Research Article

Further Results on Resistance Distance and Kirchhoff Index in Electric Networks

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In electric circuit theory, it is of great interest to compute the effective resistance between any pairs of vertices of a network, as well as the Kirchhoff index. Let $Q(G)$ be the graph obtained from G by inserting a new vertex into every edge of G and by joining by edges those pairs of these new vertices which lie on adjacent edges of G . The set of such new vertices is denoted by $I(G)$. The Q -vertex corona of G_1 and G_2 , denoted by $G_1 \odot_Q G_2$, is the graph obtained from vertex disjoint $Q(G_1)$ and $|V(G_1)|$ copies of G_2 by joining the i th vertex of $V(G_1)$ to every vertex in the i th copy of G_2 . The Q -edge corona of G_1 and G_2 , denoted by $G_1 \ominus_Q G_2$, is the graph obtained from vertex disjoint $Q(G_1)$ and $|I(G_1)|$ copies of G_2 by joining the i th vertex of $I(G_1)$ to every vertex in the i th copy of G_2 . The objective of the present work is to obtain the resistance distance and Kirchhoff index for composite networks such as Q -vertex corona and Q -edge corona networks.

1. Introduction

For decades we want to know what a graph looks like. We want to reveal the principles of the networks behaviour covered by their complex topology and dynamics. We want to learn about how the network structure evolves over time and how it affects the properties of the dynamic processes on it. Klein and Randić [1] introduced a new distance function named resistance distance based on electric network theory; the resistance distance between vertices v_i and v_j , denoted by $r_G(v_i, v_j)$, is defined to be the effective electrical resistance between them if each edge of G is replaced by a unit resistor. Resistance distance is an important character of a graph, which can imply many of its dynamic properties. The Kirchhoff index of G is the sum of resistance distances between all pairs of vertices of G . The Kirchhoff index of a network denotes the mean first-passage time measuring the efficiency of random walks; see [1]. The resistance distance and the Kirchhoff index attracted extensive attention due to its wide applications in complex network, physics, chemistry, and others. For more information on resistance distance and

Kirchhoff index of graphs, the readers are referred to [2–22] and the references therein. In view of the applications of the resistance distance and the Kirchhoff index, it is of great interest to calculate this parameter for composite networks and find possible relations between the resistance distance and Kirchhoff indexes of the original networks and those of their composite networks; see, for instance, [3].

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Let d_i be the degree of vertex i in G and let $D_G = \text{diag}(d_1, d_2, \dots, d_{|V(G)|})$ be the diagonal matrix with all vertex degrees of G as its diagonal entries. For a graph G , let A_G and B_G denote the adjacency matrix and vertex-edge incidence matrix of G , respectively. The matrix $L_G = D_G - A_G$ is called the Laplacian matrix of G , where D_G is the diagonal matrix of vertex degrees of G . We use $\mu_1(G) \geq \mu_2(G) \geq \dots \geq \mu_n(G) = 0$ to denote the eigenvalues of L_G . Let $l(G)$ be the line graph of G .

The $\{1\}$ -inverse of M is a matrix X such that $MXM = M$. If M is singular, then it has infinite many $\{1\}$ -inverses [6]. We use $M^{(1)}$ to denote any $\{1\}$ -inverse of a matrix M , and let

FIGURE 1: $C_3 \odot_Q P_2$ and $C_3 \ominus_Q P_2$.

$(M)_{uv}$ denote the (u, v) -entry of M . For a square matrix M , the group inverse of M , denoted by $M^\#$, is the unique matrix X such that $MXM = M$, $XXM = X$, and $MX = XM$. It is known that $M^\#$ exists if and only if $\text{rank}(M) = \text{rank}(M^2)$ [6, 9]. If M is real symmetric, then $M^\#$ exists and $M^\#$ is a symmetric $\{1\}$ -inverse of M . Actually, $M^\#$ is equal to the Moore-Penrose inverse of M since M is symmetric [9]. It is known that resistance distances in a connected graph G can be obtained from any $\{1\}$ -inverse of G [7, 8].

For a graph G , let $Q(G)$ be the graph obtained from G by inserting a new vertex into every edge of G and by joining by edges those pairs of these new vertices which lie on adjacent edges of G . Let $I(G)$ be the set of newly added vertices; that is, $I(G) = V(Q(G)) \setminus V(G)$. Next, we define two new operations as follows.

Let G_1 and G_2 be two vertex-disjoint graphs.

Definition 1. The Q -vertex corona of G_1 and G_2 , denoted by $G_1 \odot_Q G_2$, is the graph obtained from vertex disjoint $Q(G_1)$ and $|V(G_1)|$ copies of G_2 by joining the i th vertex of $V(G_1)$ to every vertex in the i th copy of G_2 .

Definition 2. The Q -edge corona of G_1 and G_2 , denoted by $G_1 \ominus_Q G_2$, is the graph obtained from vertex disjoint $Q(G_1)$ and $|I(G_1)|$ copies of G_2 by joining the i th vertex of $I(G_1)$ to every vertex in the i th copy of G_2 .

Note that if G_i has n_i vertices and m_i edges for $i = 1, 2$, then $G_1 \odot_Q G_2$ has $n_1 + m_1 + n_1 n_2$ vertices and $G_1 \ominus_Q G_2$ has $n_1 + m_1 + m_1 n_2$ vertices.

Let P_n and C_n denote a path and cycle with n vertices, respectively. From these definitions, Figure 1 shows the graphs $C_3 \odot_Q P_2$ and $C_3 \ominus_Q P_2$.

Bu et al. investigated resistance distance in subdivision-vertex join and subdivision-edge join of graphs [4]. Liu et al. [5] gave the resistance distance and Kirchhoff index of R -vertex join and R -edge join of two graphs. Motivated by these works, in this paper we will work on two different composite networks: Q -vertex corona and Q -edge corona networks. In fact, we will follow the techniques used in [4] but in a slightly different method in order to obtain the effective resistances and the Kirchhoff indexes of Q -vertex corona and Q -edge corona networks in terms of the same parameters on the factors.

2. Preliminaries

In this section we list some lemmas as underlying but necessary preliminaries, which will be used in the proofs of our main results.

