Research Article

Number of Forts in Iterated Logistic Mapping

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Using the theory of complete discrimination system and the computer algebra system *MAPLE V.17*, we compute the number of forts for the logistic mapping $f_\lambda(x) = \lambda x(1-x)$ on $[0, 1]$ parameterized by $\lambda \in (0, 4]$. We prove that if $0 < \lambda \leq 2$ then the number of forts does not increase under iteration and that if $\lambda > 2$ then the number of forts is not bounded under iteration. Furthermore, we focus on the case of $\lambda > 2$ and give for each $k = 1, \dots, 7$ some critical values of λ for the change of numbers of forts.

1. Introduction

Iteration is the act of repeating a process with the aim of approaching a desired goal, target, or result. In mathematical sense, for a fixed integer $n \geq 1$, the n th iterate f^n of a mapping $f : E \rightarrow E$, where E is a nonempty set, is defined recursively by

$$f^k = f \circ f^{k-1}, \quad f^0 = \text{id}, \quad (1)$$

where $\circ$ presents the composition of functions and id denote the identity mapping; that is, $\text{id}(x) \equiv x$ for all $x \in E$. Being indispensable in the computer era, iteration brings many interesting but difficult problems to mathematics. Only from one-dimensional case, one can simply notice that an iterate of a linear function of any order remains linear but the degree of a polynomial may increase drastically, which shows that the nonlinear complexity is amplified by iteration.

Actually, in the one-dimensional case, the complexity of nonlinear functions is related to nonmonotonicity. For a continuous nonmonotonic self-mapping $f : I \rightarrow I$, where I is an interval, a point $x_0 \in I$ is called a *monotone point* of f if f is strictly monotone in a neighborhood of x_0 ; otherwise, x_0 is called a *nonmonotone point* or simply a *fort* of f . Obviously, a linear function does not have a fort generically. In 1980s, Zhang and Yang (see [1]) investigated the number of forts for a class of nonmonotonic functions called *strictly piecewise*

monotone functions and simply *PM functions*, which are self-mapping on a compact interval and have at most finitely many forts each. Let $S(f)$ denote the set of all forts of f and let $N(f)$ denote the cardinality of $S(f)$. It is shown in [1, 2] that

$$0 = N(f^0) \leq N(f) \leq N(f^2) \leq \dots \leq N(f^k) \leq N(f^{k+1}) \leq \dots; \quad (2)$$

that is, the number $N(f^k)$ of forts is nondecreasing as k is increasing. One can similarly prove that (2) also holds for functions defined on the whole $\mathbb{R}$. It is easy to find nonlinear functions whose number of forts, regarded as the damages of monotonicity, increases rapidly under iteration. Consider the quadratic function

$$f(x) = 4x(1-x), \quad x \in [0, 1], \quad (3)$$

for example. Computing derivatives of f^i , $i = 1, \dots, 5$, and counting the number of real zeros with odd multiplicity for the derivatives $(f^i)'$ (as done in [3]), we get $N(f) = 1$, $N(f^2) = 3$, $N(f^3) = 7$, $N(f^4) = 15$, and $N(f^5) = 31$. From the increasing tendency, without continuing the tedious computation, we have the following question: *Does $N(f^k)$ have a bound or approach infinity as k tends to ∞ ? How can we compute the number of forts for nonmonotonic functions?*

Polynomials, a special class of nonmonotonic functions, possess the advantage that each fort of a polynomial of degree

≥ 1 is either a peak or a valley although the notion is not true in general. In this paper, we focus on the family of logistic mappings:

$$f_\lambda(x) = \lambda x(1-x), \quad x \in [0, 1], \quad (4)$$

where $\lambda \in (0, 4]$ is a parameter, which is one of the simplest polynomial mappings, and a typical example used to show chaos and some complicated dynamics, for those problems. First of all, we introduce the theory of *complete discrimination system* (see [4]) and then use it to give a method for the computation of $N(f)$ with f polynomial in Section 2. In Section 3, we employ the method in the computer algebra system *MAPLE V.17* for the family of logistic mappings. We prove in Theorem 4 that $N(f_\lambda^k) = N(f_\lambda) = 1$ for all integer $k \geq 2$ if $0 < \lambda \leq 2$ and that $N(f_\lambda^k)$ approaches ∞ as $k \rightarrow \infty$ if $2 < \lambda \leq 4$. Furthermore, for various choices of $\lambda \in (2, 4]$, we compute the number $N(f_\lambda^k)$ for each fixed $k = 2, 3, \dots, 7$ in Theorem 5.

