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Research Article

Global Solutions in the Species Competitive Chemotaxis System with Inequal Diffusion Rates

Huaihuo Cao[1,2]

[1] *College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, China*
[2] *School of Mathematics and Computer, Chizhou College, Chizhou 247000, China*

Correspondence should be addressed to Huaihuo Cao; caguhh@aliyun.com

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This paper is devoted to studying the two-species competitive chemotaxis system with signal-dependent chemotactic sensitivities and inequal diffusion rates $u_{1t} = \Delta u_1 - \nabla \cdot (u_1\chi_1(v)\nabla v) + \mu_1 u_1(1 - u_1 - a_1u_2)$, $x \in \Omega$, $t > 0$, $u_{2t} = \Delta u_2 - \nabla \cdot (u_2\chi_2(v)\nabla v) + \mu_2 u_2(1 - a_2u_1 - u_2)$, $x \in \Omega$, $t > 0$, $v_t = \tau\Delta v - \gamma v + u_1 + u_2$, $x \in \Omega$, $t > 0$, under homogeneous Neumann boundary conditions in a bounded and regular domain $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$. If the nonnegative initial date $(u_{10}, u_{20}, v_0) \in (C^1(\overline{\Omega}))^3$ and $v_0 \in (\underline{v}, \overline{v})$ where the constants $\overline{v} > \underline{v} \geq 0$, the system possesses a unique global solution that is uniformly bounded under some suitable assumptions on the chemotaxis sensitivity functions $\chi_1(v)$, $\chi_2(v)$ and linear chemical production function $-\gamma v + u_1 + u_2$.

1. Introduction

In this paper, we consider the following two-species competitive chemotaxis system with signal-dependent chemotactic sensitivities and inequal diffusion rates:

$$\begin{aligned}
u_{1t} &= \Delta u_1 - \nabla \cdot (u_1\chi_1(v)\nabla v) \\
&\quad + \mu_1 u_1 (1 - u_1 - a_1u_2), \quad x \in \Omega,\ t > 0, \\
u_{2t} &= \Delta u_2 - \nabla \cdot (u_2\chi_2(v)\nabla v) \\
&\quad + \mu_2 u_2 (1 - a_2u_1 - u_2), \quad x \in \Omega,\ t > 0, \\
v_t &= \tau\Delta v - \gamma v + u_1 + u_2, \quad x \in \Omega,\ t > 0, \\
\frac{\partial u_1}{\partial \nu} &= \frac{\partial u_2}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > 0, \\
u_1(x,0) &= u_{10}(x), \\
u_2(x,0) &= u_{20}(x), \\
v(x,0) &= v_0(x), \\
&\qquad x \in \Omega,
\end{aligned} \tag{1}$$

where Ω is a bounded and regular domain in $\mathbb{R}^n$ $(n \geq 1)$ and ν is the outward unit normal vector of the boundary $\partial\Omega$. $u_1(x,t)$ and $u_2(x,t)$ represent the populations densities, and both populations reproduce themselves and mutually compete with the other, according to the classical Lotka-Volterra kinetics [1], and the diffusion rates of the populations are 1. $v(x,t)$ denotes the concentration of the chemoattractant, and the diffusion rate of the chemical substance is strictly less than 1 (i.e., $0 < \tau < 1$). μ_1, μ_2, a_1, a_2, and γ are positive parameters, where μ_1, μ_2 are the growth coefficients and a_1, a_2 are the competitive degradation rates of population, respectively. The chemotactic sensitivity function $\chi_i(v) \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}_+) \cap C^1(\mathbb{R}_+)$ for $i = 1, 2$, which is assumed to be positive. From a biological point of view, when $\chi_i(v) > 0$, populations exhibit a tendency to move towards higher signal concentrations (chemoattraction), while conversely the choice $\chi_i(v) < 0$ leads to a model for chemorepulsion, where populations prefer to move away from the chemical in question [2]. Denote $h(u_1, u_2, v) = -\gamma v + u_1 + u_2$ representing the balance between the production of the chemical substance by the populations themselves and its natural degradation (see [3] for details).

The classical chemotaxis model was first introduced by Keller and Segel using a mathematical model of two parabolic

equations to describe the aggregation of *Dictyostelium discoideum* as well as a soil-living amoeba, in the early 1970s [4]. After the pioneering works of Keller and Segel, a large amount of chemotaxis models has been used to model the phenomena for population dynamics or gravitational collapse, among others. Winkler [5] studied the chemotaxis system

$$\begin{aligned} u_t &= \Delta u - \nabla \cdot \left(u\chi(v)\nabla v\right), \quad x \in \Omega,\ t > 0, \\ v_t &= \Delta v - v + u, \quad x \in \Omega,\ t > 0, \end{aligned} \tag{2}$$

under homogeneous Neumann boundary conditions in a smooth bounded domain $\Omega \subset \mathbb{R}^n$. It was proved that the chemotactic collapse was absented for any nonnegative initial date $u(\cdot,0) \in C^0(\overline{\Omega})$ and $v(\cdot,0) \in W^{1,r}(\Omega)$ with some $r > n$; the corresponding initial-boundary value problem possessed a unique global uniformly bounded solution. Tello and Winkler [6] considered the parabolic-parabolic-elliptic system

$$\begin{aligned} u_t &= \Delta u - \chi_1 \nabla \cdot (u\nabla w) + \mu_1 u\left(1 - u - a_1 v\right), \\ &\qquad x \in \Omega,\ t > 0, \\ v_t &= \Delta v - \chi_2 \nabla \cdot (v\nabla w) + \mu_2 v\left(1 - a_2 u - v\right), \\ &\qquad x \in \Omega,\ t > 0, \\ -\Delta w &= -\lambda w + u + v, \quad x \in \Omega,\ t > 0 \end{aligned} \tag{3}$$

under homogeneous Neumann boundary conditions in a bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ with smooth boundary. Given the suitable positive parameters a_1, a_2 and χ_1, χ_2, they showed that all solutions stabilized towards a uniquely determined spatially homogeneous positive steady state within a certain nonempty range of the logistic growth coefficients μ_1 and μ_2. Negreanu and Tello [7] studied a two-species chemotaxis system with nondiffusive chemoattractant:

$$\begin{aligned} u_t &= \Delta u - \nabla \cdot \left(u\chi_1(w)\nabla w\right) + \mu_1 u(1-u), \\ &\qquad x \in \Omega,\ t > 0, \\ v_t &= \Delta v - \nabla \cdot \left(v\chi_2(w)\nabla w\right) + \mu_2 v(1-v), \\ &\qquad x \in \Omega,\ t > 0, \\ w_t &= h(u,v,w), \quad x \in \Omega,\ t > 0, \end{aligned} \tag{4}$$

under suitable boundary and initial conditions in an n-dimensional open and bounded domain Ω for $n \geq 1$. They considered the case of positive chemosensitivities and chemical production function h increasing as the concentration of the species u, v increasing. The paper proved the global existence and uniform boundedness of solutions, and the asymptotic stability of the spatially homogeneous steady state was a consequence of the growth of h, χ_i and the size of μ_i for $i = 1, 2$.

