Research Article

Dynamical Analysis of a Computer Virus Model with Delays

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Received 5 August 2016; Accepted 29 September 2016

Academic Editor: Vincenzo Scalzo

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An SIQR computer virus model with two delays is investigated in the present paper. The linear stability conditions are obtained by using characteristic root method and the developed asymptotic analysis shows the onset of a Hopf bifurcation occurs when the delay parameter reaches a critical value. Moreover the direction of the Hopf bifurcation and stability of the bifurcating period solutions are investigated by using the normal form theory and the center manifold theorem. Finally, numerical investigations are carried out to show the feasibility of the theoretical results.

1. Introduction

The advances of information technology and the widespread popularity of computer networks have increased the interest in the computer viruses. In the past decades, many epidemic models such as SIR model [1, 2], SIRS model [3, 4], SEIR model [5–7], SEIRS model [8] and SEIQRS model [9] characterizing the spread of computer viruses in networks were investigated by many scholars. It is worthwhile to note that the dynamical models above neglect the time delay in the spreading process of computer viruses. To our knowledge, there have been some computer virus models with time delay proposed to depict the spread of a computer virus. In [10, 11], Muroya et al. studied the global stability of a delayed SIRS computer virus propagation model, respectively. In [12], Feng et al. investigated the Hopf bifurcation of a delayed SIRS viral infection model in computer networks by regarding the delay due to temporary immune period of the recovered computers as a bifurcation parameter. In [13], Dong et al. analyzed the Hopf bifurcation of a delayed SEIR computer virus model with multistate antivirus by regarding the time delay due to the period that the computers use antivirus software to clean the viruses as the bifurcation parameter. In [14], Zhang and Yang studied the Hopf bifurcation of the following SIQR computer virus model with time delay:

\[
\begin{align*}
\frac{dS(t)}{dt} &= (1-p)b - \beta S(t-\tau) I(t-\tau) - dS(t), \\
\frac{dI(t)}{dt} &= \beta S(t-\tau) I(t-\tau) - (\delta + d + \alpha_1 + \gamma) I(t), \\
\frac{dQ(t)}{dt} &= \delta I(t) - (\epsilon + d + \alpha_2) Q(t), \\
\frac{dR(t)}{dt} &= \gamma I(t) + pb + \epsilon Q(t) - dR(t),
\end{align*}
\]

where \(S(t), I(t), Q(t),\) and \(R(t)\) denote the numbers of nodes in states susceptible, infectious, quarantined, and recovered at time \(t,\) respectively. \(b, p, \alpha_1, \alpha_2, \beta, \gamma, \delta,\) and \(\epsilon\) are the parameters of system (1) and \(\tau\) is the time delay due to the latent period of the computer virus. Zhang and Yang obtained the sufficient conditions for the local stability and existence of local Hopf bifurcation by regarding the delay as a bifurcation parameter and investigated the properties of the Hopf bifurcation by using the normal form method and center manifold theory. As stated in [13], it usually takes
a period to clean the viruses in the computers infected by viruses for antivirus software. Therefore, it is reasonable to incorporate the time delay due to the period that antivirus software uses to clean viruses in the infectious and the quarantined computers into system (1). Bearing all above in mind, this paper deals with the analysis of the Hopf bifurcation of the following system with two delays:

\[
\frac{dS(t)}{dt} = (1-p) b S(t-t_1) I(t-t_1) - dS(t),
\]

\[
\frac{dI(t)}{dt} = bS(t-t_1) I(t-t_1) - \gamma I(t) - \gamma I(t-t_2),
\]

\[
\frac{dQ(t)}{dt} = \delta I(t) - (d + \alpha_2) Q(t) - \epsilon Q(t-t_2),
\]

\[
\frac{dR(t)}{dt} = \gamma I(t-t_2) + pb + \epsilon Q(t-t_2) - dR(t),
\]

where \(t_1\) is the time delay due to the latent period of the computer virus and \(t_2\) is the time delay due to the period that the antivirus software uses to clean the computer viruses in the infectious and the quarantined nodes.

This paper mainly investigates the effect of the two delays on system (2). The remainder of this paper is organized as follows. The local stability of the positive equilibrium and existence of local Hopf bifurcation are analyzed in Section 2. Further research directions are contained in Section 5.