Lemma 3 (see [7, 9]). *Let G be a connected graph. Then*

$$\begin{aligned} r_{uv}(G) &= (L_G^{(1)})_{uu} + (L_G^{(1)})_{vv} - (L_G^{(1)})_{uv} - (L_G^{(1)})_{vu} \\ &= (L_G^\#)_{uu} + (L_G^\#)_{vv} - 2(L_G^\#)_{uv}. \end{aligned} \quad (1)$$

Let 1_n denote the column vector of dimension n with all the entries equal to one. We will often use 1 to denote an all-ones column vector if the dimension can be read from the context.

Lemma 4 (see [4]). *For any graph, one has $L_G^\# 1 = 0$.*

Lemma 5 (see [17]). *Let*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad (2)$$

be a nonsingular matrix. If A and D are nonsingular, then

$$\begin{aligned} M^{-1} &= \begin{pmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{pmatrix} \\ &= \begin{pmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{pmatrix}, \end{aligned} \quad (3)$$

where $S = D - CA^{-1}B$.

For a square matrix M , let $\text{tr}(M)$ denote the trace of M .

Lemma 6 (see [10]). *Let G be a connected graph on n vertices. Then*

$$\text{Kf}(G) = n \text{tr}(L_G^{(1)}) - 1^T L_G^{(1)} 1 = n \text{tr}(L_G^\#). \quad (4)$$

Lemma 7. *Let*

$$L = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix} \quad (5)$$

be the Laplacian matrix of a connected graph. If D is nonsingular, then

$$X = \begin{pmatrix} H^\# & -H^\# BD^{-1} \\ -D^{-1} B^T H^\# & D^{-1} + D^{-1} B^T H^\# BD^{-1} \end{pmatrix} \quad (6)$$

is a symmetric $\{1\}$ -inverse of L , where $H = A - BD^{-1}B^T$.

Proof. Since $H = A - BD^{-1}B^T$ is symmetric, $H^\#$ exists and is symmetric. Since

$$L = \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} H & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & 0 \\ D^{-1}B^T & I \end{pmatrix}, \quad (7)$$

we know that

$$X = \begin{pmatrix} I & 0 \\ -D^{-1}B^T & I \end{pmatrix} \begin{pmatrix} H^\# & 0 \\ 0 & D^{-1} \end{pmatrix} \begin{pmatrix} I & -BD^{-1} \\ 0 & I \end{pmatrix} \quad (8)$$

is a symmetric $\{1\}$ -inverse of L . $\square$

Remarks 1. The above result is similar to Lemma 2.8 in [10]; this is another form of Lemma 2.8, but in the process of computing the resistance distance of graphs $G_1 \odot_Q G_2$ and $G_1 \ominus_Q G_2$, we use this formula to be more superior than Lemma 2.8 in [10].

3. Resistance Distance in Q-Vertex Coronae and Q-Edge Coronae of Two Graphs

We first give explicit resistance distance for the arbitrary two-vertex resistance distance in $G_1 \odot_Q G_2$.

Theorem 8. Let G_1 be an r_1 -regular graph on n_1 vertices and m_1 edges, and let G_2 be an arbitrary graph on n_2 vertices and m_2 edges. Then the following holds:

(i) For any $e, f \in E(G_1)$, one has

$$r_{ef}(G_1 \odot_Q G_2) = \frac{r_1}{1+r_1} r_{ef}(l(G_1)). \quad (9)$$

(ii) For any $u, v \in V(G_2)$, one has

$$\begin{aligned} r_{uv}(G_1 \odot_Q G_2) &= \left(I_{n_1} \otimes \left(L_{G_2} + I_{n_2} - \frac{1}{n_2 + r_1} j_{n_2 \times n_2} \right)^{-1} \right)_{uu} \\ &\quad + \left(I_{n_1} \otimes \left(L_{G_2} + I_{n_2} - \frac{1}{n_2 + r_1} j_{n_2 \times n_2} \right)^{-1} \right)_{vv} \\ &\quad - 2 \left(I_{n_1} \otimes \left(L_{G_2} + I_{n_2} - \frac{1}{n_2 + r_1} j_{n_2 \times n_2} \right)^{-1} \right)_{uv}. \end{aligned} \quad (10)$$

(iii) For any $u \in V(G_1)$, let $e_1, e_2, \dots, e_r$ be r edges incident to u in G_1 . For any $u \in V(G_1)$, $f \in E(G_1)$, one has

$$\begin{aligned} r_{uf}(G_1 \odot_Q G_2) &= \left(\frac{1}{r_1} I_{n_1 \times n_1} + \frac{1}{r_1(1+r_1)} RL_{l(G_1)}^\# R^T \right)_{uu} \\ &\quad + \frac{r_1}{1+r_1} (L_{l(G_1)}^\#)_{ff} \\ &\quad - \frac{2}{r_1(1+r_1)} (RL_{l(G_1)}^\#)_{uf}. \end{aligned} \quad (11)$$

(iv) For any $u, v \in V(G_1)$ ($u \neq v$), let $e_1, e_2, \dots, e_r$ (resp., $f_1, f_2, \dots, f_r$) be r edges incident to u (resp., v) in G_1 , one has

$$\begin{aligned} r_{uv}(G_1 \odot_Q G_2) &= \frac{2}{r_1} + \frac{1}{r_1(r_1+1)} (RL_{l(G_1)}^\# R^T)_{uu} \\ &\quad + \frac{1}{r_1(r_1+1)} (RL_{l(G_1)}^\# R^T)_{vv} \\ &\quad - \frac{2}{r_1+1} (RL_{l(G_1)}^\# R^T)_{uv}. \end{aligned} \quad (12)$$

(v) For any $u \in V(G_1)$, let $e_1, e_2, \dots, e_r$ be r edges incident to u in G_1 . For any $u \in V(G_1)$, $v \in V(G_2)$, one has

$$\begin{aligned} r_{uv}(G_1 \odot_Q G_2) &= \frac{1}{r_1} + \frac{1}{r_1(r_1+1)} (RL_{l(G_1)}^\# R^T)_{uu} \\ &\quad + \left(I_{n_1} \otimes \left(L_{G_2} + I_{n_2} - \frac{1}{n_2 + r_1} j_{n_2 \times n_2} \right)^{-1} \right)_{vv} \\ &\quad - 2 \left(I_{n_1} \otimes \left(L_{G_2} + I_{n_2} - \frac{1}{n_2 + r_1} j_{n_2 \times n_2} \right)^{-1} \right)_{uv}. \end{aligned} \quad (13)$$