Reviewing the recent studies, Zhang and Li [8] considered the following fully parabolic system:

$$\begin{aligned} u_t &= \Delta u - \nabla \cdot \left(u\chi_1(w)\nabla w\right) + \mu_1 u\left(1 - u - a_1 v\right), \\ &\qquad x \in \Omega,\ t > 0, \\ v_t &= \Delta v - \nabla \cdot \left(v\chi_2(w)\nabla w\right) + \mu_2 v\left(1 - a_2 u - v\right), \\ &\qquad x \in \Omega,\ t > 0, \\ w_t &= \Delta w - w + u + v, \quad x \in \Omega,\ t > 0, \end{aligned} \tag{5}$$

under homogeneous Neumann boundary conditions in a smooth bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$. By extending the method in [5] (see also [9, 10]), the first step is to estimate some associated weighted functions which depend on signal density, and the second step is to obtain L^∞-bounds of solutions from L^p-bounds using the variation-of-constants representation and a series of standard semigroup arguments (see [2, 5, 11, 12]). They proved that, if the nonnegative initial date $(u(\cdot,0), v(\cdot,0)) \in (C^0(\overline{\Omega}))^2$ and $w(\cdot,0) \in W^{1,r}(\Omega)$ for some $r > n$, the system possesses a unique global solution that is uniformly bounded under some appropriate conditions on the coefficients μ_1, μ_2 and the chemotaxis sensitivity functions $\chi_1(w)$, $\chi_2(w)$.

Inspired by the foregoing research, the main purpose of this paper is to consider the existence of global solution for the two-species competitive chemotaxis model (1) with inequal diffusion rates. This paper is organized as follows: In Section 2, we formulate the main results of this paper by means of the theorem and establish some preliminaries which are important for our proofs. In Section 3, we firstly consider the local existence of solutions and then proceed with the extensibility criterion. Finally, under some appropriate conditions, we prove that the solutions are uniformly bounded in time using an iterative method.

2. Preliminaries and Main Results

For convenience, we denote that $g_1(u_1,u_2) = 1 - u_1 - a_1 u_2$, $g_2(u_1,u_2) = 1 - a_2 u_1 - u_2$, and they are quasi-monotonic decrease in $\mathbb{R}^2_+$. Denote $h(u_1,u_2,v) = -\gamma v + u_1 + u_2$.

In order to establish the global existence and uniform boundedness of solutions to (1), we need to make some restrictive conditions throughout this paper in $h(u_1,u_2,v)$ and $\chi_i(v)$ $(i = 1, 2)$:

(i) Let the initial date v_0 satisfy $0 \leq \underline{v} < v_0 < \overline{v}$, where $\underline{v}$ and $\overline{v}$ are some positive constants.

(ii) There exist positive constants κ_1 and κ_2 such that

$$\gamma v \chi_i(v) \leq \kappa_i \quad \text{for } \underline{v} \leq v \leq \overline{v},\ i = 1, 2. \tag{6}$$

(iii) There exist positive constants κ_{01} and κ_{02} such that

$$0 < \kappa_{0i} \leq \chi_i(v) \exp\left(\int_{\underline{v}}^{v} \chi_i(s)\, ds\right) \quad \text{for } \underline{v} \leq v,\ i = 1, 2. \tag{7}$$

(iv) Assume that

$$\begin{aligned} &h(u_1, u_2, \underline{v}) \geq 0, \\ &h(u_1, u_2, \overline{v}) < 0, \\ &\qquad \text{for } 0 \leq u_1 \leq \overline{u}_1,\ 0 \leq u_2 \leq \overline{u}_2, \end{aligned} \tag{8}$$

where

$$\begin{aligned} \overline{u}_1 &:= f_1(\overline{v}) \\ &\quad \cdot \max\left\{\left(\frac{\kappa_1}{1-\tau} + \mu_1\right)(\kappa_{01} + \mu_1)^{-1},\ \|u_{10}\|_{L^\infty(\Omega)}\right\}, \\ \overline{u}_2 &:= f_2(\overline{v}) \\ &\quad \cdot \max\left\{\left(\frac{\kappa_2}{1-\tau} + \mu_2\right)(\kappa_{02} + \mu_2)^{-1},\ \|u_{20}\|_{L^\infty(\Omega)}\right\}, \end{aligned} \tag{9}$$

for $f_i(\overline{v})$ defined by

$$f_i(\overline{v}) = \exp\left(\frac{1}{1-\tau}\int_{\underline{v}}^{\overline{v}} \chi_i(s)\,ds\right), \quad i = 1, 2. \tag{10}$$

(v) Finally, for some technical reasons, we also assume that

$$\chi_i^2(v) + (1-\tau)\chi_i'(v) \leq 0, \quad i = 1, 2. \tag{11}$$

We illustrate the validity to above assumptions with the following generalized example [7].

Example 1. We take the chemosensitivity functions $\chi_i(v) = \alpha_i/(\beta_i + v)$ $(i = 1, 2)$ for positive constants α_i, β_i fulfilling

$$\begin{aligned} &\alpha_i \leq 1 - \tau, \\ &\frac{4}{\gamma} < \beta_i \leq \frac{8}{\gamma}, \\ &\qquad i = 1, 2. \end{aligned} \tag{12}$$

Clearly, (11) holds. Take a lower bound $\underline{v} = 0$ and upper bound $\overline{v}$ is to be defined later. Moreover, the initial dates u_{10} and u_{20} are satisfied as

$$\begin{aligned} \|u_{10}\|_{L^\infty(\Omega)} &\leq \left(\frac{\kappa_1}{1-\tau} + \mu_1\right)(\kappa_{01} + \mu_1)^{-1}, \\ \|u_{20}\|_{L^\infty(\Omega)} &\leq \left(\frac{\kappa_2}{1-\tau} + \mu_2\right)(\kappa_{02} + \mu_2)^{-1}. \end{aligned} \tag{13}$$

Then, consider the following.

(1) Condition (6) is equivalent to $\gamma\alpha_i(v/(\beta_i + v)) \leq \kappa_i$, so choosing $\kappa_i = \gamma\alpha_i(\overline{v}/(\beta_i + \overline{v}))$ $(i = 1, 2)$.

(2) Taking positive constants $\kappa_{0i} = \alpha_i\beta_i^{-\alpha_i}/(\beta_i + \overline{v})^{1-\alpha_i}$ $(i = 1, 2)$, then condition (7) holds since

$$\kappa_{0i} \leq \frac{\alpha_i}{\beta_i + v} e^{\alpha_i \int_0^v (1/(\beta_i + s))ds} = \frac{\alpha_i}{(\beta_i + v)^{1-\alpha_i}}\left(\frac{1}{\beta_i}\right)^{\alpha_i} \tag{14}$$

for $i = 1, 2$.