The characteristic equation of system (2) at \(D_0\) is

\[
\begin{vmatrix}
\lambda - a_{11} - b_{11} e^{-\lambda t_1} & -b_{12} e^{-\lambda t_1} & 0 & 0 \\
-b_{21} e^{-\lambda t_1} & \lambda - a_{22} - b_{22} e^{-\lambda t_1} - c_{22} e^{-\lambda t_2} & 0 & 0 \\
0 & -a_{32} & \lambda - a_{33} - c_{33} e^{-\lambda t_2} & 0 \\
0 & -c_{42} e^{-\lambda t_2} & 0 & \lambda - a_{44}
\end{vmatrix} = 0,
\]

from which we obtain

\[
\lambda^4 + A_3 \lambda^3 + A_2 \lambda^2 + A_1 \lambda + A_0 = 0,
\]

(5)

where

\[
A_0 = a_{11} a_{22} a_{33} a_{44},
A_1 = - (a_{11} a_{22} (a_{33} + a_{44}) + a_{33} a_{44} (a_{11} + a_{22})),
A_2 = a_{11} a_{22} + a_{33} a_{44} + (a_{11} + a_{22}) (a_{33} + a_{44}),
A_3 = - (a_{11} + a_{22} + a_{33} + a_{44}),
B_0 = a_{33} a_{44} (a_{11} b_{22} + a_{22} b_{11}),
B_2 = a_{11} b_{22} + a_{22} b_{11} + (a_{33} + a_{44}) (b_{11} + b_{22}),
B_1 = -(a_{33} a_{44} (b_{11} + b_{22}) + (a_{33} + a_{44}) (a_{11} b_{22} + a_{22} b_{11})),
B_3 = - (b_{11} + b_{22}),
C_0 = a_{11} a_{44} (a_{22} c_{33} + a_{33} c_{22}),
C_1 = -(c_{22} (a_{11} a_{33} + a_{11} a_{44} + a_{33} a_{44}) + c_{33} (a_{11} a_{22} + a_{11} a_{44} + a_{22} a_{44})),
C_2 = c_{22} (a_{11} + a_{33} + a_{44}) + c_{33} (a_{11} + a_{22} + a_{44}),
C_3 = -(c_{22} + c_{33}),
D_0 = a_{33} a_{44} b_{11} c_{22} + a_{44} c_{33} (a_{11} b_{22} + a_{22} b_{11}),
D_1 = a_{33} a_{44} b_{11} c_{22} + a_{44} c_{33} (a_{11} b_{22} + a_{22} b_{11}),
\]

by choosing different combination of the two delays as a bifurcation parameter. Direction of the Hopf bifurcation, stability, and period of the bifurcating periodic solutions on the center manifold are determined in Section 3. Some numerical simulations are presented to illustrate the validity of the main results in Section 4. Finally a critical analysis and further research directions are contained in Section 5.

2. Local Stability of the Positive Equilibrium and Existence of Local Hopf Bifurcation

According to the analysis in [14] we know that if \(R_0 = (1-p)b\beta/d(\delta + d\alpha_1 + \gamma) > 1\), system (2) has a unique positive equilibrium \((S_\ast, I_\ast, Q_\ast, R_\ast)\), where

\[
\begin{align*}
S_\ast &= \frac{\delta + d + \alpha_1 + \gamma}{\beta}, \\
I_\ast &= \frac{(1-p)b\beta - d(\delta + d + \alpha_1 + \gamma)}{\beta(\delta + d + \alpha_1 + \gamma)}, \\
Q_\ast &= \frac{\delta I_\ast}{\epsilon + d + \alpha_2}, \\
R_\ast &= \frac{p b + \gamma I_\ast + \epsilon Q_\ast}{d}.
\end{align*}
\]

The characteristic equation of system (2) at \(D_0\) is

\[
\begin{vmatrix}
\lambda - a_{11} - b_{11} e^{-\lambda t_1} & -b_{12} e^{-\lambda t_1} & 0 & 0 \\
-b_{21} e^{-\lambda t_1} & \lambda - a_{22} - b_{22} e^{-\lambda t_1} - c_{22} e^{-\lambda t_2} & 0 & 0 \\
0 & -a_{32} & \lambda - a_{33} - c_{33} e^{-\lambda t_2} & 0 \\
0 & -c_{42} e^{-\lambda t_2} & 0 & \lambda - a_{44}
\end{vmatrix} = 0,
\]
Case 1 ($\tau_1 = \tau_2 = 0$). When $\tau_1 = \tau_2 = 0$, (5) becomes

$$\lambda^6 + A_{13} \lambda^3 + A_{12} \lambda^2 + A_{11} \lambda + A_{10} = 0,$$

(9)