2. Preliminaries

In general, for polynomial

$$f(x) = \sum_{i=0}^n a_i x^i, \quad (5)$$

where $n \geq 2$ and $a_n \neq 0$. $N(f^k)$, $k = 1, 2, \dots$, is decided by real zeros of the derivatives $(f^k)'$.

Lemma 1 (see [3, Lemma 2.1]). *x_0 is a fort of a real polynomial f if and only if x_0 is a real zero of the derivative f' of odd multiplicity. Moreover, $N(f)$ is odd (resp., even) if the degree n of f is even (resp., odd).*

Actually, the above lemma shows how a real zero of the derivative f' can be a fort of f . Note that $(f^k)' = (f' \circ f^{k-1})(f' \circ f^{k-2}) \cdots (f' \circ f)f'$. Then the set of zeros of $(f^k)'$ is a union of the set of zeros of $f' \circ f^{k-1}$ and the set of zeros of $(f' \circ f^{k-2}) \cdots (f' \circ f)f'$. Therefore, in order to know if f^k have more forts than f^{k-1} , we need to judge if $f' \circ f^{k-1}$ have real zeros different from $(f' \circ f^{k-2}) \cdots (f' \circ f)f'$'s with odd multiplicities; the following lemma gives the answer.

Lemma 2 (see [3, Lemma 2.5]). *Let G and P be real polynomials and $G(x) \not\equiv 0$. Then the composition $G \circ P$ and the derivative P' do not have a common real zero with odd multiplicity.*

Taking $G = f'$ and $P = f^{k-1}$, by Lemmas 1 and 2, we see that $(f^k)'$ has more real zeros of odd multiplicities than $(f^{k-1})'$ if and only if $f' \circ f^{k-1}$ has real zeros with odd multiplicities. Hence, in the process of computing $N(f^k)$, we only need to find out the number of real zeros for $f' \circ f^{k-1}$ with odd multiplicities. For this reason, we first introduce some notations of the theory of complete discrimination system (see [4, 5]) which will lead us to solve this problem.

Discriminants of polynomials are useful in determining the number of zeros for polynomials. Let $\text{Discr}(f)$ denote

the *discriminant matrix* of the polynomial f , which is constructed by the Sylvester matrix of f and f' as seen in [4, Definition 1]. For each $\tau = 1, \dots, n$, let $D_\tau(f)$ denote the determinant of its submatrix formed by the first $2i$ rows and the first $2i$ columns. The n -tuple $(D_1, D_2, \dots, D_n)$ is called the *discriminant sequence* of f and the list

$$(\text{sign}(D_1), \text{sign}(D_2), \dots, \text{sign}(D_n)) \quad (6)$$

is called the *sign list* of f , where $\text{sign}(x)$ is defined to be equal to either 1 if $x > 0$, 0 if $x = 0$, or -1 if $x < 0$. Given a sign list $(s_1, s_2, \dots, s_n)$ of f , we make a new list $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$, called the *revised sign list* of f , in the following regulation:

If $(s_i, s_{i+1}, \dots, s_{i+j})$ is a section of the given list such that $s_i \neq 0$, $s_{i+1} = s_{i+2} = \dots = s_{i+j-1} = 0$, and $s_{i+j} \neq 0$, then replace the subsection $(s_{i+1}, s_{i+2}, \dots, s_{i+j-1})$ with $(\varepsilon_{i+1}, \varepsilon_{i+2}, \dots, \varepsilon_{i+j-1})$, where $\varepsilon_{i+r} = (-1)^{[(r+1)/2]} s_i$ for $r = 1, 2, \dots, j-1$; otherwise, let $\varepsilon_r = s_r$.

The following lemma tells us how to find the number of distinct zeros by means of the revised sign list.