(3) We notice that $h(0, 0, 0) = 0$ as well as

$$\begin{aligned} h(\overline{u}_1, \overline{u}_2, \overline{v}) &= \overline{u}_1 + \overline{u}_2 - \gamma\overline{v} \\ &= e^{(\alpha_1/(1-\tau))\int_0^{\overline{v}}(1/(\beta_1+s))ds}\frac{\kappa_1/(1-\tau) + \mu_1}{\kappa_{01} + \mu_1} \\ &\quad + e^{(\alpha_2/(1-\tau))\int_0^{\overline{v}}(1/(\beta_2+s))ds}\frac{\kappa_2/(1-\tau) + \mu_2}{\kappa_{02} + \mu_2} - \gamma\overline{v}. \end{aligned} \tag{15}$$

A sufficient condition for the second inequality in (8) holding is

$$\begin{aligned} e^{(\alpha_1/(1-\tau))\int_0^{\overline{v}}(1/(\beta_1+s))ds}\frac{\kappa_1/(1-\tau) + \mu_1}{\kappa_{01} + \mu_1} &< \frac{\gamma}{2}\overline{v}, \\ e^{(\alpha_2/(1-\tau))\int_0^{\overline{v}}(1/(\beta_2+s))ds}\frac{\kappa_2/(1-\tau) + \mu_2}{\kappa_{02} + \mu_2} &< \frac{\gamma}{2}\overline{v}, \end{aligned} \tag{16}$$

and, for simplicity, let us take $\mu_i = \kappa_i/(1-\tau)$ $(i = 1, 2)$ and then derive

$$\overline{v} > \max\left\{\frac{\beta_1}{(\gamma/4)\beta_1 - 1}, \frac{\beta_2}{(\gamma/4)\beta_2 - 1}\right\}. \tag{17}$$

Up to now, the above all restrictive conditions are verified, which implies that conditions (6)–(11) are sufficient to ensure the global existence of solutions.

Remark 2. In literature [7], the chemical production function $h(u_1, u_2, v) = -v + u_1 + u_2$, and the chemotactic sensitivity $\chi_i(v) = \gamma_i/(1 + \gamma_i v)$ $(i = 1, 2)$.

In this paper, the purpose is to study the global existence and uniform boundedness of solutions to (1) applying an iterative method. The main results are stated by the following theorem.

Theorem 3. *Under assumptions (6)–(11), for any initial date $(u_{10}, u_{20}, v_0) \in (C^1(\overline{\Omega}))^3$ satisfying the homogeneous Neumann boundary conditions and $u_{10} \geq 0$, $u_{20} \geq 0$, there exists a unique solution to (1):*

$$\begin{aligned} &(u_1, u_2, v) \\ &\in \left(L^p([0,\infty); W^{2,p}(\Omega)) \cap W^{1,p}([0,\infty); L^p(\Omega))\right)^3 \end{aligned} \tag{18}$$

for any $p > n$. Moreover the solution is uniformly bounded; that is,

$$\|u_1\|_{L^\infty(\Omega)} + \|u_2\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} \leq C, \tag{19}$$

where C $(<\infty)$ is positive constant.

The proof of Theorem 3 is split into several steps. Step one: we start to consider the local existence of solutions and then proceed with the extensibility criterion. Step two: under some appropriate conditions, we prove that the solutions are uniformly bounded in time.

3. Global Existence of Solutions

We will first be devoted to dealing with local-in-time existence and uniqueness of a nonnegative solution for (1). The corresponding conclusions are written by Lemma 4.

3.1. Local Existence of Solutions

Lemma 4. *Given positive initial date* $(u_{10}, u_{20}, v_0) \in (C^1(\overline{\Omega}))^3$ *and the same assumptions as in Theorem 3, there exists a small enough time* $T > 0$ *and a unique triple of nonnegative functions* $(u_1, u_2, v) \in (L^p([0,T]; W^{2,p}(\Omega)) \cap W^{1,p}([0,T]; L^p(\Omega)))^3$ *(for* $p > n$*) such that* (u_1, u_2, v) *is a solution of (1) in* $\Omega \times (0,T)$*. Moreover, we have* $v \geq \underline{v}$*.*

Proof. Introduce the change of variables $\tilde{u}_1$ and $\tilde{u}_2$ given by

$$\begin{aligned} u_1 &= f_1(v)\,\tilde{u}_1, \\ u_2 &= f_2(v)\,\tilde{u}_2, \end{aligned} \tag{20}$$

where the function $f_i(v)$ is defined by

$$f_i(v) = \exp\left\{\frac{1}{1-\tau}\int_{\underline{v}}^{v} \chi_i(s)\,ds\right\}, \quad i = 1,2. \tag{21}$$

A direct calculation yields

$$\begin{aligned} u_{1t} &= f_1'(v)\left(\tau\Delta v + h(u_1,u_2,v)\right)\tilde{u}_1 \\ &\quad + f_1(v)\,\tilde{u}_{1t}, \\ \Delta u_1 &= f_1''(v)\,\tilde{u}_1\,|\nabla v|^2 + 2f_1'(v)\,\nabla\tilde{u}_1\cdot\nabla v \\ &\quad + f_1'(v)\,\tilde{u}_1\Delta v + f_1(v)\,\Delta\tilde{u}_1, \\ \nabla\cdot\left(u_1\chi_1(v)\,\nabla v\right) &= f_1'(v)\,\tilde{u}_1\chi_1(v)\,|\nabla v|^2 \\ &\quad + f_1(v)\,\chi_1(v)\,\nabla\tilde{u}_1\cdot\nabla v \\ &\quad + f_1(v)\,\tilde{u}_1\chi_1'(v)\,|\nabla v|^2 \\ &\quad + f_1(v)\,\tilde{u}_1\chi_1(v)\,\Delta v; \end{aligned} \tag{22}$$

similarly,

$$\begin{aligned} u_{2t} &= f_2'(v)\left(\tau\Delta v + h(u_1,u_2,v)\right)\tilde{u}_2 \\ &\quad + f_2(v)\,\tilde{u}_{2t}, \\ \Delta u_2 &= f_2''(v)\,\tilde{u}_2\,|\nabla v|^2 + 2f_2'(v)\,\nabla\tilde{u}_2\cdot\nabla v \\ &\quad + f_2'(v)\,\tilde{u}_2\Delta v + f_2(v)\,\Delta\tilde{u}_2, \\ \nabla\cdot\left(u_2\chi_2(v)\,\nabla v\right) &= f_2'(v)\,\tilde{u}_2\chi_2(v)\,|\nabla v|^2 \\ &\quad + f_2(v)\,\chi_2(v)\,\nabla\tilde{u}_2\cdot\nabla v \\ &\quad + f_2(v)\,\tilde{u}_2\chi_2'(v)\,|\nabla v|^2 \\ &\quad + f_2(v)\,\tilde{u}_2\chi_2(v)\,\Delta v, \end{aligned} \tag{23}$$

where

$$\begin{aligned} f_i'(v) &= \frac{\chi_i(v)}{1-\tau} f_i(v), \\ f_i''(v) &= \left(\frac{\chi_i^2(v)}{(1-\tau)^2} + \frac{\chi_i'(v)}{1-\tau}\right) f_i(v), \\ & \qquad i = 1,2. \end{aligned} \tag{24}$$