Thus, if the condition ($H_{11}$) (9) holds, then the positive equilibrium $D_{+}$ of system (2) is locally asymptotically stable when $\tau_1 = \tau_2 = 0$.

$$\det_2 = \begin{vmatrix} A_{13} & 1 \\ A_{11} & A_{12} \end{vmatrix} > 0,$$

(10)

$$\det_3 = \begin{vmatrix} A_{13} & 1 & 0 \\ A_{11} & A_{12} & A_{13} \\ 0 & A_{10} & A_{11} \end{vmatrix} > 0.$$

(11)

$$\det_4 = \begin{vmatrix} A_{13} & 1 & 0 & 0 \\ A_{11} & A_{12} & A_{13} & 1 \\ 0 & A_{10} & A_{11} & A_{12} \\ 0 & 0 & 0 & A_{10} \end{vmatrix} > 0.$$

(12)

Case 2 ($\tau_1 > 0, \tau_2 = 0$). When $\tau_1 > 0, \tau_2 = 0$, (5) becomes

$$\lambda^4 + A_{23} \lambda^3 + A_{22} \lambda^2 + A_{21} \lambda + A_{20} = 0.$$

(13)

(14)
where

\[ m_{20} = B_{20}^2, \]
\[ m_{22} = B_{21}^2 - 2B_{20}B_{22}, \]
\[ m_{24} = B_{22}^2 - 2B_{21}B_{23}, \]
\[ m_{26} = B_{23}^2, \]
\[ n_{20} = -A_{20}B_{20}, \]
\[ n_{22} = A_{20}B_{22} - A_{21}B_{21} + A_{22}B_{20}, \]
\[ n_{24} = A_{21}B_{23} - A_{22}B_{22} + A_{23}B_{21} - B_{20}, \]
\[ n_{26} = B_{22} - A_{23}B_{23}. \]

Substituting \( \lambda(t) \) into the left side of (10) and taking the derivative with respect to \( t \), one can obtain

\[
\left[ \frac{d\lambda}{dt} \right]^{-1} = -\frac{4\lambda^3 + 3A_{23}\lambda^2 + 2A_{22}\lambda + A_{21}}{\lambda(\lambda^4 + 3A_{23}\lambda^2 + 2A_{22}\lambda^2 + A_{21}\lambda + A_{20})} + \frac{3B_{22}\lambda^2 + 2B_{21}\lambda + B_{20}}{\lambda(B_{23}\lambda^3 + B_{22}\lambda^2 + B_{21}\lambda + B_{20})} - \frac{t_1}{\lambda},
\]

Thus,

\[
\text{Re} \left[ \frac{d\lambda}{dt} \right]^{-1} = -\frac{f_i'(v_{1*})}{B_{23}^2\omega_0^2 + (B_{22}^2 - 2B_{21}B_{23})\omega_0^4 + (B_{21}^2 - 2B_{20}B_{22})\omega_0^6 + B_{20}^2},
\]

where \( v_{1*} = \omega_0^2 \) and \( f_i'(v_{1*}) = v_{1}^4 + e_{22}v_{1}^3 + e_{23}v_{1}^2 + e_{21}v_{1} + e_{20}. \)

Thus, if the condition \( (H_{22}) f_i'(v_{1*}) \neq 0 \) holds, then \( \text{Re}(d\lambda/dt) \big|_{t_1=\omega_0 t_1} \neq 0. \) According to the Hopf bifurcation theorem in [15], we have the following results.

**Theorem 1.** If the conditions \( (H_{21})-(H_{22}) \) hold, then the positive equilibrium \( D_+(S_+, I_+, Q_+, R_+) \) is asymptotically stable for \( t_1 \in [0, \tau_0) \) and system (2) undergoes a Hopf bifurcation at the positive equilibrium \( D_+(S_+, I_+, Q_+, R_+) \) when \( t_1 = \tau_{10}. \)

Case 3 \( (\tau_1 = 0, \tau_2 > 0) \). When \( \tau_1 = 0, \tau_2 > 0, (5) \) becomes

\[
\lambda^4 + A_{33}\lambda^3 + A_{32}\lambda^2 + A_{31}\lambda + A_{30} + (C_{33}\lambda^3 + C_{32}\lambda^2 + C_{31}\lambda + C_{30})e^{-\lambda t_2} + (E_{32}\lambda^2 + E_{31}\lambda + E_{30})e^{-2\lambda t_2} = 0,
\]

where

\[
\begin{align*}
A_{30} &= A_0 + B_0, \\
A_{31} &= A_1 + B_1, \\
A_{32} &= A_2 + B_2, \\
A_{33} &= A_3 + B_3,
\end{align*}
\]

\[
\begin{align*}
C_{30} &= C_0 + D_0, \\
C_{31} &= C_1 + D_1, \\
C_{32} &= C_2 + D_2, \\
C_{33} &= C_3, \\
E_{30} &= E_0 + F_0, \\
E_{31} &= E_1 + F_1, \\
E_{32} &= E_2.
\end{align*}
\]