(vi) For any $e \in E(G_1)$, $v \in V(G_2)$,

$$\begin{aligned} r_{ev}(G_1 \odot_Q G_2) &= \frac{r_1}{1+r_1} (L_{l(G_1)}^\#)_{ee} \\ &\quad + \left(I_{n_1} \otimes \left(L_{G_2} + I_{n_2} - \frac{1}{n_2 + r_1} j_{n_2 \times n_2} \right)^{-1} \right)_{vv} \\ &\quad - \frac{2}{1+r_1} (RL_{l(G_1)}^\#)_{ev}. \end{aligned} \quad (14)$$

Proof. Let B and R be the adjacency matrix of $l(G_1)$ and incidence matrix of G_1 , respectively. Then the Laplacian matrix of $G_1 \odot_Q G_2$ is as follows:

$$\begin{aligned} L(G_1 \odot_Q G_2) &= \begin{pmatrix} 2r_1 I_{m_1} - B & -R^T & 0_{m_1 \times n_1 n_2} \\ -R & (r_1 + n_2) I_{n_1 \times n_1} & -I_{n_1} \otimes 1_{n_2}^T \\ 0_{n_1 n_2 \times m_1} & -I_{n_1} \otimes 1_{n_2} & I_{n_1} \otimes (L_{G_2} + I_{n_2}) \end{pmatrix}. \end{aligned} \quad (15)$$

Let $A = 2r_1 I_{m_1} - B$, $B = \begin{pmatrix} -R^T & 0_{m_1 \times n_1 n_2} \end{pmatrix}$, $B^T = \begin{pmatrix} -R & 0_{n_1 n_2 \times m_1} \end{pmatrix}$, and

$$D = \begin{pmatrix} (r_1 + n_2) I_{n_1 \times n_1} & -I_{n_1} \otimes 1_{n_2}^T \\ -I_{n_1} \otimes 1_{n_2} & I_{n_1} \otimes (L_{(G_2)} + I_{n_2}) \end{pmatrix}. \quad (16)$$

Note that $RR^T = D_{G_1} + A_{G_1}$. Let $R_1 = I_{n_1} \otimes (L_{(G_2)} + I_{n_2}) - (-I_{n_1} \otimes 1_{n_2})((r_1 + n_2) I_{n_1 \times n_1})^{-1}(-I_{n_1} \otimes 1_{n_2}^T) = I_{n_1} \otimes (L_{G_2} + I_{n_2} - (1/(n_2 + r_1)) j_{n_2 \times n_2})^{-1}$. By Lemma 5, we have

$$D^{-1} = \begin{pmatrix} \frac{1}{r_1} I_{n_1 \times n_1} & \frac{1}{r_1} (I_{n_1} \otimes 1_{n_2}^T) \\ \frac{1}{r_1} (I_{n_1} \otimes 1_{n_2}) & I_{n_1} \otimes \left(L_{G_2} + I_{n_2} - \frac{1}{n_2 + r_1} j_{n_2 \times n_2} \right)^{-1} \end{pmatrix}. \quad (17)$$

Now we are ready to calculate $\{1\}$ -inverse of $L(G_1 \odot_Q G_2)$.

First we begin with the calculation about H .

Let $Q = I_{n_1} \otimes (L_{G_2} + I_{n_2} - (1/(n_2 + r_1)) j_{n_2 \times n_2})$; then

$$\begin{aligned} H &= 2r_1 I_{m_1} - B - \begin{pmatrix} -R^T & 0_{m_1 \times n_1 n_2} \end{pmatrix} \begin{pmatrix} \frac{1}{r_1} I_{n_1} & \frac{1}{r_1} (I_{n_1} \otimes 1_{n_2}^T) \\ \frac{1}{r_1} (I_{n_1} \otimes 1_{n_2}) & Q^{-1} \end{pmatrix} \begin{pmatrix} -R \\ 0_{n_1 n_2 \times m_1} \end{pmatrix} \\ &= 2r_1 I_{m_1} - B - \begin{pmatrix} -\frac{1}{r_1} R^T & -\frac{1}{r_1} R^T (I_{n_1} \otimes 1_{n_2}^T) \end{pmatrix} \begin{pmatrix} -R \\ 0_{n_1 n_2 \times m_1} \end{pmatrix} = 2r_1 I_{m_1} - B - \frac{1}{r_1} (2I_{m_1} + B) = \frac{1+r_1}{r_1} L_{l(G_1)}. \end{aligned} \quad (18)$$

By Lemma 7, we have $H^\# = (r_1/(1+r_1))L_{l(G_1)}^\#$.

Next according to Lemma 7, we calculate $-H^\#BD^{-1}$ and $-D^{-1}B^TH^\#$:

$$-H^\#BD^{-1} = -\frac{r_1}{1+r_1}$$

$$\cdot L_{l(G_1)}^\# \begin{pmatrix} -R^T & 0_{m_1 \times n_1 n_2} \end{pmatrix} \begin{pmatrix} \frac{1}{r_1} I_{n_1} & \frac{1}{r_1} (I_{n_1} \otimes 1_{n_2}^T) \\ \frac{1}{r_1} (I_{n_1} \otimes 1_{n_2}) & Q^{-1} \end{pmatrix}$$

$$= -\frac{r_1}{1+r_1} L_{l(G_1)}^\# \begin{pmatrix} -\frac{1}{r_1} R^T & -\frac{1}{r_1} R^T (I_{n_1} \otimes 1_{n_2}^T) \end{pmatrix} = \frac{1}{1+r_1}$$

$$\cdot (L_{l(G_1)}^\# R^T \quad L_{l(G_1)}^\# R^T (I_{n_1} \otimes 1_{n_2}^T)),$$

$$-D^{-1}B^TH^\#$$

$$= -\begin{pmatrix} \frac{1}{r_1} I_{n_1} & \frac{1}{r_1} (I_{n_1} \otimes 1_{n_2}^T) \\ \frac{1}{r_1} (I_{n_1} \otimes 1_{n_2}) & Q^{-1} \end{pmatrix} \begin{pmatrix} -R \\ 0_{n_1 n_2 \times m_1} \end{pmatrix} \frac{r_1}{1+r_1}$$

$$\cdot L_{l(G_1)}^\# = \begin{pmatrix} \frac{1}{r_1} R \\ \frac{1}{r_1} (I_{n_1} \otimes 1_{n_2}) R \end{pmatrix} \frac{r_1}{1+r_1} L_{l(G_1)}^\# = \frac{1}{1+r_1}$$

$$\cdot \begin{pmatrix} RL_{l(G_1)}^\# \\ (I_{n_1} \otimes 1_{n_2}) RL_{l(G_1)}^\# \end{pmatrix}.$$

(19)

We are ready to compute $D^{-1}B^TH^\#BD^{-1}$.