Substitute (20), (22), and (23) into (1) and denote that

$$\begin{aligned} \mathscr{L}_i(v)\,\tilde{u}_i &= \tilde{u}_{it} - \Delta\tilde{u}_i - \frac{1+\tau}{1-\tau}\chi_i(v)\,\nabla v\cdot\nabla\tilde{u}_i, \\ G_i(\tilde{u}_1,\tilde{u}_2,v) &= -\frac{x_i(v)}{1-\tau} h\left(f_1(v)\,\tilde{u}_1, f_2(v)\,\tilde{u}_2, v\right) \\ &\quad + \mu_i g_i\left(f_1(v)\,\tilde{u}_1, f_2(v)\,\tilde{u}_2\right), \end{aligned} \tag{25}$$

for $i = 1,2$. Then system (1) becomes

$$\begin{aligned} &\mathscr{L}_1(v)\,\tilde{u}_1 \\ &\quad = \tilde{u}_1 G_1(\tilde{u}_1,\tilde{u}_2,v) \\ &\qquad + \tilde{u}_1\frac{\tau}{(1-\tau)^2}\left(\chi_1^2(v) + (1-\tau)\,\chi_1'(v)\right)|\nabla v|^2, \\ &\qquad\qquad x\in\Omega,\ t>0, \\ &\mathscr{L}_2(v)\,\tilde{u}_2 \\ &\quad = \tilde{u}_2 G_2(\tilde{u}_1,\tilde{u}_2,v) \\ &\qquad + \tilde{u}_2\frac{\tau}{(1-\tau)^2}\left(\chi_2^2(v) + (1-\tau)\,\chi_2'(v)\right)|\nabla v|^2, \\ &\qquad\qquad x\in\Omega,\ t>0, \\ &v_t = \tau\Delta v + h\left(f_1(v)\,\tilde{u}_1, f_2(v)\,\tilde{u}_2, v\right), \quad x\in\Omega,\ t>0, \end{aligned} \tag{26}$$

with boundary conditions $\partial\tilde{u}_1/\partial\nu = \partial\tilde{u}_2/\partial\nu = \partial v/\partial\nu = 0$, $x \in \partial\Omega$, $t > 0$, and initial conditions $\tilde{u}_1(x,0) := \tilde{u}_{10}(x) = u_{10}(x)/f_1(v_0(x))$, $\tilde{u}_2(x,0) := \tilde{u}_{20}(x) = u_{20}(x)/f_2(v_0(x))$, $v(x,0) = v_0(x)$, $x \in \Omega$.

For sufficiently small $T > 0$, in the space $[L^p([0,T]; W^{2,p}(\Omega)) \cap W^{1,p}([0,T]; L^p(\Omega))]^2$ (for $p > n$), we implement a fixed point argument. Taking $v \in C([0,T]; C^1(\overline{\Omega}))$ satisfying $v \in (\underline{v}, \overline{v})$ and $|\nabla v| < C$, we define $\tilde{u}_1, \tilde{u}_2$ in $(0,T)$ as the unique solution to the question

$$\begin{aligned} &\mathscr{L}_1(v)\,\tilde{u}_1 \\ &\quad = \tilde{u}_1 G_1(\tilde{u}_1,\tilde{u}_2,v) \\ &\qquad + \tilde{u}_1\frac{\tau}{(1-\tau)^2}\left(\chi_1^2(v) + (1-\tau)\,\chi_1'(v)\right)|\nabla v|^2 \\ &\qquad\qquad \text{in } \Omega\times(0,T), \\ &\mathscr{L}_2(v)\,\tilde{u}_2 \\ &\quad = \tilde{u}_2 G_2(\tilde{u}_1,\tilde{u}_2,v) \\ &\qquad + \tilde{u}_2\frac{\tau}{(1-\tau)^2}\left(\chi_2^2(v) + (1-\tau)\,\chi_2'(v)\right)|\nabla v|^2 \\ &\qquad\qquad \text{in } \Omega\times(0,T); \end{aligned} \tag{27}$$

for the details, we refer the reader to [13, Remark 48.3]. Notice that the right-hand side terms of (27) are the multiplicative for $\widetilde{u}_1$ and $\widetilde{u}_2$, applying the maximum principle to verify that both $\widetilde{u}_1$ and $\widetilde{u}_2$ are nonnegative.

Since condition (11), we easily see that the regular functions $\widetilde{u}_1$ and $\widetilde{u}_2$ satisfy

$$\begin{aligned} \mathscr{L}_1(v)\widetilde{u}_1 &\le \widetilde{u}_1 G_1(\widetilde{u}_1, \widetilde{u}_2, v), \\ \mathscr{L}_2(v)\widetilde{u}_2 &\le \widetilde{u}_2 G_2(\widetilde{u}_1, \widetilde{u}_2, v), \end{aligned} \tag{28}$$

and the right-hand side terms of (27) are quasi-decreased; looking upon the lower-solution $\underline{u}_1 = \underline{u}_2 = 0$, one may construct upper-solutions $\overline{u}_1$ and $\overline{u}_2$ such that

$$\begin{aligned} &0 \le u_1 \le \overline{u}_1, \\ &0 \le u_2 \le \overline{u}_2, \\ &t < T. \end{aligned} \tag{29}$$

It follows that we apply Theorem 2.1 of [7] to get a unique nonnegative solution.

We solve the parabolic equation

$$v_t = \tau\Delta v + h\left(f_1(v)\widetilde{u}_1, f_2(v)\widetilde{u}_2, v\right). \tag{30}$$

Thanks to the essential estimations of parabolic equations and the embedding theorems, we apply the Schauder fixed point theorem to obtain the local existence of solution $v(x,t)$. The smoothness of $h(f_1(v)\widetilde{u}_1, f_2(v)\widetilde{u}_2, v)$ ensures the uniqueness of solution $v(x,t)$.

