 & 1 + \delta c \end{pmatrix}. \quad (14)$$

So, $\lambda_1 = 1$, $\lambda_2 = 1 + \delta c$; then $p_0(0, 0)$ is unstable.

Now, we state a definition and a lemma same as in [28, 29].

Definition 1. A fixed point (x, y) is called

- (i) sink if $|\lambda_1| < 1$ and $|\lambda_2| < 1$, and it is asymptotically stable;
- (ii) source if $|\lambda_1| > 1$ and $|\lambda_2| > 1$, and it is locally unstable;

(iii) saddle if $|\lambda_1| > 1$ and $|\lambda_2| < 1$ (or $|\lambda_1| < 1$ and $|\lambda_2| > 1$);

(iv) nonhyperbolic if either $|\lambda_1| = 1$ or $|\lambda_2| = 1$.

Lemma 2. Let $F(\lambda) = \lambda^2 + P\lambda + Q$. Suppose that $F(1) > 0$ and λ_1, λ_2 are roots of $F(\lambda) = 0$. Then, one has the following:

- (i) $|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $F(-1) > 0$ and $Q < 1$;
- (ii) $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$) if and only if $F(-1) < 0$;
- (iii) $|\lambda_1| > 1$ and $|\lambda_2| > 1$ if and only if $F(-1) > 0$ and $Q > 1$;
- (iv) $\lambda_1 = -1$ and $|\lambda_2| \neq 1$ if and only if $F(-1) = 0$ and $P \neq 0, 2$;
- (v) $|\lambda_1|$ and $|\lambda_2|$ are complex and $|\lambda_1| = -1$, $|\lambda_2| = -1$ if and only if $P^2 - 4Q < 0$ and $Q = 1$.

Proposition 3. The fixed point $p_1(k, 0)$ is source if $\delta > 2/a + 2A_1/ak$, saddle if $0 < \delta < 2/a + 2A_1/ak$, and nonhyperbolic if $\delta = 2/a + 2A_1/ak$. In this condition, we assume $A_1 > -k$. That is, if $-k < A_1 < 0$, it is weak Allee effect, and if $A_1 > 0$, it is strong Allee effect.

One can see that when $\delta = 2/a + 2A_1/ak$, one of the eigen values of fixed point $p_1(k, 0)$ is -1 and magnitude of other is not equal to 1. Thus the flip bifurcation occurs when a parameter changes in small neighborhood of $\delta = 2/a + 2A_1/ak$.

Proposition 4. There exist different topological types of $p_2(0, c/d)$ for possible parameters.

- (i) $p_2(0, c/d)$ is sink if $0 < \delta < \min\{2/c, 2dl/bc\}$.
- (ii) $p_2(0, c/d)$ is source if $\delta > \max\{2/c, 2dl/bc\}$.
- (iii) $p_2(0, c/d)$ is nonhyperbolic if $\delta = 2/c$ or $\delta = 2dl/bc$.

(iv) $p_2(0, c/d)$ is saddle for all value of the parameters, except for values which lie in (i) to (iii).

In this condition, we can see that the system at p_2 is not affected by the Allee effect threshold value; that is, bifurcation parameter δ has nothing to do with A_1 .

The term (iii) of Proposition 4 implies that the parameters lie in the set

$$F_A = \left\{ (a, b, c, d, k, l, m, \delta, A_1), \delta = \frac{2}{c}, \delta \neq \frac{2dl}{bc}, a, b, c, d, k, l, m, \delta, A_1 > 0 \right\}. \quad (15)$$

If the term (iii) of Proposition 4 holds, then one of the eigen values of fixed points $p_2(0, c/d)$ is -1 and the magnitude of the other is not equal to 1. The point $p_2(0, c/d)$ undergoes flip bifurcation when the parameter changes in small neighborhood of F_A .

Last condition, the characteristic equation of the Jacobian matrix J of system (9) at the fixed point $p_3(x^*, y^*)$ is written as

$$\lambda^2 + p(x^*, y^*)\lambda + q(x^*, y^*) = 0, \quad (16)$$

where

$$p(x^*, y^*) = -2 - G\delta,$$

$$q(x^*, y^*) = 1 + G\delta + H\delta^2,$$

$$G = \frac{ax^{*2} + 2A_1ax^*}{(A_1 + x^*)^2} - \frac{2ax^{*3} + 3A_1ax^{*2}}{k(A_1 + x^*)^2} - \frac{bly^*}{(x^* + l)^2} + c + \frac{mbx^*}{x^* + l} - 2dy^*, \quad (17)$$

$$H = \left[\frac{ax^{*2} + 2A_1ax^*}{(A_1 + x^*)^2} - \frac{2ax^{*3} + 3A_1ax^{*2}}{k(A_1 + x^*)^2} - \frac{bly^*}{(x^* + l)^2} \right] \left[c + \frac{mbx^*}{x^* + l} - 2dy^* \right] + \frac{mb^2lx^*y^*}{(x^* + l)^3}.$$

Now

$$F(\lambda) = \lambda^2 - (2 + G\delta)\lambda + (1 + G\delta + H\delta^2). \quad (18)$$

Therefore

$$F(1) = H\delta^2, \quad (19)$$

$$F(-1) = 4 + 2G\delta + H\delta^2.$$

Using Lemma 2, we get the following proposition.