Let $K = I_{n_1} \otimes 1_{n_2}$, $Q = I_{n_1} \otimes (L_{G_2} + I_{n_2} - (1/(n_2 + r_1)) j_{n_2 \times n_2})^{-1}$; then

$$D^{-1}B^TH^\#BD^{-1} = -\frac{1}{1+r_1}$$

$$\cdot \begin{pmatrix} RL_{l(G_1)}^\# \\ KRL_{l(G_1)}^\# \end{pmatrix} \begin{pmatrix} -R^T & 0_{m_1 \times n_1 n_2} \end{pmatrix} \begin{pmatrix} \frac{1}{r_1} I_{n_1 \times n_1} & \frac{1}{r_1} K^T \\ \frac{1}{r_1} K & Q^{-1} \end{pmatrix}$$

$$= -\frac{1}{1+r_1} \begin{pmatrix} RL_{l(G_1)}^\# \\ KRL_{l(G_1)}^\# \end{pmatrix} \begin{pmatrix} -\frac{1}{r_1} R^T & -\frac{1}{r_1} R^T K^T \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{r_1(1+r_1)} RL_{l(G_1)}^\# R^T & \frac{1}{r_1(1+r_1)} RL_{l(G_1)}^\# R^T K^T \\ \frac{1}{r_1(1+r_1)} KRL_{l(G_1)}^\# R^T & \frac{1}{r_1(1+r_1)} KRL_{l(G_1)}^\# R^T K^T \end{pmatrix}.$$

Based on Lemmas 5 and 7, the following matrix

$$N = \begin{pmatrix} \frac{r_1}{1+r_1} L_{l(G_1)}^\# & \frac{1}{1+r_1} L_{l(G_1)}^\# R^T & \frac{1}{1+r_1} L_{l(G_1)}^\# R^T K^T \\ \frac{1}{1+r_1} RL_{l(G_1)}^\# & \frac{1}{r_1} I_{n_1 \times n_1} + \frac{1}{r_1(1+r_1)} RL_{l(G_1)}^\# R^T & \frac{1}{r_1(1+r_1)} RL_{l(G_1)}^\# R^T K^T + \frac{1}{r_1} K^T \\ \frac{1}{1+r_1} KRL_{l(G_1)}^\# & \frac{1}{r_1} K + \frac{1}{r_1(1+r_1)} KRL_{l(G_1)}^\# R^T & Q^{-1} + \frac{1}{r_1(1+r_1)} KRL_{l(G_1)}^\# R^T K^T \end{pmatrix} \quad (21)$$

is a symmetric $\{1\}$ -inverse of $L(G_1 \odot_Q G_2)$, where $K = I_{n_1} \otimes 1_{n_2}$ and $Q = I_{n_1} \otimes (L_{G_2} + I_{n_2} - (1/(n_2 + r_1))j_{n_2 \times n_2})$.

For any $e, f \in E(G_1)$, by Lemma 3 and (21), we have

$$r_{ef}(G_1 \odot_Q G_2) = \frac{r_1}{1 + r_1} r_{ef}(l(G_1)). \quad (22)$$

For any $u, v \in V(G_2)$, by Lemma 3 and (21), we have

$$\begin{aligned} r_{uv}(G_1 \odot_Q G_2) &= \left(I_{n_1} \otimes \left(L_{G_2} + I_{n_2} - \frac{1}{n_2 + r_1} j_{n_2 \times n_2} \right)^{-1} \right)_{uu} \\ &\quad + \left(I_{n_1} \otimes \left(L_{G_2} + I_{n_2} - \frac{1}{n_2 + r_1} j_{n_2 \times n_2} \right)^{-1} \right)_{vv} \\ &\quad - 2 \left(I_{n_1} \otimes \left(L_{G_2} + I_{n_2} - \frac{1}{n_2 + r_1} j_{n_2 \times n_2} \right)^{-1} \right)_{uv}. \end{aligned} \quad (23)$$

For any $u \in V(G_1)$, let $e_1, e_2, \dots, e_r$ be r edges incident to u in G_1 . For any $u \in V(G_1)$, $f \in E(G_1)$, by Lemma 3 and (21), we have

$$\begin{aligned} r_{uf}(G_1 \odot_Q G_2) &= \left(\frac{1}{r_1} I_{n_1 \times n_1} + \frac{1}{r_1(1 + r_1)} RL_{l(G_1)}^\# R^T \right)_{uu} \\ &\quad + \frac{r_1}{1 + r_1} (L_{l(G_1)}^\#)_{ff} - \frac{2}{1 + r_1} (RL_{l(G_1)}^\#)_{uf}. \end{aligned} \quad (24)$$

For any $u, v \in V(G_1)$ ($u \neq v$), let $e_1, e_2, \dots, e_r$ (resp., $f_1, f_2, \dots, f_r$) be r edges incident to u (resp., v) in G_1 . By Lemma 3 and (21), we have

$$\begin{aligned} r_{uv}(G_1 \odot_Q G_2) &= \frac{2}{r_1} + \frac{1}{r_1(r_1 + 1)} (RL_{l(G_1)}^\# R^T)_{uu} \\ &\quad + \frac{1}{r_1(r_1 + 1)} (RL_{l(G_1)}^\# R^T)_{vv} \\ &\quad - \frac{2}{r_1(r_1 + 1)} (RL_{l(G_1)}^\# R^T)_{uv}. \end{aligned} \quad (25)$$