Multiplying by \( e^{\lambda t_2}, (20) \) becomes

\[
C_{33}\lambda^3 + C_{32}\lambda^2 + C_{31}\lambda + C_{30} + (\lambda^4 + A_{33}\lambda^3 + A_{32}\lambda^2 + A_{31}\lambda + A_{30})e^{\lambda t_2} + (E_{32}\lambda^2 + E_{31}\lambda + E_{30})e^{2\lambda t_2} = 0.
\]

Let \( \lambda = i\omega_2 (\omega_2 > 0) \) be the root of \( (22) \); then

\[
\begin{align*}
(\omega_2^4 - (A_{32} + E_{32})\omega_2^2 + A_{30} - E_{30}) \sin t_2 \omega_2 \\
+ ((A_{31} + E_{31}) \omega_2 - A_{33}\omega_2^3) \cos t_2 \omega_2 = C_{33}\omega_2^3 - C_{31}\omega_2,
\end{align*}
\]

\[
\begin{align*}
(\omega_2^4 - (A_{32} + E_{32})\omega_2^2 + A_{30} + E_{30}) \cos t_2 \omega_2 \\
- ((A_{31} - E_{31}) \omega_2 - A_{33}\omega_2^3) \sin t_2 \omega_2 = C_{32}\omega_2^2 - A_{30}.
\end{align*}
\]

It follows that

\[
\begin{align*}
\cos t_2 \omega_2 &= \frac{n_{30}\omega_2^6 + 2n_{34}\omega_2^4 + 2n_{32}\omega_2^2 + n_{30}}{\omega_2^6 + 2n_{34}\omega_2^4 + 2n_{32}\omega_2^2 + n_{30}},
\sin t_2 \omega_2 &= \frac{n_{32}\omega_2^7 + n_{34}\omega_2^5 + n_{32}\omega_2^3 + n_{30}\omega_2}{\omega_2^6 + 2n_{34}\omega_2^4 + 2n_{32}\omega_2^2 + n_{30}},
\end{align*}
\]

where

\[
\begin{align*}
m_{30} &= A_{30}^2 - E_{30}^2, \\
m_{32} &= A_{31}^2 - E_{31}^2 - 2A_{30}(A_{32} + E_{32}), \\
m_{34} &= (A_{32} + E_{32})^2 - 2A_{31}A_{33} + 2A_{30}, \\
m_{36} &= A_{33}^2 - 2(A_{32} + E_{32}), \\
n_{30} &= (E_{30} - A_{30})C_{30}, \\
n_{31} &= (A_{31} + E_{31})C_{30} - (A_{30} + E_{30})C_{31}, \\
n_{32} &= (A_{32} + E_{32})C_{30} - (A_{31} - E_{31})C_{31} + (A_{30} - E_{30})C_{32},
\end{align*}
\]
\[ n_{33} = (A_{30} + E_{30})C_{33} - (A_{31} + E_{31})C_{32} + (A_{32} + E_{32})C_{31} - C_{30}, \]
\[ n_{34} = (A_{31} - E_{31})C_{33} - (A_{32} + E_{32})C_{32} - A_{33}C_{31}, \]
\[ n_{35} = A_{33}C_{32} - C_{31} - (A_{32} + E_{32})C_{33}, \]
\[ n_{36} = C_{32} - A_{33}C_{33}, \]
\[ n_{37} = C_{33}. \]