Proposition 5. *There exist different topological types of $p_3(x^*, y^*)$ for all possible parameters.*

(i) $p_3(x^*, y^*)$ is a sink if either condition (1) or (2) holds:

- (1) $G^2 - 4H \geq 0$ and $0 < \delta < (-G - \sqrt{G^2 - 4H})/H$,
- (2) $G^2 - 4H < 0$ and $0 < \delta < -G/H$.

(ii) $p_3(x^*, y^*)$ is a source if either condition (1) or (2) holds:

- (1) $G^2 - 4H \geq 0$ and $\delta > (-G + \sqrt{G^2 - 4H})/H$,
- (2) $G^2 - 4H < 0$ and $\delta > -G/H$.

(iii) $p_3(x^*, y^*)$ is nonhyperbolic if either condition (1) or (2) holds:

- (1) $G^2 - 4H \geq 0$ and $\delta = (-G \pm \sqrt{G^2 - 4H})/H$,
- (2) $G^2 - 4H < 0$ and $\delta = -G/H$.

(iv) $p_3(x^*, y^*)$ is saddle for all values of the parameters, except for values which lie in (i) to (iii).

If the term (1) in (iii) of Proposition 5 holds, then one of the eigen values of the fixed point $p_3(x^*, y^*)$ is -1 and the magnitude of the other is not equal to 1. The term (1) in (iii) of Proposition 5 may be written as follows:

$$F_{B1} = \left\{ (a, b, c, d, k, l, m, \delta, A_1) : \delta = \frac{-G - \sqrt{G^2 - 4H}}{H}, G^2 - 4H \geq 0, a, b, c, d, k, l, m, \delta, A_1 > 0 \right\}, \quad (20)$$

$$F_{B2} = \left\{ (a, b, c, d, k, l, m, \delta, A_1) : \delta = \frac{-G + \sqrt{G^2 - 4H}}{H}, G^2 - 4H \geq 0, a, b, c, d, k, l, m, \delta, A_1 > 0 \right\}.$$

Then a flip bifurcation will occur in the neighborhood of F_{B1} or F_{B2} .

If the term (2) in (iii) of Proposition 5 holds, then the eigen values of the fixed point $p_3(x^*, y^*)$ are a pair of complex conjugate numbers with modulus 1. The term (2) in (iii) of Proposition 5 may be written as follows:

$$H_B = \left\{ (a, b, c, d, k, l, m, \delta, A_1) : \delta = -\frac{G}{H}, G^2 - 4H < 0, a, b, c, d, k, l, m, \delta, A_1 > 0 \right\}. \quad (21)$$

One can see that a Hopf bifurcation will occur in the neighborhood of H_B .

3. Bifurcation Behavior

In this section, we study the flip bifurcation and Hopf bifurcation at the fixed point $p_3(x^*, y^*)$.

3.1. Flip Bifurcation. We consider system (9) with arbitrary parameter $(a, b, c, d, k, l, m, \delta_1, A_1) \in F_{B1}$, which is described as follows:

$$\begin{aligned} x &\longrightarrow x + \delta_1 x \left[\left(1 - \frac{x}{k}\right) \frac{ax}{A_1 + x} - \frac{by}{x + l} \right], \\ y &\longrightarrow y + \delta_1 y \left(c + \frac{mbx}{x + l} - dy \right), \end{aligned} \quad (22)$$

$p_3(x^*, y^*)$ is fixed point of system (22), where x^*, y^* are given by (9), and

$$\delta_1 = \frac{-G - \sqrt{G^2 - 4H}}{H}. \quad (23)$$

The eigen values of $p_3(x^*, y^*)$ are $\lambda_1 = -1$, $\lambda_2 = 3 + G\delta_1$, with $|\lambda_2| \neq 1$ by Proposition 5.

Consider the perturbation of (22) as below:

$$\begin{aligned} x &\longrightarrow x + (\delta_1 + \delta^*) x \left[\left(1 - \frac{x}{k}\right) \frac{ax}{A_1 + x} - \frac{by}{x + l} \right], \\ y &\longrightarrow y + (\delta_1 + \delta^*) y \left(c + \frac{mbx}{x + l} - dy \right), \end{aligned} \quad (24)$$

where $|\delta^*| \ll 1$ is a limited perturbation parameter.

Let $u = x - x^*$ and $v = y - y^*$.

After the transformation of the fixed point $p_3(x^*, y^*)$ of system (24) to the point $(0, 0)$, we obtain

$$\begin{pmatrix} u \\ v \end{pmatrix} \longrightarrow \begin{pmatrix} a_{11}u + a_{12}v + a_{13}u^2 + a_{14}uv + b_{11}\delta^*u + b_{12}\delta^*v + b_{13}\delta^*u^2 + b_{14}\delta^*uv + O(|u|, |v|, |\delta^*|)^3 \\ a_{21}u + a_{22}v + a_{23}u^2 + a_{24}uv + a_{25}v^2 + b_{21}\delta^*u + b_{22}\delta^*v + b_{23}\delta^*u^2 + b_{24}\delta^*uv + b_{25}\delta^*v^2 + O(|u|, |v|, |\delta^*|)^3 \end{pmatrix}, \quad (25)$$