For any $u \in V(G_1)$, let $e_1, e_2, \dots, e_r$ be r edges incident to u in G_1 . For any $u \in V(G_1)$, $v \in V(G_2)$, by Lemma 3 and (21), we have

$$\begin{aligned} r_{uv}(G_1 \odot_Q G_2) &= \frac{1}{r_1} + \frac{1}{r_1(r_1 + 1)} (RL_{l(G_1)}^\# R^T)_{uu} \\ &\quad + \left(I_{n_1} \otimes \left(L_{G_2} + I_{n_2} - \frac{1}{n_2 + r_1} j_{n_2 \times n_2} \right)^{-1} \right)_{vv} \\ &\quad - 2 \left(I_{n_1} \otimes \left(L_{G_2} + I_{n_2} - \frac{1}{n_2 + r_1} j_{n_2 \times n_2} \right)^{-1} \right)_{uv}. \end{aligned} \quad (26)$$

For any $e \in E(G_1)$, $v \in V(G_2)$, by Lemma 3 and (21), we have

$$\begin{aligned} r_{ev}(G_1 \odot_Q G_2) &= \frac{r_1}{1 + r_1} (L_{l(G_1)}^\#)_{ee} \\ &\quad + \left(I_{n_1} \otimes \left(L_{G_2} + I_{n_2} - \frac{1}{n_2 + r_1} j_{n_2 \times n_2} \right)^{-1} \right)_{vv} \\ &\quad - \frac{2}{1 + r_1} (RL_{l(G_1)}^\#)_{ev}. \end{aligned} \quad (27)$$

□

Next it will give the formulae for resistance distance of graph $G_1 \odot_Q G_2$ when G_1 is a regular graph as follows.

Theorem 9. Let G_1 be an r_1 -regular graph with n_1 vertices, and let G_2 be an arbitrary graph with n_2 vertices. Then the following holds:

(i) For any $e, f \in E(G_1)$, one has

$$r_{ef}(G_1 \odot_Q G_2) = \frac{r_1}{1 + r_1} r_{ef}(l(G_1)). \quad (28)$$

(ii) For any $u, v \in V(G_2)$, one has

$$\begin{aligned} r_{uv}(G_1 \odot_Q G_2) &= \left(I_{m_1} \otimes (L_{(G_2)} + I_{n_2})^{-1} \right)_{uu} \\ &\quad + \left(I_{m_1} \otimes (L_{(G_2)} + I_{n_2})^{-1} \right)_{vv} \\ &\quad - 2 \left(I_{m_1} \otimes (L_{(G_2)} + I_{n_2})^{-1} \right)_{uv}. \end{aligned} \quad (29)$$

(iii) For any $u \in V(G_1)$, let $e_1, e_2, \dots, e_r$ be r edges incident to u in G . For any $u \in V(G_1)$, $f \in E(G_1)$, one has

$$\begin{aligned} r_{uf}(G_1 \odot_Q G_2) &= \left(\frac{1}{r_1} I_{n_1 \times n_1} + \frac{1}{r_1(1 + r_1)} RL_{l(G_1)}^\# R^T \right)_{uu} \\ &\quad + \frac{r_1}{1 + r_1} (L_{l(G_1)}^\#)_{ff} - \frac{2}{1 + r_1} (RL_{l(G_1)}^\#)_{uf}. \end{aligned} \quad (30)$$

(iv) For any $u, v \in V(G_1)$ ($u \neq v$), let $e_1, e_2, \dots, e_r$ (resp., $f_1, f_2, \dots, f_r$) be r edges incident to u (resp., v) in G_1 ; one has

$$\begin{aligned} r_{uv}(G_1 \odot_Q G_2) &= \frac{2}{r_1} + \frac{1}{r_1(r_1 + 1)} (RL_{l(G_1)}^\# R^T)_{uu} \\ &\quad + \frac{1}{r_1(r_1 + 1)} (RL_{l(G_1)}^\# R^T)_{vv} \\ &\quad - \frac{2}{r_1(r_1 + 1)} (RL_{l(G_1)}^\# R^T)_{uv}. \end{aligned} \quad (31)$$

(v) For any $u \in V(G_1)$, let $e_1, e_2, \dots, e_r$ be r edges incident to u in G_1 . For any $u \in V(G_1)$, $v \in V(G_2)$, one has

$$\begin{aligned} r_{uv}(G_1 \ominus_Q G_2) &= \frac{1}{r_1} + \frac{1}{r_1(r_1 + 1)} (RL_{l(G_1)}^\# R^T)_{uu} \\ &\quad + (I_{m_1} \otimes (L_{(G_2)} + I_{n_2})^{-1})_{vv} \\ &\quad - 2(I_{m_1} \otimes (L_{(G_2)} + I_{n_2})^{-1})_{uv}. \end{aligned} \quad (32)$$

(vi) For any $e \in E(G_1)$, $v \in V(G_2)$, one has

$$\begin{aligned} r_{ev}(G_1 \ominus_Q G_2) &= \frac{r_1}{1 + r_1} (L_{l(G_1)}^\#)_{ee} \\ &\quad + (I_{m_1} \otimes (L_{(G_2)} + I_{n_2})^{-1})_{vv} \\ &\quad - \frac{2}{1 + r_1} (RL_{l(G_1)}^\#)_{ev}. \end{aligned} \quad (33)$$

Proof. Let B and R be the adjacency matrix of $l(G_1)$ and incidence matrix of G_1 , respectively. Then the Laplacian matrix of $G_1 \ominus_Q G_2$ is as follows:

$$\begin{aligned} L(G_1 \ominus_Q G_2) &= \begin{pmatrix} (2r_1 + n_2)I_{m_1} - B & -R^T & -I_{m_1} \otimes 1_{n_2}^T \\ -R & r_1 I_{n_1} & 0_{n_1 \times n_2 m_1} \\ -I_{m_1} \otimes 1_{n_2} & 0_{n_2 m_1 \times n_1} & I_{m_1} \otimes (L_{(G_2)} + I_{n_2}) \end{pmatrix}. \end{aligned} \quad (34)$$

Now we are ready to calculate a $\{1\}$ -inverse of $L(G_1 \ominus_Q G_2)$.