To prove $v \ge \underline{v}$, in view of assumption (8) and the monotone of $h(u_1, u_2, v)$, we have $0 \le h(0,0,\underline{v}) \le h(u_1, u_2, \underline{v})$. It follows that $\underline{v}$ is a lower-solution for the equation

$$\begin{aligned} v_t - \tau\Delta v &= h(u_1, u_2, v) \quad \text{in } \Omega \times (0,T), \\ \frac{\partial v}{\partial \nu} &= 0 \quad \text{on } \partial\Omega \times (0,T), \\ v(x,0) &= v_0(x) \quad \text{in } \Omega. \end{aligned} \tag{31}$$

The proof of Lemma 4 is completed. □

About the above solution (u_1, u_2, v), we have the following extensibility criterion: *The solution is extended to the interval* $(0, T_{\max})$, *where* $T_{\max}$ *has the following property:*

$$\begin{aligned} &\text{If } T_{\max} < \infty, \\ &\text{then } \limsup_{t \to T_{\max}} \left(\|u_1(\cdot,t)\|_{L^\infty(\Omega)} + \|u_2(\cdot,t)\|_{L^\infty(\Omega)}\right. \\ &\quad \left. + \|v(\cdot,t)\|_{L^\infty(\Omega)}\right) = \infty. \end{aligned} \tag{32}$$

3.2. Uniform Boundedness of Solutions. To obtain some a priori estimates, we need some technical lemmas. The following L^1-estimate of solution is first given.

Lemma 5. *For all* $t \in (0, T_{max})$, *the solutions to (1) satisfy the following estimates:*

$$\begin{aligned} \|u_1\|_{L^1(\Omega)} &\le \max\left\{|\Omega|, \|u_{10}\|_{L^1(\Omega)}\right\} \le M, \\ \|u_2\|_{L^1(\Omega)} &\le \max\left\{|\Omega|, \|u_{20}\|_{L^1(\Omega)}\right\} \le M, \\ \|v\|_{L^1(\Omega)} &\le \max\left\{\|v_0\|_{L^1(\Omega)}, 2M\gamma^{-1}\right\}, \end{aligned} \tag{33}$$

where $M = \max\{|\Omega|, \|u_{10}\|_{L^1(\Omega)}, \|u_{20}\|_{L^1(\Omega)}\}$.

Proof. Integrating the first equation of (1) over Ω, we have

$$\begin{aligned} \frac{d}{dt}\int_\Omega u_1 dx &= \mu_1 \int_\Omega \left(u_1 - u_1^2 - a_1 u_1 u_2\right) dx \\ &\le \mu_1 \int_\Omega \left(u_1 - u_1^2\right) dx. \end{aligned} \tag{34}$$

Using Cauchy inequality yields

$$\begin{aligned} \frac{d}{dt}\int_\Omega u_1 dx &\le \mu_1 \int_\Omega (1 - u_1)\, dx \\ &\le \mu_1 |\Omega| - \mu_1 \int_\Omega u_1 dx. \end{aligned} \tag{35}$$

By the Gronwall Lemma, we derive

$$\|u_1(x,t)\|_{L^1(\Omega)} \le \max\left\{|\Omega|, \|u_{10}(x)\|_{L^1(\Omega)}\right\} \le M; \tag{36}$$

a similar estimate holds for u_2, which leads to

$$\|u_2(x,t)\|_{L^1(\Omega)} \le \max\left\{|\Omega|, \|u_{20}(x)\|_{L^1(\Omega)}\right\} \le M. \tag{37}$$

Integrating the third equation of (1) over Ω, we have

$$\begin{aligned} \frac{d}{dt}\int_\Omega v\, dx &= \int_\Omega (-\gamma v + u_1 + u_2)\, dx \\ &= -\gamma \int_\Omega v\, dx + \int_\Omega (u_1 + u_2)\, dx \\ &\le -\gamma \int_\Omega v\, dx + 2M; \end{aligned} \tag{38}$$

it implies

$$\|v\|_{L^1(\Omega)} \le \max\left\{\|v_0\|_{L^1(\Omega)}, 2M\gamma^{-1}\right\}. \tag{39}$$

This proves the lemma. □

Lemma 6. *Letting $p > 1$ and under assumption (11), then the following estimates hold:*

$$\begin{aligned}\frac{d}{dt}\int_\Omega u_1^p f_{1p}^{1-p}dx &\le \frac{1-p}{1-\tau((p-1)/p)}\int_\Omega u_1^p\chi_1(v) f_{1p}^{1-p}h(u_1,u_2,v)\,dx \\ &\quad + p\mu_1\int_\Omega u_1^p f_{1p}^{1-p}g_1(u_1,u_2)\,dx,\end{aligned} \tag{40}$$

$$\begin{aligned}\frac{d}{dt}\int_\Omega u_2^p f_{2p}^{1-p}dx &\le \frac{1-p}{1-\tau((p-1)/p)}\int_\Omega u_2^p\chi_2(v) f_{2p}^{1-p}h(u_1,u_2,v)\,dx \\ &\quad + p\mu_2\int_\Omega u_2^p f_{2p}^{1-p}g_2(u_1,u_2)\,dx,\end{aligned} \tag{41}$$

where $f_{ip}(v)$ is defined by

$$f_{ip}(v) = \exp\left(\frac{1}{1-\tau((p-1)/p)}\int_{\underline{v}}^{v}\chi_i(s)\,ds\right), \quad i = 1,2. \tag{42}$$

Proof. It is easy to check

$$\begin{aligned}f_{1p}'(v) &= \frac{1}{1-\tau((p-1)/p)}\chi_1(v) f_{1p}(v), \\ f_{2p}'(v) &= \frac{1}{1-\tau((p-1)/p)}\chi_2(v) f_{2p}(v).\end{aligned} \tag{43}$$

Then, for any $p > 1$, we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega u_1^p f_{1p}^{1-p}dx &= p\int_\Omega u_1^{p-1}u_{1t}f_{1p}^{1-p}dx + (1-p) \\ &\quad\cdot\int_\Omega u_1^p f_{1p}^{-p}f_{1p}'v_t dx = p\int_\Omega u_1^{p-1}f_{1p}^{1-p}\Big[\Delta u_1 - \nabla \\ &\quad\cdot(u_1\chi_1(v)\nabla v) + \frac{\tau(1-p)}{p-\tau(p-1)}u_1\chi_1(v)\Delta v\Big]dx \\ &\quad + \frac{1-p}{1-\tau((p-1)/p)}\int_\Omega u_1^p f_{1p}^{1-p}\chi_1(v) \\ &\quad\cdot h(u_1,u_2,v)\,dx \\ &\quad + p\mu_1\int_\Omega u_1^p f_{1p}^{1-p}g_1(u_1,u_2)\,dx := I_1 + I_2 + I_3.\end{aligned} \tag{44}$$

Clearly, if $I_1 \le 0$, then (40) is a consequence of (44). We in the following verify that the result holds.