where

$$\begin{aligned} a_{11} &= 1 + \delta_1 \left[\frac{(2A_1 + x^*)ax^*}{(A_1 + x^*)^2} - \frac{(3A_1x^{*2} + 2x^{*3})a}{k(A_1 + x^*)^2} \right. \\ &\quad \left. - \frac{by^*}{x^* + l} + \frac{bx^*y^*}{(x^* + l)^2} \right], \\ a_{12} &= -\frac{\delta_1 bx^*}{x^* + l}, \\ a_{13} &= \delta_1 \left[\frac{aA_1^2}{(A_1 + x^*)^3} - \frac{(ax^{*3} + 3A_1x^* + 3A_1^2x^*)}{k(A_1 + x^*)^3} \right. \\ &\quad \left. + \frac{by^*}{(x^* + l)^2} - \frac{bx^*y^*}{(x^* + l)^3} \right], \\ a_{14} &= \delta_1 \left[\frac{bx^*}{(x^* + l)^2} - \frac{b}{x^* + l} \right], \\ a_{15} &= \frac{(2A_1 + x^*)ax^*}{(A_1 + x^*)^2} - \frac{(3A_1x^{*2} + 2x^{*3})a}{k(A_1 + x^*)^2} \\ &\quad - \frac{by^*}{x^* + l} + \frac{bx^*y^*}{(x^* + l)^2}, \\ a_{16} &= -\frac{bx^*}{x^* + l}, \\ a_{17} &= \frac{aA_1^2}{(A_1 + x^*)^3} - \frac{ax^{*3} + 3A_1x^* + 3A_1^2x^*}{k(A_1 + x^*)^3} \end{aligned}$$

$$\begin{aligned} &+ \frac{by^*}{(x^* + l)^2} - \frac{bx^*y^*}{(x^* + l)^3}, \\ b_{14} &= \frac{bx^*}{(x^* + l)^2} - \frac{b}{x^* + l}, \\ a_{21} &= \frac{\delta_1 mbly^*}{(x^* + l)^2}, \\ a_{22} &= 1 - \delta_1 dy^*, \\ a_{23} &= -\frac{\delta_1 mbly^*}{(x^* + l)^3}, \\ a_{24} &= \frac{\delta_1 mbl}{(x^* + l)^2}, \\ a_{25} &= -\delta_1 d, \\ b_{21} &= \frac{mbly^*}{(x^* + l)^2}, \\ b_{22} &= -dy^*, \\ b_{23} &= -\frac{mbly^*}{(x^* + l)^3}, \\ b_{24} &= \frac{mbl}{(x^* + l)^2}, \\ b_{25} &= -d. \end{aligned} \quad (26)$$

Consider the following translation:

$$\begin{pmatrix} u \\ v \end{pmatrix} = T \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}, \quad (27)$$

where

$$T = \begin{pmatrix} a_{11} & a_{12} \\ -1 - a_{11} & \lambda_2 - a_{11} \end{pmatrix},$$

$$T^{-1} = \begin{pmatrix} \frac{\lambda_2 - a_{11}}{a_{12}(\lambda_2 + 1)} & -\frac{1}{\lambda_2 + 1} \\ \frac{1 + a_{11}}{a_{12}(\lambda_2 + 1)} & \frac{1}{\lambda_2 + 1} \end{pmatrix}, \quad (28)$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = T \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ -1 - a_{11} & \lambda_2 - a_{11} \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}.$$

Taking T^{-1} on both sides of (16), we get

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} + \begin{pmatrix} f(u, v, \delta^*) \\ g(u, v, \delta^*) \end{pmatrix}, \quad (29)$$

where

$$f(u, v, \delta^*) = \frac{[a_{13}(\lambda_2 - a_{11}) - a_{12}a_{23}]u^2}{a_{12}(\lambda_2 + 1)} + \frac{[a_{14}(\lambda_2 - a_{11}) - a_{12}a_{24}]uv}{a_{12}(\lambda_2 + 1)} - \frac{a_{12}a_{25}v^2}{a_{12}(\lambda_2 + 1)} + \frac{[b_{11}(\lambda_2 - a_{11}) - a_{12}b_{21}]\delta^*u}{a_{12}(\lambda_2 + 1)} + \frac{[b_{12}(\lambda_2 - a_{11}) - a_{12}b_{22}]\delta^*v}{a_{12}(\lambda_2 + 1)} + \frac{[b_{13}(\lambda_2 - a_{11}) - a_{12}b_{23}]\delta^*u^2}{a_{12}(\lambda_2 + 1)} + \frac{[b_{14}(\lambda_2 - a_{11}) - a_{12}b_{24}]\delta^*uv}{a_{12}(\lambda_2 + 1)} - \frac{a_{12}b_{25}\delta^*v^2}{a_{12}(\lambda_2 + 1)} + 0(|u|, |v|, |\delta^*|)^3, \quad (30)$$