First we are ready to calculate $H^\#$.

Let $M = I_{m_1} \otimes 1_{n_2}$, $P = I_{m_1} \otimes (L_{(G_2)} + I_{n_2})$, then

$$\begin{aligned} H &= (2r_1 + n_2)I_{m_1} - B \\ &\quad - (-R^T \ -M^T) \begin{pmatrix} r_1 I_{n_1} & 0_{n_1 \times n_2 m_1} \\ 0_{n_2 m_1 \times n_1} & P \end{pmatrix}^{-1} \begin{pmatrix} -R \\ -M \end{pmatrix} \\ &= (2r_1 + n_2)I_{m_1} - B \\ &\quad - \left(-\frac{1}{r_1} R^T \ -M^T P^{-1} \right) \begin{pmatrix} -R \\ -M \end{pmatrix} \\ &= (2r_1 + n_2)I_{m_1} - B - \frac{1}{r_1} R^T R - n_2 I_{m_1} \\ &= \frac{1 + r_1}{r_1} L_{l(G_1)}. \end{aligned} \quad (35)$$

By Lemma 7, we have $H^\# = (r_1/(1 + r_1))L_{l(G_1)}^\#$.

Next we compute $-H^\# B D^{-1}$ and $-D^{-1} B^T H^\#$:

$$\begin{aligned} -H^\# B D^{-1} &= -\frac{r_1}{1 + r_1} \\ &\quad \cdot L_{l(G_1)}^\# (-R^T \ -I_{m_1} \otimes 1_{n_2}^T) \begin{pmatrix} \frac{1}{r_1} I_{n_1} & 0_{n_1 \times n_2 m_1} \\ 0_{n_2 m_1 \times n_1} & I_{m_1} \otimes (L_{(G_2)} + I_{n_2})^{-1} \end{pmatrix} \\ &= -\frac{r_1}{1 + r_1} L_{l(G_1)}^\# \begin{pmatrix} -\frac{1}{r_1} R^T & -I_{m_1} \otimes 1_{n_2}^T \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{r_1 + 1} L_{l(G_1)}^\# R^T & \frac{r_1}{r_1 + 1} L_{l(G_1)}^\# (I_{m_1} \otimes 1_{n_2}^T) \end{pmatrix}, \\ -D^{-1} B^T H^\# &= -\begin{pmatrix} \frac{1}{r_1} I_{n_1} & 0_{n_1 \times n_2 m_1} \\ 0_{n_2 m_1 \times n_1} & I_{m_1} \otimes (L_{(G_2)} + I_{n_2})^{-1} \end{pmatrix} \begin{pmatrix} -R \\ -I_{m_1} \otimes 1_{n_2} \end{pmatrix} \frac{r_1}{1 + r_1} \\ &\quad \cdot L_{l(G_1)}^\# = \frac{r_1}{1 + r_1} \begin{pmatrix} \frac{1}{r_1} R \\ I_{m_1} \otimes 1_{n_2} \end{pmatrix} L_{l(G_1)}^\# \\ &= \begin{pmatrix} \frac{1}{r_1 + 1} R L_{l(G_1)}^\# \\ \frac{r_1}{r_1 + 1} (I_{m_1} \otimes 1_{n_2}) L_{l(G_1)}^\# \end{pmatrix}. \end{aligned} \quad (36)$$

We are ready to compute $D^{-1} B^T H^\# B D^{-1}$:

$$\begin{aligned} D^{-1} B^T H^\# B D^{-1} &= -\begin{pmatrix} \frac{1}{r_1 + 1} R L_{l(G_1)}^\# \\ \frac{r_1}{r_1 + 1} M L_{l(G_1)}^\# \end{pmatrix} (-R^T \ -M^T) \begin{pmatrix} \frac{1}{r_1} I_{n_1} & 0_{n_1 \times n_2 m_1} \\ 0_{n_2 m_1 \times n_1} & P^{-1} \end{pmatrix} \\ &= -\begin{pmatrix} \frac{1}{r_1 + 1} R L_{l(G_1)}^\# \\ \frac{r_1}{r_1 + 1} M L_{l(G_1)}^\# \end{pmatrix} \begin{pmatrix} -\frac{1}{r_1} R^T & -M^T P^{-1} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{r_1(r_1 + 1)} R L_{l(G_1)}^\# R^T & \frac{1}{r_1 + 1} R L_{l(G_1)}^\# M^T \\ \frac{1}{r_1 + 1} M L_{l(G_1)}^\# R^T & \frac{r_1}{r_1 + 1} M L_{l(G_1)}^\# M^T \end{pmatrix}. \end{aligned} \quad (37)$$

Based on Lemmas 5 and 7, the following matrix

$$N = \begin{pmatrix} \frac{r_1}{1 + r_1} L_{l(G_1)}^\# & \frac{1}{r_1 + 1} L_{l(G_1)}^\# R^T & \frac{r_1}{r_1 + 1} L_{l(G_1)}^\# M^T \\ \frac{1}{1 + r_1} R L_{l(G_1)}^\# & \frac{1}{r_1} I_{n_1} + \frac{1}{r_1(r_1 + 1)} R L_{l(G_1)}^\# R^T & \frac{1}{r_1 + 1} R L_{l(G_1)}^\# M^T \\ \frac{1}{1 + r_1} M L_{l(G_1)}^\# & \frac{1}{r_1 + 1} M L_{l(G_1)}^\# R^T & P^{-1} + \frac{r_1}{r_1 + 1} M L_{l(G_1)}^\# M^T \end{pmatrix} \quad (38)$$

is a symmetric $\{1\}$ -inverse of $L(G_1 \ominus_Q G_2)$, where $M = I_{m_1} \otimes 1_{n_2}$ and $P = I_{m_1} \otimes (L_{(G_2)} + I_{n_2})$.