Then, we can get
\[ \omega_2^{16} + e_{32}\omega_2^{14} + e_{34}\omega_2^{12} + e_{35}\omega_2^{10} + e_{36}\omega_2^8 + e_{37}\omega_2^6 + e_{32}\omega_2^4 + e_{31}\omega_2^2 + e_{30} = 0, \]

with
\[ e_{30} = m_{30}^2 - n_{30}^2, \]
\[ e_{31} = 2m_{30}n_{32} - 2n_{30}n_{32} - n_{31}^2, \]
\[ e_{32} = m_{32}^2 - n_{32}^2 + 2m_{30}m_{32} - 2n_{30}n_{32} - 2n_{31}n_{33}, \]
\[ e_{33} = 2m_{30}m_{33} + 2m_{32}m_{33} - 2n_{30}n_{36} - 2n_{31}n_{35}, \]
\[ e_{34} = 2n_{32}n_{34} - n_{33}^2, \]
\[ e_{35} = 2m_{32} + 2m_{34}n_{36} - 2n_{33}n_{37} - 2n_{34}n_{35} - n_{35}^2, \]
\[ e_{36} = n_{36}^2 - n_{32}^2 + 2m_{34} - 2n_{35}n_{37}, \]
\[ e_{37} = 2n_{36} - n_{32}^2. \]

Let \( \omega_2^2 = \nu_2; \) (26) becomes
\[ \omega_2^8 + e_{37}\nu_2^6 + e_{36}\nu_2^4 + e_{35}\nu_2^2 + e_{34}\nu_2^4 + e_{33}\nu_2^6 + e_{32}\nu_2^8 + e_{31}\nu_2^{10} + e_{30}\ = 0. \]

Differentiating the two sides of (22) regarding \( \tau_2, \) we obtain
\[ \left[ \frac{d\lambda}{d\tau_2} \right]^{-1} = \frac{P_{30}(\lambda)}{Q_{30}(\lambda)} - \frac{\tau_2}{\lambda}, \]

where
\[ P_{30}(\lambda) = 3C_{33}\lambda^2 + 2C_{32}\lambda + C_{31} + (2E_{32} + E_{31})e^{-\lambda\tau_2} + (4\lambda^3 + 3A_{33}\lambda^2 + 2A_{32}\lambda + A_{31})e^{\lambda\tau_2}, \]
\[ Q_{30}(\lambda) = (E_{33}\lambda^3 + E_{31}\lambda^2 + E_{30}\lambda)e^{-\lambda\tau_2} - \left( \lambda^5 + A_{33}\lambda^4 + A_{32}\lambda^3 + A_{31}\lambda^2 + A_{30}\lambda \right)e^{\lambda\tau_2}, \]

where
\[ P_{3R} = (A_{31} + E_{31} - 3A_{33}\omega_2^2)\cos\tau_2\omega_20 - 2(A_{32} - E_{32})\omega_20 - 2A_{30}\omega_20 + C_{31} - 3C_{33}\omega_2^2, \]
\[ P_{3L} = (A_{31} - E_{31} - 3A_{33}\omega_2^2)\sin\tau_2\omega_20 - 2(A_{32} + E_{32})\omega_20 - 2A_{30}\omega_20 + 2C_{32}\omega_20, \]
\[ Q_{3R} = ((A_{31} - E_{31})\omega_2^2 - A_{33}\omega_2^4)\cos\tau_2\omega_20 + (\omega_2^5 - (A_{32} + E_{32})\omega_2^3 + (A_{30} + E_{30})\omega_20) \cdot \sin\tau_2\omega_20, \]
\[ Q_{3L} = ((A_{31} + E_{31})\omega_2^2 - A_{33}\omega_2^4)\sin\tau_2\omega_20 - (\omega_2^5 + (A_{32} - E_{32})\omega_2^3 + (A_{30} - E_{30})\omega_20) \cdot \cos\tau_2\omega_20. \]

Therefore, if condition (H_{32}) P_{3R}Q_{3R} + P_{3L}Q_{3L} \neq 0 holds, then \( \text{Re}[d\lambda/d\tau_2] \neq 0 \) for some \( \omega_2 \). Thus, we have the following results according to the Hopf bifurcation theorem in [15].

**Theorem 2.** If conditions (H_{31})-(H_{32}) hold, then the positive equilibrium \( D_1(S_1, L_1, Q_1, R_1) \) is asymptotically stable for \( \tau_2 \in [0, \tau_{20}] \) and system (2) undergoes a Hopf bifurcation at the positive equilibrium \( D_1(S_1, L_1, Q_1, R_1) \) when \( \tau_2 = \tau_{20} \).