$$g(u, v, \delta^*) = \frac{[a_{13}(1 + a_{11}) + a_{12}a_{23}]u^2}{a_{12}(\lambda_2 + 1)} + \frac{[a_{14}(1 + a_{11}) + a_{12}a_{24}]uv}{a_{12}(\lambda_2 + 1)} + \frac{a_{12}a_{25}v^2}{a_{12}(\lambda_2 + 1)} + \frac{[b_{11}(1 + a_{11}) + a_{12}b_{21}]\delta^*u}{a_{12}(\lambda_2 + 1)} + \frac{[b_{12}(1 + a_{11}) + a_{12}b_{22}]\delta^*v}{a_{12}(\lambda_2 + 1)} + \frac{[b_{13}(1 + a_{11}) + a_{12}b_{23}]\delta^*u^2}{a_{12}(\lambda_2 + 1)} + \frac{[b_{14}(1 + a_{11}) + a_{12}b_{24}]\delta^*uv}{a_{12}(\lambda_2 + 1)} + \frac{a_{12}b_{25}\delta^*v^2}{a_{12}(\lambda_2 + 1)} + 0(|u|, |v|, |\delta^*|)^3, \quad (31)$$

$$u = a_{12}(\tilde{x} + \tilde{y}),$$

$$v = -(1 + a_{11})\tilde{x} + (\lambda_2 - a_{11})\tilde{y}.$$

Apply the center manifold theorem to (22) at the origin in the limited neighborhood of $\delta^* = 0$. The center manifold $W^c(0, 0)$ can be approximately presented as

$$W^c(0, 0) = \{(\tilde{x}, \tilde{y}) : \tilde{y} = a_0\delta^* + a_1\tilde{x}^2 + a_2\tilde{x}\delta^* + a_3\delta^{*2} + O(|\tilde{x}| + |\delta^*|)^3\}, \quad (32)$$

where $O(|\tilde{x}| + |\delta^*|)^3$ is a function with at least third orders in variables $(\tilde{x}, \delta^*)$.

By simple calculations for center manifold, we have

$$a_0 = 0,$$

$$a_1 = \frac{[a_{13}(1 + a_{11}) + a_{12}a_{23}]a_{12} - [a_{14}(1 + a_{11}) + a_{12}a_{24}](1 + a_{11}) + a_{25}(1 + a_{11})^2}{1 - \lambda_2^2}, \quad (33)$$

$$a_2 = -\frac{[b_{11}(1 + a_{11}) + a_{12}b_{21}]a_{12} + [b_{12}(1 + a_{11}) + a_{12}b_{22}](1 + a_{11})}{a_{12}(1 + \lambda_2)^2},$$

$$a_3 = 0.$$

Now, consider the map restricted to the center manifold $W^c(0, 0)$ as below:

$$f: \tilde{x} \longrightarrow -\tilde{x} + h_1 \tilde{x}^2 + h_2 \tilde{x} \delta^* + h_3 \tilde{x}^2 \delta^* + h_4 \tilde{x} (\delta^*)^2 + h_5 \tilde{x}^3 + O((|\tilde{x}| + |\delta^*|)^4), \quad (34)$$

where

$$\begin{aligned} h_1 &= \frac{[a_{13}(\lambda_2 - a_{11}) - a_{12}a_{23}]a_{12}}{\lambda_2 + 1} \\ &\quad - \frac{[a_{14}(\lambda_2 - a_{11}) - a_{12}a_{24}](1 + a_{11})}{\lambda_2 + 1} \\ &\quad - \frac{a_{12}a_{25}(1 + a_{11})^2}{a_{12}(\lambda_2 + 1)}, \\ h_2 &= \frac{[b_{11}(\lambda_2 - a_{11}) - a_{12}b_{21}]}{\lambda_2 + 1} \\ &\quad - \frac{[b_{12}(\lambda_2 - a_{11}) - a_{12}b_{22}](1 + a_{11})}{(\lambda_2 + 1)a_{12}}, \\ h_3 &= \frac{[a_{13}(\lambda_2 - a_{11}) - a_{12}a_{23}]2a_2a_{12}}{\lambda_2 + 1} \\ &\quad + \frac{[a_{14}(\lambda_2 - a_{11}) - a_{12}a_{24}](\lambda_2 - 2a_{11} - 1)a_2}{\lambda_2 + 1} \\ &\quad + \frac{2a_{25}(1 + a_{11})(\lambda_2 - a_{11})a_2}{\lambda_2 + 1} \\ &\quad + \frac{[b_{11}(\lambda_2 - a_{11}) - a_{12}b_{21}]a_1}{\lambda_2 + 1} \\ &\quad + \frac{[b_{12}(\lambda_2 - a_{11}) - a_{12}b_{22}](\lambda_2 - a_{11})a_1}{a_{12}(\lambda_2 + 1)} \\ &\quad + \frac{[b_{13}(\lambda_2 - a_{11}) - a_{12}b_{23}]a_{12}}{\lambda_2 + 1} \\ &\quad - \frac{[b_{14}(\lambda_2 - a_{11}) - a_{12}b_{24}](1 + a_{11})}{\lambda_2 + 1} \\ &\quad - \frac{b_{25}(1 + a_{11})^2}{\lambda_2 + 1}, \\ h_4 &= \frac{[b_{11}(\lambda_2 - a_{11}) - a_{12}b_{21}]a_2}{\lambda_2 + 1} \\ &\quad + \frac{[b_{12}(\lambda_2 - a_{11}) - a_{12}b_{22}](\lambda_2 - a_{11})a_2}{a_{12}(\lambda_2 + 1)}, \\ h_5 &= \frac{[a_{13}(\lambda_2 - a_{11}) - a_{12}a_{23}]2a_{12}a_1}{\lambda_2 + 1} \\ &\quad + \frac{[a_{14}(\lambda_2 - a_{11}) - a_{12}a_{24}](\lambda_2 - 2a_{11} - 1)a_1}{\lambda_2 + 1} \\ &\quad + \frac{2a_{25}(1 + a_{11})(\lambda_2 - a_{11})a_1}{\lambda_2 + 1}. \end{aligned} \quad (35)$$