In fact

$$\begin{aligned}I_1 &= p\int_\Omega u_1^{p-1}f_{1p}^{1-p}\Big[\Delta u_1 - \nabla\cdot(u_1\chi_1(v)\nabla v) \\ &\quad + \frac{\tau(1-p)}{p-\tau(p-1)}(\nabla\cdot(u_1\chi_1(v)\nabla v) - \nabla(u_1\chi_1(v)) \\ &\quad\cdot\nabla v)\Big]dx = p\int_\Omega u_1^{p-1}f_{1p}^{1-p}\Big[\nabla \\ &\quad\cdot\left(f_{1p}\nabla\left(u_1 f_{1p}^{-1}\right)\right) - \frac{\tau(1-p)}{p-\tau(p-1)}\left(\chi_1(v)\nabla u_1\cdot\nabla v\right. \\ &\quad\left.+ u_1\chi_1'(v)|\nabla v|^2\right)\Big]dx = p(1-p) \\ &\quad\cdot\int_\Omega u_1^{p-2}f_{1p}^{2-p}\Big[f_{1p}\left|\nabla\left(u_1 f_{1p}^{-1}\right)\right|^2 \\ &\quad - \frac{\tau u_1\left(\chi_1(v)\nabla u_1\cdot\nabla v + u_1\chi_1'(v)|\nabla v|^2\right)}{f_{1p}(p-\tau(p-1))}\Big]dx \\ &= p(1-p)\int_\Omega u_1^{p-2}f_{1p}^{3-p}\left|\nabla\left(u_1 f_{1p}^{-1}\right)\right|^2 dx \\ &\quad + \frac{\tau p(p-1)}{p-\tau(p-1)}\int_\Omega u_1^{p-1}f_{1p}^{1-p}\chi_1(v)\nabla u_1\cdot\nabla v\,dx \\ &\quad + \frac{\tau p(p-1)}{p-\tau(p-1)}\int_\Omega u_1^p f_{1p}^{1-p}\chi_1'(v)|\nabla v|^2 dx := J_1 \\ &\quad + J_2 + J_3.\end{aligned} \tag{45}$$

Since $\nabla u_1 = \nabla(f_{1p}(u_1 f_{1p}^{-1}))$, multiplying both sides by $\chi_1(v)\nabla v$, we get

$$\begin{aligned}&\chi_1(v)\nabla u_1\cdot\nabla v \\ &\quad = f_{1p}\chi_1(v)\nabla\left(u_1 f_{1p}^{-1}\right)\cdot\nabla v \\ &\quad\quad + \frac{1}{1-\tau((p-1)/p)}u_1\chi_1^2(v)|\nabla v|^2,\end{aligned} \tag{46}$$

and by ε-Yöung's inequality

$$\begin{aligned}&\chi_1(v)\nabla\left(u_1 f_{1p}^{-1}\right)\cdot\nabla v \\ &\quad \le \frac{p-\tau(p-1)}{\tau}\frac{f_{1p}}{u_1}\left|\nabla\left(u_1 f_{1p}^{-1}\right)\right|^2 \\ &\quad\quad + \frac{\tau}{p-\tau(p-1)}\frac{u_1}{f_{1p}}\chi_1^2(v)|\nabla v|^2.\end{aligned} \tag{47}$$

Combining (46) and (47) and substituting it into J_2, we derive

$$\begin{aligned}J_2 &\le p(p-1)\int_\Omega u_1^{p-2}f_{1p}^{3-p}\left|\nabla\left(u_1 f_{1p}^{-1}\right)\right|^2 dx \\ &\quad + \frac{\tau p(p-1)(\tau+p)}{(p-\tau(p-1))^2}\int_\Omega u_1^p f_{1p}^{1-p}\chi_1^2(v)|\nabla v|^2 dx.\end{aligned} \tag{48}$$

Inserting (48) into (45) yields

$$\begin{aligned} I_1 \le{}& \frac{\tau p(p-1)}{p-\tau(p-1)} \int_\Omega \left(\frac{\tau+p}{(p-\tau(p-1))} \chi_1^2(v) \right. \\ &\left. + \chi_1'(v) \right) u_1^p f_{1p}^{1-p} |\nabla v|^2 dx = \frac{\tau p(p-1)}{p-\tau(p-1)} \\ &\cdot \int_\Omega \left[\frac{1}{(1-\tau)} \left(1 - \frac{\tau^2}{\tau+(1-\tau)p} \right) \chi_1^2(v) \right. \\ &\left. + \chi_1'(v) \right] u_1^p f_{1p}^{1-p} |\nabla v|^2 dx \le \frac{\tau p(p-1)}{p-\tau(p-1)} \\ &\cdot \int_\Omega \left(\frac{1}{(1-\tau)} \chi_1^2(v) + \chi_1'(v) \right) u_1^p f_{1p}^{1-p} |\nabla v|^2 dx; \end{aligned} \tag{49}$$

we deduce from assumption (11) that $I_1 \le 0$ for any $p > 1$, and the proof ends. In the same way, we achieve (41). This completes the proof of Lemma 6. □

In view of $v \in C((0, T_{\max}); C^1(\Omega))$ and initial date $\underline{v} < v_0 < \overline{v}$, we consider $T^* > 0$ defined by

$$T^* := \begin{cases} T_{\max}, & \text{if } v \le \overline{v} \text{ in } (0, T_{\max}), \\ T_v, & \text{if } \|v\|_{L^\infty(\Omega_{T_{\max}})} > \overline{v}, \end{cases} \tag{50}$$

where $T_v < T_{\max}$ satisfies that $\|v\|_{L^\infty(\Omega_{T_v})} \le \overline{v}$ and $\|v\|_{L^\infty(\Omega_{T_v+\delta})} > \overline{v}$ for any $\delta > 0$.

Lemma 7. *Letting $p > 1$ and for any $t < T^*$, then the solutions to (1) satisfy*

$$\begin{aligned} &\frac{1}{p-1} \frac{d}{dt} \int_\Omega u_1^p f_{1p}^{1-p} dx \\ &\quad \le -(\kappa_{01} + \mu_1) \int_\Omega u_1^{p+1} f_{1p}^{-p} dx \\ &\qquad + \left(\frac{\kappa_1}{1-\tau} + \frac{p}{p-1} \mu_1 \right) \int_\Omega u_1^p f_{1p}^{1-p} dx, \end{aligned} \tag{51}$$

$$\begin{aligned} &\frac{1}{p-1} \frac{d}{dt} \int_\Omega u_2^p f_{2p}^{1-p} dx \\ &\quad \le -(\kappa_{02} + \mu_2) \int_\Omega u_2^{p+1} f_{2p}^{-p} dx \\ &\qquad + \left(\frac{\kappa_2}{1-\tau} + \frac{p}{p-1} \mu_2 \right) \int_\Omega u_2^p f_{2p}^{1-p} dx. \end{aligned} \tag{52}$$

Proof. In order to prove Lemma 7, we consider the terms I_2 and I_3 appearing in the proof of Lemma 6.