Lemma 3 (see [4, Theorem 1]). *Let f be a real polynomial and suppose that the number of the sign changes in the revised sign list of f is ν . Then the number of pairs of distinct conjugate complex zeros of f equals ν . Furthermore, if the number of nonvanishing members in the revised sign list is l , then F has $l - 2\nu$ distinct real zeros.*

Now, we are ready to apply the above lemmas to detail the process in computing $N(f^k)$, $k = 1, 2, \dots$. First of all, we compute f^k and $f' \circ f^{k-1}$ and the discriminant sequence for $f' \circ f^{k-1}$. Secondly, under algebraic relations among coefficients a_i 's in the discriminant sequence, discuss the sign of each component of the discriminant sequence and list the sign lists. Then, compute the revised sign lists for $f' \circ f^{k-1}$ through the sign lists. According to the revised sign lists, we find out the number of real zeros of $f' \circ f^{k-1}$ with odd multiplicities and finally obtain $N(f^k)$.

The above idea can be implemented in the computer algebra system *MAPLE V.17*, and we will use this method for the logistic mappings up to iteration index $k = 7$ in next section.

3. Number of Forts

In this section, we first draw a conclusion for the logistic mappings which describe that the numbers of forts can be preserved or approach ∞ as λ varies under iteration and then compute $N(f_\lambda^k)$ for $f_\lambda(x)$ up to iteration index $k = 7$ with different choice of λ .

Theorem 4. *The logistic mappings f_λ defined in (4) have $N(f_\lambda^k) = N(f_\lambda) = 1$ for all integer $k \geq 2$ if and only if $0 < \lambda \leq 2$. Otherwise, $N(f_\lambda^k)$ approaches ∞ as $k \rightarrow \infty$.*

TABLE 1: The number of forts of f^i , $i = 1, 2, \dots, 7$.

Ranges of λ	$N(f_\lambda)$	$N(f_\lambda^2)$	$N(f_\lambda^3)$	$N(f_\lambda^4)$	$N(f_\lambda^5)$	$N(f_\lambda^6)$	$N(f_\lambda^7)$
$\lambda_{1,0} < \lambda \leq \lambda_{2,1}$	1	1	1	1	1	1	1
$\lambda_{2,1} < \lambda \leq \lambda_{3,2}$	1	3	5	7	9	11	13
$\lambda_{3,2} < \lambda \leq \lambda_{3,4}$	1	3	7	13	21	31	43
$\lambda_{3,4} < \lambda \leq \lambda_{3,6}$	1	3	7	13	23	37	57
$\lambda_{3,6} < \lambda \leq \lambda_{3,7}$	1	3	7	13	23	37	59
$\lambda_{3,7} < \lambda \leq \lambda_{4,3}$	1	3	7	13	23	39	65
$\lambda_{4,3} < \lambda \leq \lambda_{4,9}$	1	3	7	15	29	53	93
$\lambda_{4,9} < \lambda \leq \lambda_{4,10}$	1	3	7	15	29	53	95
$\lambda_{4,10} < \lambda \leq \lambda_{4,11}$	1	3	7	15	29	55	101
$\lambda_{4,11} < \lambda \leq \lambda_{5,5}$	1	3	7	15	29	55	103
$\lambda_{5,5} < \lambda \leq \lambda_{5,12}$	1	3	7	15	31	61	117
$\lambda_{5,12} < \lambda \leq \lambda_{6,8}$	1	3	7	15	31	61	119
$\lambda_{6,8} < \lambda \leq \lambda_{7,13}$	1	3	7	15	31	63	125
$\lambda_{7,13} < \lambda \leq 4$	1	3	7	15	31	63	127

Proof. In order to obtain the condition for $N(f_\lambda^k) = N(f_\lambda) = 1$, from the method mentioned in the end of Section 2, we need to compute $N(f_\lambda^2)$. Simple computation shows that

$$f'_\lambda \circ f_\lambda = 2\lambda x^2 - 2\lambda x + 1. \quad (7)$$

Then computing the discriminant sequence of (7), we have

$$(\lambda^2, \lambda^3(\lambda - 2)). \quad (8)$$

From (8), if $0 < \lambda < 2$, the revised sign list is $(1, -1)$, implying that $f'_\lambda \circ f_\lambda$ has one pair of complex zeros, which leads to $N(f_\lambda^2) = 1$; if $\lambda = 2$, the revised sign list is $(1, 0)$, implying that $f'_\lambda \circ f_\lambda$ has a double real zero, which leads to $N(f_\lambda^2) = 1$; therefore, $N(f_\lambda) = N(f_\lambda^k) = 1$ if and only if $0 < \lambda \leq 2$.