According to flip bifurcation, the discriminatory quantities γ_1 and γ_2 are given by

$$\begin{aligned} \gamma_1 &= \left(\frac{\partial^2 f}{\partial \tilde{x} \partial \delta^*} + \frac{1}{2} \frac{\partial f}{\partial \delta^*} \frac{\partial^2 f}{\partial \tilde{x}^2} \right) \bigg|_{(0,0)}, \\ \gamma_2 &= \left(\frac{1}{6} \frac{\partial^3 f}{\partial \tilde{x}^3} + \left(\frac{1}{2} \frac{\partial^2 f}{\partial \tilde{x}^2} \right)^2 \right) \bigg|_{(0,0)}. \end{aligned} \quad (36)$$

After simple calculations, we obtain $\gamma_1 = h_2$ and $\gamma_2 = h_5 + h_1^2$.

Analyzing above and the flip bifurcation conditions discussed in [30], we write the following theorem.

Theorem 6. *If $\gamma_2 \neq 0$ and parameter δ^* alters in the limiting region of the point $(0, 0)$, then system (10) passes through flip bifurcation at the point $p_3(x^*, y^*)$. Also, the period-2 points that bifurcations from fixed point $p_3(x^*, y^*)$ are stable (resp., unstable) if $\gamma_2 > 0$ (resp., $\gamma_2 < 0$).*

3.2. Hopf Bifurcation. Consider system (7) with arbitrary parameter $(a, b, c, k, l, m, \delta_2, A_1) \in H_B$. This is described as follows:

$$\begin{aligned} x &\longrightarrow x + \delta_2 x \left[\left(1 - \frac{x}{k} \right) \frac{ax}{A_1 + x} - \frac{by}{x + l} \right], \\ y &\longrightarrow y + \delta_2 y \left(c + \frac{mbx}{x + l} - dy \right). \end{aligned} \quad (37)$$

$p_3(x^*, y^*)$ is the fixed point of system (37), where x^*, y^* are given by (3) and $\delta_2 = -G/H$.

Consider the perturbation of (37) as given below:

$$\begin{aligned} x &\longrightarrow x + (\delta_2 + \delta) x \left[\left(1 - \frac{x}{k} \right) \frac{ax}{A_1 + x} - \frac{by}{x + l} \right], \\ y &\longrightarrow y + (\delta_2 + \delta) y \left(c + \frac{mbx}{x + l} - dy \right), \end{aligned} \quad (38)$$

where $|\delta| \ll 1$ is limited perturbation parameter.

The characteristic equation of map (38) at $p_3(x^*, y^*)$ is given by $\lambda^2 + p(\delta)\lambda + q(\delta) = 0$, where

$$\begin{aligned} p(\delta) &= -2 - G\delta, \\ q(\delta) &= 1 + G\delta + H\delta^2. \end{aligned} \quad (39)$$

Since the parameter $(a, b, c, k, l, m, \delta_2, A_1) \in H_B$, the eigen values of $p_3(x^*, y^*)$ are a pair of complex conjugate numbers $\bar{\lambda}$ and λ with modulus 1 by Proposition 5, where

$$\bar{\lambda}, \lambda = \frac{-p(\delta) \mp i\sqrt{4q(\delta) - p^2(\delta)}}{2}. \quad (40)$$

Therefore

$$\bar{\lambda}, \lambda = 1 + \frac{G(\delta_2 + \delta)}{2} \mp \frac{i(\delta_2 + \delta)\sqrt{4H - G^2}}{2}. \quad (41)$$

Now we have

$$|\lambda| = q(\delta)^{1/2}, \quad (42)$$

$$l = \frac{d|\lambda|}{d\delta} \Big|_{\delta=0} = -\frac{G}{2} > 0.$$

When δ varies in limited neighborhood of $\delta = 0$, then $\bar{\lambda}, \lambda = \alpha \pm i\beta$, where

$$\alpha = 1 + \frac{\delta_2 G}{2}, \quad (43)$$

$$\beta = \frac{\delta_2 \sqrt{4H - G^2}}{2}.$$

Hopf bifurcation requires that when $\delta = 0$, then $\bar{\lambda}^n, \lambda^n \neq 1$ ($n = 1, 2, 3, 4$) which is equivalent to $p(0) \neq -2, 0, 1, 2$.

Since the parameter $(a, b, c, d, l, m, \delta_2, A_1) \in H_B$, therefore $p(0) \neq -2, 2$. It is the only requirement that $p(0) \neq 0, 1$, which follows that $G^2 \neq 2H, 3H$. Let $u = x - x^*$ and $v = y - y^*$.

After the transformation of the fixed points $p_3(x^*, y^*)$ of system (38) to the points $(0, 0)$, we have

$$\begin{pmatrix} u \\ v \end{pmatrix} \rightarrow \begin{pmatrix} a_{11}u + a_{12}v + a_{13}u^2 + a_{14}uv + 0(|u|, |v|)^3 \\ a_{21}u + a_{22}v + a_{23}u^2 + a_{24}uv + a_{25}v^2 + 0(|u|, |v|)^3 \end{pmatrix}, \quad (44)$$

where $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{24}, a_{25}$ are given in (26) by substituting δ_2 for $\delta_2 + \delta$.