By the nonnegativity for u_1, u_2 and assumption (6), we obtain

$$-h(u_1, u_2, v) \le -u_1 + \frac{\kappa_1}{\chi_1(v)}. \tag{53}$$

Multiplying (53) by $u_1^p f_{1p}^{1-p} \chi_1(v)$ and integrating over Ω yield

$$\begin{aligned} &-\int_\Omega u_1^p f_{1p}^{1-p} \chi_1(v) h(u_1, u_2, v) dx \\ &\quad \le -\int_\Omega u_1^{p+1} f_{1p}^{1-p} \chi_1(v) dx + \kappa_1 \int_\Omega u_1^p f_{1p}^{1-p} dx. \end{aligned} \tag{54}$$

By the monotonic property of exponential function and assumption (7), we have

$$0 < \kappa_{01} \le \chi_1(v) \exp\left(\int_{\underline{v}}^{v} \chi_1(s) ds \right) \le \chi_1(v) f_{1p}(v). \tag{55}$$

It follows that (54) becomes

$$\begin{aligned} &-\int_\Omega u_1^p f_{1p}^{1-p} \chi_1(v) h(u_1, u_2, v) dx \\ &\quad \le -\kappa_{01} \int_\Omega u_1^{p+1} f_{1p}^{-p} dx + \kappa_1 \int_\Omega u_1^p f_{1p}^{1-p} dx. \end{aligned} \tag{56}$$

Multiplying (56) by $(p-1)/(1-\tau((p-1)/p))$, we get

$$\begin{aligned} I_2 \le{}& \frac{p-1}{1-\tau((p-1)/p)} \left(-\kappa_{01} \int_\Omega u_1^{p+1} f_{1p}^{-p} dx \right. \\ &\left. + \kappa_1 \int_\Omega u_1^p f_{1p}^{1-p} dx \right) \le (p-1) \\ &\cdot \left(-\kappa_{01} \int_\Omega u_1^{p+1} f_{1p}^{-p} dx + \frac{\kappa_1}{1-\tau} \int_\Omega u_1^p f_{1p}^{1-p} dx \right). \end{aligned} \tag{57}$$

For the term I_3, we easily achieve

$$\begin{aligned} I_3 &\le p\mu_1 \left(\int_\Omega u_1^p f_{1p}^{1-p} dx - \int_\Omega u_1^{p+1} f_{1p}^{-p} dx \right) \\ &\le (p-1) \mu_1 \left(\int_\Omega u_1^p f_{1p}^{1-p} dx - \int_\Omega u_1^{p+1} f_{1p}^{-p} dx \right) \\ &\quad + \mu_1 \int_\Omega u_1^p f_{1p}^{1-p} dx. \end{aligned} \tag{58}$$

Combining (44), (57), and (58) leads to

$$\begin{aligned} &\frac{d}{dt} \int_\Omega u_1^p f_{1p}^{1-p} dx \le (p-1) \\ &\quad \cdot \left(-\kappa_{01} \int_\Omega u_1^{p+1} f_{1p}^{-p} dx + \frac{\kappa_1}{1-\tau} \int_\Omega u_1^p f_{1p}^{1-p} dx \right) \\ &\quad + (p-1) \mu_1 \left(\int_\Omega u_1^p f_{1p}^{1-p} dx - \int_\Omega u_1^{p+1} f_{1p}^{-p} dx \right) \\ &\quad + \mu_1 \int_\Omega u_1^p f_{1p}^{1-p} dx, \end{aligned} \tag{59}$$

which implies that (51) holds. The same arguments are used to achieve (52). The proof ends. □

To establish the uniform boundedness of solutions, we apply an iterative method based on the L^p-norm of solutions inspired by the Moser-Alikakos type iterations (see [10, 13]).

Lemma 8. *For any $t < T^*$, then the solutions to (1) are bounded by*

$$\begin{aligned}\|u_1\|_{L^\infty(\Omega)} &\le f_1(\overline{v}) \\ &\cdot \max\left\{\left(\frac{\kappa_1}{1-\tau}+\mu_1\right)(\kappa_{01}+\mu_1)^{-1},\ \|u_{10}\|_{L^\infty(\Omega)}\right\},\end{aligned} \tag{60}$$

$$\begin{aligned}\|u_2\|_{L^\infty(\Omega)} &\le f_2(\overline{v}) \\ &\cdot \max\left\{\left(\frac{\kappa_2}{1-\tau}+\mu_2\right)(\kappa_{02}+\mu_2)^{-1},\ \|u_{20}\|_{L^\infty(\Omega)}\right\}.\end{aligned} \tag{61}$$

Proof. Denote $\int_\Omega u_1^p f_{1p}^{1-p}dx$ by $\Psi(t)$. Taking arbitrary positive constant $\rho > 0$, then it holds that

$$\begin{aligned}\Psi(t) &= \int_{f_{1p}\le\rho(\kappa_{01}+\mu_1)u_1} u_1^p f_{1p}^{1-p}dx \\ &\quad + \int_{f_{1p}>\rho(\kappa_{01}+\mu_1)u_1} u_1^p f_{1p}^{1-p}dx \\ &\le \rho(\kappa_{01}+\mu_1)\int_{f_{1p}\le\rho(\kappa_{01}+\mu_1)u_1} u_1^{p+1} f_{1p}^{-p}dx \\ &\quad + (\rho(\kappa_{01}+\mu_1))^{1-p}\int_{f_{1p}>\rho(\kappa_{01}+\mu_1)u_1} u_1 dx \\ &\le \rho(\kappa_{01}+\mu_1)\int_\Omega u_1^{p+1} f_{1p}^{-p}dx \\ &\quad + (\rho(\kappa_{01}+\mu_1))^{1-p}\int_\Omega u_1 dx.\end{aligned} \tag{62}$$

Hence

$$\begin{aligned}&-(\kappa_{01}+\mu_1)\int_\Omega u_1^{p+1} f_{1p}^{-p}dx \\ &\quad \le -\frac{1}{\rho}\Psi(t) + \rho^{-p}(\kappa_{01}+\mu_1)^{1-p}\int_\Omega u_1 dx.\end{aligned} \tag{63}$$

Substituting (63) into (51), we have

$$\begin{aligned}\frac{1}{p-1}\frac{d}{dx}\Psi(t) &\le \left(\frac{\kappa_1}{1-\tau}+\frac{p}{p-1}\mu_1-\frac{1}{\rho}\right)\Psi(t) \\ &\quad + \rho^{-p}(\kappa_{01}+\mu_1)^{1-p}\int_\Omega u_1 dx.\end{aligned} \tag{64}$$