Without loss of generality, we can turn the general form $f_\lambda(x) = -\lambda x^2 + \lambda x$ into

$$f_\lambda(x) = -\lambda \left(x - \frac{1}{2}\right)^2 + \frac{\lambda}{4}. \quad (9)$$

So the vertex of the parabola is

$$(x_0, y_0) := \left(\frac{1}{2}, \frac{\lambda}{4}\right). \quad (10)$$

If $2 < \lambda \leq 4$, then f_λ and a diagonal line intersect at two points ξ_1 and ξ_2 , where $\xi_1 < x_0 < \xi_2$. Obviously, ξ_1 and ξ_2 are fixed points of f_λ ; $\xi_1 = 0$ and $\xi_2 = (\lambda - 1)/\lambda$ and f_λ is strictly increasing on the subinterval $[\xi_1, x_0]$. Thus,

$$\begin{aligned} \xi_1 < f_\lambda^{-k-1}(x_0) < f_\lambda^{-k}(x_0) < x_0, \\ \forall k = 1, 2, \dots, \quad \lim_{k \rightarrow +\infty} f_\lambda^{-k}(x_0) = \xi_1. \end{aligned} \quad (11)$$

Since

$$S(f_\lambda^k) = S(f_\lambda^{k-1}) \cup \{x \in [0, 1] : f_\lambda^{k-1} \in S(f_\lambda)\}, \quad (12)$$

we get

$$\begin{aligned} \mathcal{S}(f_\lambda^k) \setminus \mathcal{S}(f_\lambda^{k-1}) &= \mathcal{S}(f_\lambda \circ f_\lambda^{k-1}) \setminus \mathcal{S}(f_\lambda^{k-1}) \\ &= (\mathcal{S}(f_\lambda^{k-1}) \\ &\cup \{x \in [0, 1] : f_\lambda^{k-1}(x) \in \mathcal{S}(f_\lambda)\}) \setminus \mathcal{S}(f_\lambda^{k-1}) \\ &= \{x \in [0, 1] : f_\lambda^{k-1}(x) \in \mathcal{S}(f_\lambda)\} \setminus \mathcal{S}(f_\lambda^{k-1}) \\ &\neq \emptyset \quad \forall k \in \mathbb{N}^+, \end{aligned} \quad (13)$$

which implies that $N(f_\lambda^k)$ approaches ∞ as $k \rightarrow \infty$. This completes the proof. $\square$

Theorem 4 shows that the number $N(f_\lambda^k)$ approaches ∞ as $k \rightarrow \infty$ for each fixed $\lambda \in (2, 4]$. It is also interesting to see for each fixed k how the number $N(f_\lambda^k)$ varies as the parameter λ changes in $(2, 4]$. The following theorem shows the change of numbers $N(f_\lambda^k)$ as λ varies for each $k = 2, \dots, 7$ (but larger k can be considered if the computational capacity of our computer is better). It gives a sequence of parameter values at which new forts arise.

Theorem 5. $N(f_\lambda^2) = 3$ for all $\lambda \in (2, 4]$ and $N(f_\lambda^3) = 5$ and 7 for $\lambda \in (2, \lambda_{3,2}]$ and $\lambda \in (\lambda_{3,2}, 4]$, respectively, where $\lambda_{3,2} := \sqrt{5} + 1 \approx 3.236067977$. For more details, with the convenient notations $\lambda_{1,0} := 0$ and $\lambda_{2,1} := 2$, numbers $N(f_\lambda^k)$, $k = 1, 2, \dots, 7$, are given in Table 1, where