Now, we discuss the normal form of (44) when $\delta = 0$.

Consider the translation as below:

$$\begin{pmatrix} u \\ v \end{pmatrix} = T \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}, \quad (45)$$

where

$$T = \begin{pmatrix} a_{12} & 0 \\ \alpha - a_{11} & -\beta \end{pmatrix},$$

$$T^{-1} = \begin{pmatrix} \frac{1}{\alpha - a_{11}} & 0 \\ \frac{a_{12}}{\beta a_{12}} & -\frac{1}{\beta} \end{pmatrix}, \quad (46)$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = T \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} a_{12} & 0 \\ \alpha - a_{11} & -\beta \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}.$$

Taking T^{-1} on both sides of (44), we get

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \rightarrow \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} + \begin{pmatrix} \tilde{f}(\tilde{x}, \tilde{y}) \\ \tilde{g}(\tilde{x}, \tilde{y}) \end{pmatrix}, \quad (47)$$

where

$$\begin{aligned} \tilde{f}(\tilde{x}, \tilde{y}) &= \frac{a_{13}}{a_{12}}u^2 + \frac{a_{14}}{a_{12}uv} + O(|u|, |v|)^3, \\ \tilde{g}(\tilde{x}, \tilde{y}) &= \frac{[a_{13}(\alpha - a_{11}) - a_{12}a_{23}]}{a_{12}\beta}u^2 \\ &\quad + \frac{[a_{14}(\alpha - a_{11}) - a_{12}a_{24}]}{a_{12}\beta}uv - \frac{a_{25}}{\beta}v^2 \\ &\quad + O(|u|, |v|)^3, \\ u &= a_{12}\tilde{x}, \\ v &= (\alpha - a_{11})\tilde{x} - \beta\tilde{y}. \end{aligned} \quad (48)$$

Now

$$\begin{aligned} \tilde{f}_{\tilde{x}\tilde{x}} &= 2a_{12}a_{13} + 2a_{14}(\alpha - a_{11}), \\ \tilde{f}_{\tilde{x}\tilde{y}} &= -a_{14}\beta, \\ \tilde{f}_{\tilde{y}\tilde{y}} &= 0, \\ \tilde{f}_{\tilde{x}\tilde{x}\tilde{x}} &= 0, \\ \tilde{f}_{\tilde{x}\tilde{x}\tilde{y}} &= 0, \\ \tilde{f}_{\tilde{x}\tilde{y}\tilde{y}} &= 0, \\ \tilde{f}_{\tilde{y}\tilde{y}\tilde{y}} &= 0, \\ \tilde{g}_{\tilde{x}\tilde{x}} &= \frac{2a_{12}}{\beta} [a_{13}(\alpha - a_{11}) - a_{12}a_{23}] \\ &\quad + \frac{2(\alpha - a_{11})}{\beta} [a_{14}(\alpha - a_{11}) - a_{12}a_{24}] \\ &\quad - \frac{2a_{25}}{\beta} (\alpha - a_{11})^2, \\ \tilde{g}_{\tilde{x}\tilde{y}} &= -[a_{14}(\alpha - a_{11}) - a_{12}a_{24}] + 2a_{25}(\alpha - a_{11}), \\ \tilde{g}_{\tilde{y}\tilde{y}} &= -2a_{25}\beta, \\ \tilde{g}_{\tilde{x}\tilde{x}\tilde{x}} &= 0, \\ \tilde{g}_{\tilde{x}\tilde{x}\tilde{y}} &= 0, \\ \tilde{g}_{\tilde{x}\tilde{y}\tilde{y}} &= 0, \\ \tilde{g}_{\tilde{y}\tilde{y}\tilde{y}} &= 0. \end{aligned} \quad (49)$$

According to Hopf bifurcation, the discriminatory quantity s is given by

$$\begin{aligned} s &= -\operatorname{Re} \left[\frac{(1 - 2\bar{\lambda})\bar{\lambda}^2}{1 - \lambda} \varphi_{11}\varphi_{20} \right] - \frac{1}{2} \|\varphi_{11}\|^2 - \|\varphi_{02}\|^2 \\ &\quad + \operatorname{Re}(\bar{\lambda}\varphi_{21}), \end{aligned} \quad (50)$$

FIGURE 1: Largest Lyapunov exponents and bifurcation diagram of system (25) for $a = 0.8$, $b = 0.8$, $c = 0.4$, $d = 0.2$, $k = 5$, $l = 2$, and $A_1 = 0.18$ with the initial value of $(x, y) = (1, 1)$ and δ covering $[4.6, 6.6]$.