For any $e, f \in E(G_1)$, by Lemma 3 and (38), we have

$$r_{ef}(G_1 \ominus_Q G_2) = \frac{r_1}{1+r_1} r_{ef}(l(G_1)). \quad (39)$$

For any $u, v \in V(G_2)$, by Lemma 3 and (38), we have

$$\begin{aligned} r_{uv}(G_1 \ominus_Q G_2) &= \left(I_{m_1} \otimes (L_{(G_2)} + I_{n_2})^{-1} \right)_{uu} \\ &\quad + \left(I_{m_1} \otimes (L_{(G_2)} + I_{n_2})^{-1} \right)_{vv} \\ &\quad - 2 \left(I_{m_1} \otimes (L_{(G_2)} + I_{n_2})^{-1} \right)_{uv}. \end{aligned} \quad (40)$$

For any $u \in V(G_1)$, let $e_1, e_2, \dots, e_r$ be r edges incident to u in G . For any $u \in V(G_1)$, $f \in E(G_1)$, by Lemma 3 and (38), we have

$$\begin{aligned} r_{uf}(G_1 \ominus_Q G_2) &= \left(\frac{1}{r_1} I_{n_1 \times n_1} + \frac{1}{r_1(1+r_1)} RL_{l(G_1)}^\# R^T \right)_{uu} \\ &\quad + \frac{r_1}{1+r_1} (L_{l(G_1)}^\#)_{ff} \\ &\quad - \frac{2}{1+r_1} (RL_{l(G_1)}^\#)_{uf}. \end{aligned} \quad (41)$$

For any $u, v \in V(G_1)$ ($u \neq v$), let $e_1, e_2, \dots, e_r$ (resp., $f_1, f_2, \dots, f_r$) be r edges incident to u (resp., v) in G_1 . By Lemma 3 and (38), we have

$$\begin{aligned} r_{uv}(G_1 \ominus_Q G_2) &= \frac{2}{r_1} + \frac{1}{r_1(1+r_1)} (RL_{l(G_1)}^\# R^T)_{uu} \\ &\quad + \frac{1}{r_1(1+r_1)} (RL_{l(G_1)}^\# R^T)_{vv} \\ &\quad - \frac{2}{r_1(1+r_1)} (RL_{l(G_1)}^\# R^T)_{uv}. \end{aligned} \quad (42)$$

For any $u \in V(G_1)$, let $e_1, e_2, \dots, e_r$ be r edges incident to u in G_1 . For any $u \in V(G_1)$, $v \in V(G_2)$, by Lemma 3 and (38), we have

$$\begin{aligned} r_{uv}(G_1 \ominus_Q G_2) &= \frac{1}{r_1} + \frac{1}{r_1(1+r_1)} (RL_{l(G_1)}^\# R^T)_{uu} \\ &\quad + \left(I_{m_1} \otimes (L_{(G_2)} + I_{n_2})^{-1} \right)_{vv} \\ &\quad - 2 \left(I_{m_1} \otimes (L_{(G_2)} + I_{n_2})^{-1} \right)_{uv}. \end{aligned} \quad (43)$$

For any $e \in E(G_1)$, $v \in V(G_2)$, by Lemma 3 and (38), we have

$$\begin{aligned} r_{ev}(G_1 \ominus_Q G_2) &= \frac{r_1}{1+r_1} (L_{l(G_1)}^\#)_{ee} \\ &\quad + \left(I_{m_1} \otimes (L_{(G_2)} + I_{n_2})^{-1} \right)_{vv} \\ &\quad - \frac{2}{1+r_1} (RL_{l(G_1)}^\#)_{ev}. \end{aligned} \quad (44)$$

□

4. Kirchhoff Index in Q -Vertex Corona and Q -Edge Corona of Two Graphs

Theorem 10. Let G_1 be an r_1 -regular graph on n_1 vertices and m_1 edges, and let G_2 be an arbitrary graph on n_2 vertices and m_2 edges. Then the following holds:

$$\begin{aligned} \text{Kf}(G_1 \ominus_Q G_2) &= (n_1 + m_1 + n_1 n_2) \left(\frac{r_1}{1+r_1} \text{tr}(L_{l(G_1)}^\#) \right. \\ &\quad + \frac{n_1}{r_1} I_{n_1} + \frac{1}{r_1(1+r_1)} \text{tr}(RL_{l(G_1)}^\# R^T) \\ &\quad + n_1 \sum_{i=1}^{n_2} \frac{1}{\mu_i(G_2) + 1} + \frac{1}{r_1(r_1 + 1)} \text{tr}[(I_{n_1} \otimes 1_{n_2}) \\ &\quad \cdot RL_{l(G_1)}^\# R^T (I_{n_1} \otimes 1_{n_2}^T)] \Big) - \frac{n_1}{r_1} - 3n_1 n_2. \end{aligned} \quad (45)$$

Proof. Let $L_{G_1 \ominus_Q G_2}^{(1)}$ be the symmetric $\{1\}$ -inverse of $L_{G_1 \ominus_Q G_2}$. Then

$$\begin{aligned} \text{tr}(L_{G_1 \ominus_Q G_2}^{(1)}) &= \frac{r_1}{1+r_1} \text{tr}(L_{l(G_1)}^\#) \\ &\quad + \text{tr} \left(\frac{1}{r_1} I_{n_1} + \frac{1}{r_1(1+r_1)} RL_{l(G_1)}^\# R^T \right) \\ &\quad + \text{tr} \left(I_{n_1} \otimes \left(L_{G_2} + I_{n_2} - \frac{1}{n_2 + r_1} j_{n_2 \times n_2} \right)^{-1} \right) \\ &\quad + \frac{1}{r_1(r_1 + 1)} \text{tr}[(I_{n_1} \otimes 1_{n_2}) RL_{l(G_1)}^\# R^T (I_{n_1} \otimes 1_{n_2}^T)]. \end{aligned} \quad (46)$$

Note that the eigenvalues of $(L_{G_2} + I_{n_2} - (1/(n_2 + r_1)) j_{n_2 \times n_2})$ are $\mu_1(G_2) + 1, \mu_2(G_2) + 1, \dots, \mu_{n_2}(G_2) + 1$. Then

$$\begin{aligned} &\text{tr} \left(I_{n_1} \otimes \left(L_{G_2} + I_{n_2} - \frac{1}{n_2 + r_1} j_{n_2 \times n_2} \right)^{-1} \right) \\ &= n_1 \sum_{i=1}^{n_2} (\mu_i(G_2) + 1)^{-1} = n_1 \sum_{i=1}^{n_2} \frac{1}{\mu_i(G_2) + 1}. \end{aligned} \quad (47)$$