$$\lambda_{4,3} \approx 3.831874056,$$

$$\lambda_{3,4} \approx 3.498561699,$$

$$\lambda_{5,5} \approx 3.960270127,$$

$$\lambda_{3,6} \approx 3.627557530,$$

$$\lambda_{3,7} \approx 3.738914913,$$

$$\lambda_{6,8} \approx 3.990267047,$$

$$\begin{aligned}
\lambda_{4,9} &\approx 3.844568792, \\
\lambda_{4,10} &\approx 3.905706470, \\
\lambda_{4,11} &\approx 3.937536445, \\
\lambda_{5,12} &\approx 3.977766422, \\
\lambda_{7,13} &\approx 3.997583118.
\end{aligned} \tag{14}$$

Proof. By (8), if $2 < \lambda \leq 4$, the revised sign list of $f'_\lambda \circ f_\lambda$ is $(1, 1)$, implying that it has two distinct real zeros, which shows that $N(f^2) = 3$.

Furthermore, in order to obtain $N(f^3_\lambda)$, we compute

$$\begin{aligned}
f'_\lambda \circ f^2_\lambda &= 2\lambda^3 x^4 - 4\lambda^3 x^3 + 2\lambda^3 x^2 + 2\lambda^2 x^2 - 2\lambda^2 x \\
&\quad + 1.
\end{aligned} \tag{15}$$

As shown in Section 2, we give the discriminant sequence for (15):

$$\begin{aligned}
&(\lambda^6, \lambda^{11}(\lambda - 2), \lambda^{14}(\lambda - 2)^2, \lambda^{15}(\lambda^2 - 2\lambda - 4) \\
&\quad \cdot (\lambda - 2)^3).
\end{aligned} \tag{16}$$

Then, the revised sign list for (16) is

- (i) $(1, 1, 1, 1)$, if $\lambda_{3,2} < \lambda \leq 4$, which implies that $f'_\lambda \circ f^2_\lambda$ has 4 distinct simple real zeros;
- (ii) $(1, 1, 1, 0)$, if $\lambda = \lambda_{3,2}$, which implies that $f'_\lambda \circ f^2_\lambda$ has 3 distinct real zeros, 2 of which are simple zeros and the remaining one is a double zero;
- (iii) $(1, 1, 1, -1)$, if $\lambda_{2,1} < \lambda < \lambda_{3,2}$, which implies that $f'_\lambda \circ f^2_\lambda$ has one pair of complex zeros and 2 distinct simple real zeros.

Here $\lambda_{2,1} = 2$ and $\lambda_{3,2} = \sqrt{5} + 1$, as defined in the theorem. $\lambda_{3,2}$ is the real zero of $g(\lambda) := \lambda^2 - 2\lambda - 4$ in $(2, 4]$. By Lemmas 1 and 2, $N(f^3_\lambda) = 5$ if $\lambda_{2,1} < \lambda \leq \lambda_{3,2}$ and $N(f^3_\lambda) = 7$ if $\lambda_{3,2} < \lambda \leq 4$.

Similarly, compute

$$\begin{aligned}
f'_\lambda \circ f^3_\lambda &= 2\lambda^7 x^8 - 8\lambda^7 x^7 + (12\lambda^7 + 4\lambda^6) x^6 \\
&\quad + (-8\lambda^7 - 12\lambda^6) x^5 \\
&\quad + (2\lambda^7 + 12\lambda^6 + 2\lambda^5 + 2\lambda^4) x^4 \\
&\quad + (-4\lambda^6 - 4\lambda^5 - 4\lambda^4) x^3 \\
&\quad + (2\lambda^5 + 2\lambda^4 + 2\lambda^3) x^2 - 2\lambda^3 x + 1.
\end{aligned} \tag{17}$$