FIGURE 2: The portraits for several values of $A_1 = 0.18$, $\delta = 6.45$.

where

$$\begin{aligned}
 \varphi_{20} &= \frac{1}{8} [(\tilde{f}_{\bar{x}\bar{x}} - \tilde{f}_{\bar{y}\bar{y}} + 2\tilde{g}_{\bar{x}\bar{y}}) \\
 &\quad + i(\tilde{g}_{\bar{x}\bar{x}} - \tilde{g}_{\bar{y}\bar{y}} - 2\tilde{f}_{\bar{x}\bar{y}})], \\
 \varphi_{11} &= \frac{1}{4} [(\tilde{f}_{\bar{x}\bar{x}} + \tilde{f}_{\bar{y}\bar{y}}) + i(\tilde{g}_{\bar{x}\bar{x}} + \tilde{g}_{\bar{y}\bar{y}})], \\
 \varphi_{02} &= \frac{1}{8} [(\tilde{f}_{\bar{x}\bar{x}} - \tilde{f}_{\bar{y}\bar{y}} - 2\tilde{g}_{\bar{x}\bar{y}}) \\
 &\quad + i(\tilde{g}_{\bar{x}\bar{x}} - \tilde{g}_{\bar{y}\bar{y}} + 2\tilde{f}_{\bar{x}\bar{y}})], \\
 \varphi_{21} &= \frac{1}{16} [(\tilde{f}_{\bar{x}\bar{x}\bar{x}} + \tilde{f}_{\bar{x}\bar{y}\bar{y}} + \tilde{g}_{\bar{x}\bar{x}\bar{y}} + \tilde{g}_{\bar{y}\bar{y}\bar{y}}) \\
 &\quad + i(\tilde{g}_{\bar{x}\bar{x}\bar{x}} - \tilde{g}_{\bar{x}\bar{y}\bar{y}} - \tilde{f}_{\bar{x}\bar{x}\bar{y}} - \tilde{f}_{\bar{y}\bar{y}\bar{y}})].
 \end{aligned} \tag{51}$$

Analyzing above and Hopf bifurcation conditions discussed in [30], we write the theorem as below.

Theorem 7. *If the condition (38) holds, $s \neq 0$, and the parameter δ alters in the limited region of the point $(0, 0)$,*

then system (38) passes through a Hopf bifurcation at the point $p_3(x^, y^*)$. Moreover, if $s < 0$ (resp., $s > 0$), then an attracting (resp., repelling) invariant closed curve bifurcates from the fixed point $p_3(x^*, y^*)$ for $\delta > 0$ (resp., $\delta < 0$).*

4. Numerical Simulations

To verify the theoretical analysis, we draw the bifurcation diagrams, largest Lyapunov exponents, and phase portraits for system (22). These diagrams show the complete dynamical behavior and the global stability of the system at the fixed points. The bifurcation behavior of system (22) is discussed in the following cases.

Case 1. On the one hand, if we take $a = 0.8$, $b = 0.8$, $c = 0.4$, $d = 0.2$, $k = 5$, $l = 2$, $m = 0.1$, and $A_1 = 0.18$ with the initial value of $(x, y) = (1, 1)$ and δ covering $[4.6, 6.6]$, then, we observe that at the fixed point $(2.0126, 2.2006)$ a flip bifurcation appears at $\delta = 4.792$. In this case $\gamma_1 = -0.3528$, $\gamma_2 = 2.857$ (see Figure 1).

From Figure 2, we can see that the fixed point of system (9) is stable for $\delta < 4.792$ and loses its stability at $\delta = 4.792$ and there appears period doubling bifurcation for $\delta > 4.792$.

FIGURE 3: The portraits for several values of $A_1 = 0.18$, $\delta = 6.4$. FIGURE 4: Largest Lyapunov exponents and bifurcation diagram of system (25) for $a = 0.8$, $b = 0.8$, $c = 0.4$, $d = 0.2$, $k = 5$, $l = 2$, and $A_1 = 0.03$ with the initial value of $(x, y) = (1, 1)$ and δ covering $[4.6, 5.8]$.

The phase portraits show that there are orbits of periods 2, 4, 8 for $\delta \in [5, 6.15]$ and chaotic sets at $\delta = 6.45$ and $\delta = 6.4$ (see Figures 2 and 3).

On the other hand, if we take $a = 0.8$, $b = 0.8$, $c = 0.4$, $d = 0.2$, $k = 5$, $l = 2$, $m = 0.1$, and $A_1 = 0.03$ with the initial value of $(x, y) = (1, 1)$ and δ covering $[4.6, 5.8]$, then, we observe that at the fixed point $(2.5025, 2.2223)$ a flip bifurcation appears at $\delta = 4.846$. In this case $\gamma_1 = -0.175$, $\gamma_2 = 2.2893$ (see Figure 4).

From Figure 5, we can see that the fixed point of system (9) is stable for $\delta < 4.846$ and loses its stability at $\delta = 4.846$ and there appears period doubling bifurcation for $\delta > 4.846$. The phase portraits show that there are orbits of periods 2, 4, 8 for $\delta \in [5, 5.6]$ and chaotic sets at $\delta = 5.7$ and $\delta = 5.75$.

But, from the former paper in [17], we know when $\delta = 4.87985$ loses its stability without Allee effect. So, we can see Allee effect changes the value of δ .

Case 2. First, we take $a = 0.8$, $b = 0.8$, $c = 0.4$, $d = 0.2$, $k = 5$, $l = 3$, $m = 0.1$, and $A_1 = 0.18$ with initial value of $(x, y) = (1, 1)$ and δ covering $[5.3, 5.37]$ (see Figure 6). We can see that at the fixed point $(3.0927, 2.203)$ a Hopf bifurcation emerges

at $\delta = 5.335$. We can see that the fixed point of system (9) is stable $\delta < 5.335$ and loses its stability at $\delta = 5.335$, and an invariant circle appears at $\delta > 5.335$ (see Figures 7 and 8). Here $\alpha = -0.8171$, $\beta = 0.37345$, and $s = -0.911$.