So

$$\begin{aligned} \text{tr}(L_{G_1 \ominus_Q G_2}^{(1)}) &= \frac{r_1}{1+r_1} \text{tr}(L_{l(G_1)}^\#) + \frac{n_1}{r_1} I_{n_1} \\ &\quad + \frac{1}{r_1(1+r_1)} \text{tr}(RL_{l(G_1)}^\# R^T) + n_1 \sum_{i=1}^{n_2} \frac{1}{\mu_i(G_2) + 1} \\ &\quad + \frac{1}{r_1(r_1 + 1)} \text{tr}[(I_{n_1} \otimes 1_{n_2}) RL_{l(G_1)}^\# R^T (I_{n_1} \otimes 1_{n_2}^T)]. \end{aligned} \quad (48)$$

Next, we calculate $1^T(L_{G_1 \odot_Q G_2}^{(1)})1$. Since, by Lemma 4, $L_G^\# 1 = 0$, then

$$\begin{aligned} 1^T(L_{G_1 \odot_Q G_2}^{(1)})1 &= 1^T \left(\frac{1}{r_1} I_{n_1} + \frac{1}{r_1(1+r_1)} RL_{l(G_1)}^\# R^T \right) \\ &\quad \cdot 1 + \frac{1}{r_1(1+r_1)} 1^T \left[RL_{l(G_1)}^\# R^T (I_{n_1} \otimes 1_{n_2}^T) \right. \\ &\quad \left. + \frac{1}{r_1} (I_{n_1} \otimes 1_{n_2}^T) \right] 1 + 1^T \left[\frac{1}{r_1} (I_{n_1} \otimes 1_{n_2}) \right. \\ &\quad \left. + \frac{1}{r_1(1+r_1)} (I_{n_1} \otimes 1_{n_2}) RL_{l(G_1)}^\# R^T \right] 1 + 1^T \left(Q^{-1} \right. \\ &\quad \left. + \frac{1}{r_1(1+r_1)} (I_{n_1} \otimes 1_{n_2}) RL_{l(G_1)}^\# R^T (I_{n_1} \otimes 1_{n_2}^T) \right) 1, \end{aligned} \quad (49)$$

where $Q = L_{G_2} + I_{n_2} - (1/(n_2 + r_1))j_{n_2 \times n_2}$. Since $1^T R = R^T 1 = 2 \cdot 1$, then

$$\begin{aligned} 1^T RL_{l(G_1)}^\# R^T 1 &= 1^T (RL_{l(G_1)}^\# R^T (I_{m_1} \otimes 1_{n_2}^T)) 1 \\ &= 1^T (I_{n_1} \otimes 1_{n_2}^T) RL_{l(G_1)}^\# R^T 1 \\ &= 1^T (I_{n_1} \otimes 1_{n_2}) RL_{l(G_1)}^\# R^T (I_{n_1} \otimes 1_{n_2}^T) 1 \\ &= 0. \end{aligned} \quad (50)$$

Let $T = 1_{n_1 n_2}^T (I_{n_1} \otimes (L_{G_2} + I_{n_2} - (1/(n_2 + r_1))j_{n_2 \times n_2})^{-1}) 1_{n_2 n_1}$; then

$$\begin{aligned} T &= (1_{n_2}^T \ 1_{n_2}^T \ \cdots \ 1_{n_2}^T) \begin{pmatrix} Q^{-1} & & \\ & Q^{-1} & \\ & & \ddots \\ & & & Q^{-1} \end{pmatrix} \begin{pmatrix} 1_{n_2} \\ 1_{n_2} \\ \vdots \\ 1_{n_2} \end{pmatrix} \\ &= \left(1_{n_2}^T \left(L_{G_2} + I_{n_2} - \frac{1}{n_2 + r_1} j_{n_2 \times n_2} \right)^{-1} \cdots 1_{n_2}^T \left(L_{G_2} + I_{n_2} - \frac{1}{n_2 + r_1} j_{n_2 \times n_2} \right)^{-1} \right) \begin{pmatrix} 1_{n_2} \\ 1_{n_2} \\ \vdots \\ 1_{n_2} \end{pmatrix} \\ &= n_1 1_{n_2}^T \left(L_{G_2} + I_{n_2} - \frac{1}{n_2 + r_1} j_{n_2 \times n_2} \right)^{-1} 1_{n_2} = n_1 n_2. \end{aligned} \quad (51)$$

So

$$1^T(L_{G_1 \odot_Q G_2}^{(1)})1 = \frac{n_1}{r_1} + 3n_1 n_2. \quad (52)$$

Lemma 6 implies that

$$\begin{aligned} \text{Kf}(L_{G_1 \odot_Q G_2}^{(1)}) &= (n_1 + m_1 + n_1 n_2) \text{tr}(L_{G_1}^\#) \\ &\quad - 1^T(L_{G_1 \odot_Q G_2}^{(1)})1. \end{aligned} \quad (53)$$

Then plugging $\text{tr}(L_{G_1 \odot_Q G_2}^{(1)})$ and $1^T(L_{G_1 \odot_Q G_2}^{(1)})1$ into the equation above, we obtain the required result. $\square$

Theorem 11. Let G_1 be an r_1 -regular graph with n_1 vertices, and let G_2 be an arbitrary graph with n_2 vertices. Then

$$\begin{aligned} \text{Kf}(G_1 \odot_Q G_2) &= (n_1 + m_1 + m_1 n_2) \left(\frac{r_1}{1+r_1} \text{tr}(L_{l(G_1)}^\#) \right. \\ &\quad \left. + \frac{n_2}{r_1} + \frac{1}{r_1(r_1+1)} \text{tr}(RL_{G_1}^\# R^T) \right. \\ &\quad \left. + m_1 \sum_{i=1}^{n_2} \frac{1}{\mu_i(G_2) + 2} \right. \\ &\quad \left. + \frac{1}{1+r_1} \text{tr}[(I_{m_1} \otimes 1_{n_2}) L_{l(G_1)}^\# (I_{m_1} \otimes 1_{n_2}^T)] \right) - \frac{n_1}{r_1} \\ &\quad - n_1 n_2. \end{aligned} \quad (54)$$

Proof. The proof is similar to those of Theorem 10, omitted. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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