Then we obtain the discriminant sequence for $f'_\lambda \circ f^3_\lambda$:

$$\begin{aligned}
&(\lambda^{14}, \lambda^{27}(\lambda - 2), \lambda^{38}(\lambda - 2)^2, \lambda^{47}(\lambda^2 - 2\lambda - 4) \\
&\quad \cdot (\lambda - 2)^3, \lambda^{54}(\lambda^2 - 2\lambda - 4) \\
&\quad \cdot (\lambda - 2)^4, \lambda^{59}(\lambda(\lambda - 2)(\lambda^2 - 2\lambda - 4) - 16) \\
&\quad \cdot (\lambda - 2)^5, \lambda^{62}(\lambda^2 - 2\lambda - 4) \\
&\quad \cdot (\lambda(\lambda - 2)(\lambda^2 - 2\lambda - 4) - 16) \\
&\quad \cdot (\lambda - 2)^6, \lambda^{63}(\lambda(\lambda - 2)(\lambda^2 - 2\lambda - 4)^2 - 64) \\
&\quad \cdot (\lambda^2 - 2\lambda - 4)^2(\lambda - 2)^7).
\end{aligned} \tag{18}$$

Hence,

- (i) for $\lambda_{2,1} < \lambda < \lambda_{3,2}$, the revised sign list for $f'_\lambda \circ f^3_\lambda$ is $(1, 1, 1, -1, -1, -1, 1, -1)$, implying that $f'_\lambda \circ f^3_\lambda$ has 3 pairs of complex zeros and 2 distinct simple real zeros;
- (ii) for $\lambda = \lambda_{3,2}$, the revised sign list for $f'_\lambda \circ f^3_\lambda$ is $(1, 1, 1, -1, -1, -1, 0, 0)$ and $f'_\lambda \circ f^3_\lambda$ can be factorized as

$$\begin{aligned}
f'_\lambda \circ f^3_\lambda &= \frac{1}{64} (13\sqrt{5} + 29) \\
&\quad \cdot (32x^2 - 32x + 4\sqrt{5} + 12\sqrt{2} - 4\sqrt{10} - 4) \\
&\quad \cdot \left(x - \frac{1}{2} - \frac{1}{4}\sqrt{6 - 2\sqrt{5} + 6\sqrt{2} - 2\sqrt{10}}\right) \\
&\quad \cdot \left(x - \frac{1}{2} + \frac{1}{4}\sqrt{6 - 2\sqrt{5} + 6\sqrt{2} - 2\sqrt{10}}\right) \\
&\quad \cdot (4x - 3 + \sqrt{5})^2 (-4x + 1 + \sqrt{5})^2,
\end{aligned} \tag{19}$$

implying that $f'_\lambda \circ f^3_\lambda$ has a pair of complex zeros and 4 distinct real zeros and two of the 4 distinct real zeros are simple and the rest are both double zeros;

- (iii) for $\lambda_{3,2} < \lambda < 1 + (3 + 2\sqrt{5})^{1/2}$, where $1 + (3 + 2\sqrt{5})^{1/2}$ is the real zero of $h(\lambda) := \lambda(\lambda - 2)(\lambda^2 - 2\lambda - 4) - 16$ in $(\lambda_{3,2}, 4]$, the revised sign list for $f'_\lambda \circ f^3_\lambda$ is $(1, 1, 1, 1, 1, -1, -1, -1)$, implying that $f'_\lambda \circ f^3_\lambda$ has a pair of complex zeros and 6 distinct simple real zeros;
- (iv) for $\lambda = 1 + (3 + 2\sqrt{5})^{1/2}$, the revised sign list for $f'_\lambda \circ f^3_\lambda$ is $(1, 1, 1, 1, 1, -1, -1, -1)$, implying that $f'_\lambda \circ f^3_\lambda$ has a pair of complex zeros and 6 distinct simple real zeros;
- (v) for $1 + (3 + 2\sqrt{5})^{1/2} < \lambda < \lambda_{4,3}$, where

$$\begin{aligned}
&\lambda_{4,3} \\
&:= \frac{\sqrt{3}}{3} \left(\frac{2(100 + 12\sqrt{69})^{2/3} + 11(100 + 12\sqrt{69})^{1/3} + 8}{(100 + 12\sqrt{69})^{1/3}} \right)^{1/2} \\
&\quad + 1 \approx 3.831874056
\end{aligned} \tag{20}$$

is the real zero of $p(\lambda) := \lambda(\lambda - 2)(\lambda^2 - 2\lambda - 4)^2 - 64$ in $(1 + (3 + 2\sqrt{5})^{1/2}, 4]$, the revised sign list for $f' \circ f^3$ is $(1, 1, 1, 1, 1, 1, -1)$, implying that $f'_\lambda \circ f_\lambda^3$ has a pair of complex zeros and 6 distinct simple real zeros;