Take $1/\rho > \kappa_1/(1-\tau) + (p/(p-1))\mu_1$. Noticing Lemma 5 and initial date $u_{10} \in C^1(\overline{\Omega})$, we apply the Gronwall Lemma to obtain

$$\begin{aligned}\Psi(t) &\le \max\left\{\rho^{-p}(\kappa_{01}+\mu_1)^{1-p}\right. \\ &\quad \left.\cdot\left(\frac{1}{\rho}-\frac{\kappa_1}{1-\tau}-\frac{p}{p-1}\mu_1\right)^{-1}\int_\Omega u_1 dx,\ \Psi(0)\right\};\end{aligned} \tag{65}$$

taking limits leads to

$$\lim_{p\to\infty}\Psi^{1/p}(t) \le \max\left\{\rho^{-1}(\kappa_{01}+\mu_1)^{-1},\ \|u_{10}\|_{L^\infty(\Omega)}\right\}. \tag{66}$$

Since

$$\Psi(t) = \int_\Omega u_1^p f_{1p}^{1-p}(v)\,dx \ge f_{1p}^{1-p}(\overline{v})\int_\Omega u_1^p dx, \tag{67}$$

combining (66) and (67) and taking limits as $\rho^{-1} \to \kappa_1/(1-\tau) + \mu_1$, it is found that (60) holds. The same processes are used to prove (61), which ends the proof of Lemma 8. □

Remark 9. In the above arguments, we used well such fact: Let Ω be bounded and regular domain of $\mathbb{R}^n$. If $\|u\|_{L^p(\Omega)} \le C$, where C is some positive constant and independent of p, then $\|u\|_{L^p(\Omega)} \to \|u\|_{L^\infty(\Omega)}$ as $p \to \infty$.

To end the arguments of uniform boundedness of solutions, it follows to prove that $T^* = T_{\max}$ and $T_{\max} = +\infty$ by contradiction.

As in Lemmas 4–8, we define $\widehat{u}_1, \widehat{u}_2$ and $\widehat{v}$ for the unique solutions in $(0, \widehat{T}_{\max})$ to

$$\begin{aligned}\widehat{u}_{1t} &= \Delta\widehat{u}_1 - \nabla\cdot(\widehat{u}_1\chi_1(\widehat{v})\nabla v) + \mu_1\widehat{u}_1 g_1(\widehat{u}_1,\widehat{u}_2), \\ &\qquad x\in\Omega,\ t>0, \\ \widehat{u}_{2t} &= \Delta\widehat{u}_2 - \nabla\cdot(\widehat{u}_2\chi_2(\widehat{v})\nabla\widehat{v}) + \mu_2 u_2 g_2(\widehat{u}_1,\widehat{u}_2), \\ &\qquad x\in\Omega,\ t>0, \\ \widehat{v}_t &= \tau\Delta\widehat{v} + \widehat{h}(\widehat{u}_1,\widehat{u}_2,\widehat{v}), \quad x\in\Omega,\ t>0, \\ \frac{\partial\widehat{u}_1}{\partial v} &= \frac{\partial\widehat{u}_2}{\partial v} = \frac{\partial\widehat{v}}{\partial v} = 0, \quad x\in\partial\Omega,\ t>0, \\ \widehat{u}_1(x,0) &= u_{10}(x), \\ \widehat{u}_2(x,0) &= u_{20}(x), \\ \widehat{v}(x,0) &= v_0(x), \\ &\qquad x\in\Omega,\end{aligned} \tag{68}$$

where $\widehat{T}_{\max}$ is defined in the same fashion as (32) and the truncation $\widehat{h}(u_1,u_2,v)$ is defined as follows:

$$\widehat{h}(u_1,u_2,v) := \begin{cases} h(u_1,u_2,v), & \text{if } v\le\overline{v}, \\ h(u_1,u_2,\overline{v}), & \text{if } v>\overline{v}.\end{cases} \tag{69}$$

Then it is easy to see that

$$\begin{aligned}\|\widehat{u}_1\|_{L^\infty(\Omega)} &\le f_1(\overline{v}) \\ &\quad\cdot\max\left\{\left(\frac{\kappa_1}{1-\tau}+\mu_1\right)(\kappa_{01}+\mu_1)^{-1},\ \|u_{10}\|_{L^\infty(\Omega)}\right\}, \\ \|\widehat{u}_2\|_{L^\infty(\Omega)} &\le f_2(\overline{v}) \\ &\quad\cdot\max\left\{\left(\frac{\kappa_2}{1-\tau}+\mu_2\right)(\kappa_{02}+\mu_2)^{-1},\ \|u_{20}\|_{L^\infty(\Omega)}\right\},\end{aligned} \tag{70}$$

as far as $\widehat{v} \le \overline{v}$.

By analogy with T^*, we define

$$\widehat{T}^* := \begin{cases}\widehat{T}_{\max}, & \text{if } \widehat{v}\le\overline{v} \text{ in } (0,\widehat{T}_{\max}), \\ \widehat{T}_{\widehat{v}}, & \text{if } \|\widehat{v}\|_{L^\infty(\Omega_{\widehat{T}_{\max}})} > \overline{v},\end{cases} \tag{71}$$

where $\widehat{T}_{\widehat{v}} < \widehat{T}_{\max}$ satisfies that $\|\widehat{v}\|_{L^\infty(\Omega_{\widehat{T}_{\widehat{v}}})} \leq \overline{v}$ and $\|\widehat{v}\|_{L^\infty(\Omega_{\widehat{T}_{\widehat{v}}+\delta})} > \overline{v}$ for any $\delta > 0$. To prove $\widehat{T}^* = \widehat{T}_{\max}$, we suppose

$$\widehat{T}^* < \widehat{T}_{\max}; \tag{72}$$

then $\widehat{h}(\widehat{u}_1, \widehat{u}_2, \widehat{v}) \leq \widehat{h}(\overline{u}_1, \overline{u}_2, \overline{v}) = h(\overline{u}_1, \overline{u}_2, \overline{v}) < 0$. It follows to easily check that $\overline{v}$ is a strict upper-solution to

$$\begin{aligned} \widehat{v}_t - \tau\Delta\widehat{v} &= \widehat{h}\left(\widehat{u}_1, \widehat{u}_2, \widehat{v}\right) \quad \text{in } \Omega \times \left(0, \widehat{T}^*\right), \\ \frac{\partial \widehat{v}}{\partial \nu} &= 0 \quad \text{on } \partial\Omega \times \left(0, \widehat{T}^*\right), \\ \widehat{v}(x, 0) &= v_0(x) \quad \text{in } \Omega. \end{aligned} \tag{73}$$

This implies $\widehat{v}(x,t) < \overline{v}$, $\forall x \in \Omega$, $t \in (0, \widehat{T}^*)$, which contradicts (72) and proves $\widehat{T}^* = \widehat{T}_{\max}$ as well as $\widehat{T}_{\max} = \infty$.

Thanks to $\widehat{v} \leq \overline{v}$, we show that $(\widehat{u}_1, \widehat{u}_2, \widehat{v})$ is also a solution to (6). So we have

$$T^* = T_{\max} \geq \widehat{T}_{\max} = \infty. \tag{74}$$

The proof of Theorem 3 is completed.

Competing Interests

The author declares that they have no competing interests.

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