Second, we take $a = 0.8$, $b = 0.8$, $c = 0.4$, $d = 0.2$, $k = 5$, $l = 3$, $m = 0.1$, and $A_1 = 0.03$ with initial value of $(x, y) = (1, 1)$ and δ covering $[4.87, 4.95]$ (see Figure 9). We can see that at the fixed point $(3.206, 2.207)$ a Hopf bifurcation emerges at $\delta = 4.8795$. We can see that the fixed point of system (9) is stable $\delta < 4.8795$ and loses its stability at $\delta = 4.8795$, and an invariant circle appears at $\delta > 4.8795$ (see Figures 10 and 11). Here $\alpha = -0.9523$, $\beta = 0.3096$, and $s = -0.9172$.

5. Conclusions

In recent years, the Allee effect has drawn considerable attention in almost every aspect of ecology and conservation. The consequences of Allee effect become more and more significant. In this paper, based on the discrete-time predator-prey system with predator partially dependent on prey and crowding effect of predator, we investigated the impact of Allee

FIGURE 5: The portraits for several values of $A_1 = 0.03$, $\delta = 5.7$, and $\delta = 5.75$.

FIGURE 6: Largest Lyapunov exponents and bifurcation diagram of system (25) for $a = 0.8$, $b = 0.8$, $c = 0.4$, $d = 0.2$, $k = 5$, $l = 3$, and $A_1 = 0.18$ with the initial value of $(x, y) = (1, 1)$ and δ covering $[5.32, 5.39]$.

FIGURE 7: The portraits for several values of $A_1 = 0.18$, $\delta = 5.3$, and $\delta = 5.335$.

FIGURE 8: The portraits for several values of $A_1 = 0.18$, $\delta = 5.34$, and $\delta = 5.37$.

FIGURE 9: Largest Lyapunov exponents and bifurcation diagram of system (25) for $a = 0.8$, $b = 0.8$, $c = 0.4$, $d = 0.2$, $k = 5$, $l = 3$, and $A_1 = 0.03$ with the initial value of $(x, y) = (1, 1)$ and δ covering $[4.8, 5.1]$.

FIGURE 10: The portraits for several values of $A_1 = 0.03$, $\delta = 4.875$, and $\delta = 4.8795$.

FIGURE 11: The portraits for several values of $A_1 = 0.03$, $\delta = 4.885$, and $\delta = 5$.

effect on the stability of a predator-prey system when the species is subject to Allee effect in the closed first quadrant R_+^2 .

By analyzing system (9), we found that the system is asymptotically stable at the $p_2(0, c/d)$, $p_3(x^*, y^*)$ at the interior of the first quadrant for any parameter values. Ecologically speaking, this means that there only exists predator at $p_2(0, c/d)$, in which δ satisfies $0 < \delta < \min\{2/c, 2dl/bc\}$, and the prey and predator species coexist at $p_3(x^*, y^*)$, which should satisfy either condition (1): $G^2 - 4H \geq 0$ and $0 < \delta < (-G - \sqrt{G^2 - 4H})/H$ or condition (2): $G^2 - 4H < 0$ and $0 < \delta < -G/H$.

It is shown that system (9) with Allee effect can go through flip bifurcation and Hopf bifurcation at the neighborhood of the equilibrium point. Because one of the eigen values of fixed point $p_1(k, 0)$ is -1 and magnitude of other is not equal to 1 when $\delta = 2/a + 2A_1/ak$, a flip bifurcation occurs when a parameter changes in small neighborhood of $\delta = 2/a + 2A_1/ak$. The point $p_2(0, c/d)$ undergoes flip bifurcation when the parameter changes in small neighborhood of $F_A = \{(a, b, c, d, k, l, m, \delta, A_1) : \delta = 2/c, \delta \neq 2dl/bc, a, b, c, d, k, l, m, \delta, A_1 > 0\}$. Finally, at the positive equilibrium point $p_3(x^*, y^*)$, by our analysing a flip bifurcation will occur in the neighborhood of $F_{B1} = \{(a, b, c, d, k, l, m, \delta, A_1) : \delta = (-G - \sqrt{G^2 - 4H})/H, G^2 - 4H \geq 0 \text{ and } a, b, c, d, k, l, m, \delta, A_1 > 0\}$ or $F_{B2} = \{(a, b, c, d, k, l, m, \delta, A_1) : \delta = (-G + \sqrt{G^2 - 4H})/H, G^2 - 4H \geq 0 \text{ and } a, b, c, d, k, l, m, \delta, A_1 > 0\}$. In contrast, the system will get through a Hopf bifurcation at the neighborhood of $H_B = \{(a, b, c, d, k, l, m, \delta, A_1) : \delta = -G/H, G^2 - 4H < 0 \text{ and } a, b, c, d, k, l, m, \delta, A_1 > 0\}$.

In the paper, we assume the parameter $A_1 > 0$; that is, we only consider the strong Allee effect. When $A_1 < 0$, it is the weak Allee effect, and it is not the same as the $A_1 > 0$. This is the situation we have to consider in the future.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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