(vi) for $\lambda = \lambda_{4,3}$, the revised sign list for $f'_\lambda \circ f_\lambda^3$ is $(1, 1, 1, 1, 1, 1, 0)$, implying that $f'_\lambda \circ f_\lambda^3$ has 7 distinct real zeros, one of which is a double zero but the rest are all simple;

(vii) for $\lambda_{4,3} < \lambda \leq 4$, the revised sign list for $f'_\lambda \circ f_\lambda^3$ is $(1, 1, 1, 1, 1, 1, 1)$, implying that $f'_\lambda \circ f_\lambda^3$ has 8 distinct simple real zeros.

It follows that $N(f_\lambda^4) = 7$ if $\lambda_{2,1} < \lambda \leq \lambda_{3,2}$, $N(f_\lambda^4) = 13$ if $\lambda_{3,2} < \lambda \leq \lambda_{4,3}$, and $N(f_\lambda^4) = 15$ if $\lambda_{4,3} < \lambda \leq 4$.

We similarly compute $f'_\lambda \circ f_\lambda^4$ and obtain the discriminant sequence

$$\begin{aligned} & (\lambda^{30}, \lambda^{59}(\lambda - 2), \lambda^{86}(\lambda - 2)^2, \lambda^{111}(\lambda^2 - 2\lambda - 4)(\lambda - 2)^3, \lambda^{134}(\lambda^2 - 2\lambda - 4)(\lambda - 2)^4, \lambda^{155}(\lambda^4 - 4\lambda^3 \\ & + 8\lambda - 16)(\lambda - 2)^5, \lambda^{174}(\lambda^2 - 2\lambda - 4)(\lambda^4 - 4\lambda^3 \\ & + 8\lambda - 16)(\lambda - 2)^6, \lambda^{191}(\lambda^6 - 6\lambda^5 + 4\lambda^4 + 24\lambda^3 \\ & - 16\lambda^2 - 32\lambda - 64)(\lambda^2 - 2\lambda - 4)^2(\lambda - 2)^7, \\ & \lambda^{206}(\lambda^6 - 6\lambda^5 + 4\lambda^4 + 24\lambda^3 - 16\lambda^2 - 32\lambda - 64) \\ & \cdot (\lambda^2 - 2\lambda - 4)^2(\lambda - 2)^8, \lambda^{219}(\lambda^2 - 2\lambda - 4)(\lambda^{10} \\ & - 10\lambda^9 + 28\lambda^8 + 16\lambda^7 - 160\lambda^6 + 64\lambda^5 + 192\lambda^4 \\ & + 384\lambda^3 - 256\lambda^2 - 1024\lambda + 1024)(\lambda - 2)^9, \\ & \lambda^{230}(\lambda^4 - 4\lambda^3 + 8\lambda - 16)(\lambda^{10} - 10\lambda^9 + 28\lambda^8 \\ & + 16\lambda^7 - 160\lambda^6 + 64\lambda^5 + 192\lambda^4 + 384\lambda^3 - 256\lambda^2 \\ & - 1024\lambda + 1024)(\lambda - 2)^{10}, \lambda^{239}(\lambda^2 - 2\lambda - 4)(\lambda^8 \\ & - 8\lambda^7 + 16\lambda^6 + 16\lambda^5 - 64\lambda^4 + 128\lambda - 256)(\lambda^4 \\ & - 4\lambda^3 + 8\lambda - 16)^2(\lambda - 2)^{11}, \lambda^{246}(\lambda^8 - 8\lambda^7 + 16\lambda^6 \\ & + 16\lambda^5 - 64\lambda^4 + 128\lambda - 256)(\lambda^2 - 2\lambda - 4)^2(\lambda^4 \\ & - 4\lambda^3 + 8\lambda - 16)^2(\lambda - 2)^{12}, \lambda^{251}(\lambda^4 - 4\lambda^3 + 8\lambda \\ & - 16)(\lambda^{14} - 14\lambda^{13} + 72\lambda^{12} - 136\lambda^{11} - 144\lambda^{10} \\ & + 1088\lambda^9 - 1600\lambda^8 - 640\lambda^7 + 4096\lambda^6 - 512\lambda^5 \\ & - 4096\lambda^4 - 12288\lambda^3 + 16384\lambda^2 + 8192\lambda - 16384) \\ & \cdot (\lambda^2 - 2\lambda - 4)^3(\lambda - 2)^{13}, \lambda^{254}(\lambda^6 - 6\lambda^5 + 4\lambda^4 \end{aligned}$$

$$\begin{aligned} & + 24\lambda^3 - 16\lambda^2 - 32\lambda - 64)(\lambda^{14} - 14\lambda^{13} + 72\lambda^{12} \\ & - 136\lambda^{11} - 144\lambda^{10} + 1088\lambda^9 - 1600\lambda^8 - 640\lambda^7 \\ & + 4096\lambda^6 - 512\lambda^5 - 4096\lambda^4 - 12288\lambda^3 + 16384\lambda^2 \\ & + 8192\lambda - 16384)(\lambda^2 - 2\lambda - 4)^4(\lambda - 2)^{14}, \\ & \lambda^{255}(\lambda^{12} - 12\lambda^{11} + 48\lambda^{10} - 40\lambda^9 - 192\lambda^8 + 384\lambda^7 \\ & + 64\lambda^6 - 1024\lambda^4 - 512\lambda^3 + 2048\lambda^2 + 4096)(\lambda^6 \\ & - 6\lambda^5 + 4\lambda^4 + 24\lambda^3 - 16\lambda^2 - 32\lambda - 64)^2(\lambda^2 - 2\lambda \\ & - 4)^5(\lambda - 2)^{15}). \end{aligned}$$

(21)

Similar discussion gives $N(f_\lambda^5)$ for various λ as shown in Table 1.

It is more complicated to compute discriminant sequences of $f'_\lambda \circ f_\lambda^5$ and $f'_\lambda \circ f_\lambda^6$ because the two discriminant sequences contain 32 and 64 components, respectively, and the biggest component in the discriminant sequence of $f'_\lambda \circ f_\lambda^5$ is a polynomial in the single variable λ of degree 1152 with 130 terms. Using a similar discussion as for $f'_\lambda \circ f_\lambda^k$, $k = 1, 2, 3, 4$, we obtain parameter values $\lambda_{3,6}$, $\lambda_{3,7}$, $\lambda_{6,8}$, $\lambda_{4,9}$, $\lambda_{4,10}$, $\lambda_{4,11}$, $\lambda_{5,12}$, and $\lambda_{7,13}$ as well as the numbers $N(f_\lambda^6)$ and $N(f_\lambda^7)$ on intervals between them as shown in Table 1. This completes the proof. $\square$

Although we are not able to compute for all k those parameter values for changes of $N(f_\lambda^k)$ in Theorem 5, those data of Table 1 for $k = 1, 2, \dots, 7$ show that the number $N(f_\lambda^k)$ can reach its maximum $2^k - 1$ if $\lambda > \lambda_{k,l}$, $l = 0, 1, 2, 3, 5, 8, 13$. We naturally have the following.

Question 1. Does $N(f_\lambda^k)$ reach the maximum $2^k - 1$ for any integer $k \geq 1$?

Additionally, the well-known Feigenbaum sequence (see [6, 7]) is $\lambda_1 = 3$, $\lambda_2 \approx 3.449490$, $\lambda_3 \approx 3.544090$, $\lambda_4 \approx 3.564407$, $\lambda_5 \approx 3.568750$, $\lambda_6 = 3.569690, \dots$, at each of which a period-doubling bifurcation happens in the logistic mapping. This suggests the following.

Question 2. Is there any relation between our sequence $\{\lambda_{k,l}\}$ and the Feigenbaum sequence $\{\lambda_k\}$?

A related work can be found from [8], but the question is not answered yet.

Competing Interests

The authors declare that they have no competing interests.

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