**Case 1:** \( \tau_1 > 0, \tau_2 > 0 \) and \( \tau_2 \in (0, \tau_{20}) \). Let \( \lambda = i\omega' \) (\( \omega' > 0 \)) be the root of (5); then we obtain
\[ G_{41}(\omega')\sin\tau_1\omega'_1 + G_{42}(\omega')\cos\tau_1\omega'_1 = G_{43}(\omega'), \]
\[ G_{41}(\omega')\cos\tau_1\omega'_1 - G_{42}(\omega')\sin\tau_1\omega'_1 = G_{44}(\omega'). \]
where
\[
G_{41}(\omega_1) = B_1 \omega_1 - B_3 (\omega_1)^3 + D_1 \omega_1 \cos \tau_2 \omega_1^1 \\
- (D_0 - D_2 (\omega_1)^2) \sin \tau_2 \omega_1^1 \\
+ F_1 \omega_1^1 \cos 2\tau_2 \omega_1^1 - F_0 \sin 2\tau_2 \omega_1^1,
\]
\[
G_{42}(\omega_1) = B_1 - B_3 (\omega_1)^2 + D_1 \omega_1 \sin \tau_2 \omega_1^1 \\
+ (D_0 - D_2 (\omega_1)^2) \cos \tau_2 \omega_1^1 \\
+ F_1 \omega_1^1 \sin 2\tau_2 \omega_1^1 - F_0 \cos 2\tau_2 \omega_1^1,
\]
\[
G_{43}(\omega_1) = A_2 (\omega_1)^2 - (\omega_1)^4 - A_0 \\
- (C_1 \omega_1^1 - C_3 (\omega_1)^3) \sin \tau_2 \omega_1^1 \\
- (C_0 - C_2 (\omega_1)^2) \cos \tau_2 \omega_1^1 \\
- E_1 \omega_1^1 \sin 2\tau_2 \omega_1^1 \\
- (E_0 - E_2 (\omega_1)^2) \cos 2\tau_2 \omega_1^1,
\]
\[
G_{44}(\omega_1) = A_3 (\omega_1)^3 - A_4 \omega_1^1 \\
- (C_1 \omega_1^1 - C_3 (\omega_1)^3) \cos \tau_2 \omega_1^1 \\
+ (C_0 - C_2 (\omega_1)^2) \sin \tau_2 \omega_1^1 \\
- E_1 \omega_1^1 \cos 2\tau_2 \omega_1^1 \\
+ (E_0 - E_2 (\omega_1)^2) \sin 2\tau_2 \omega_1^1.
\]

Then, we can obtain the following equation with respect to \(\omega_1^1\):
\[
f_{40}(\omega_1^1) + 2f_{41}(\omega_1^1) \cos \tau_2 \omega_1^1 + 2f_{42}(\omega_1^1) \sin \tau_2 \omega_1^1 \\
+ 2f_{43}(\omega_1^1) \cos 2\tau_2 \omega_1^1 + 2f_{44}(\omega_1^1) \sin 2\tau_2 \omega_1^1 = 0,
\]  
(35)

where
\[
f_{40}(\omega_1^1) = (\omega_1^1)^8 + (A_3^2 - B_3^2 + C_3^2 - 2 A_2) (\omega_1^1)^6 \\
+ (A_2^2 - B_2^2 + C_2^2 - D_2^2 + E_2^2 - 2 A_1 A_3 + 2 B_1 B_3) (\omega_1^1)^4 \\
+ (2 C_3 A_3 + 2 A_0) (\omega_1^1)^2 + (A_1^2 - B_1^2 + C_1^2 - D_1^2) (\omega_1^1) \\
+ E_1^2 - F_1^2 - 2 A_0 A_1 + 2 B_0 B_2 - 2 C_0 C_2 + 2 D_0 D_2 \\
- 2 E_0 E_2) (\omega_1^1)^2 + A_0^2 - B_0^2 + C_0^2 - D_0^2 + E_0^2 - F_0^2,
\]
\[
f_{41}(\omega_1^1) = (A_3 C_3 - C_2) (\omega_1^1)^6 + (A_2 C_2 - A_1 C_3 \\
- A_1 C_1 - B_2 D_2 + B_3 D_1 + C_2 E_2 - C_3 E_1 + C_0) \\
\cdot (\omega_1^1)^4 + (A_1 C_1 - A_0 C_2 - A_2 C_0 + B_0 D_2 - B_1 D_1 \\
+ B_2 D_0 - C_0 E_2 + C_1 E_1 - C_2 E_0 - D_1 F_1 + D_2 F_0) \\
\cdot (\omega_1^1)^2 + A_0 C_0 - B_0 F_0 - D_0 F_0,
\]
\[
f_{42}(\omega_1^1) = -C_3 (\omega_1^1)^7 + (A_3 C_3 - A_2 C_2 + B_3 D_2 \\
- C_3 E_2 + C_1) (\omega_1^1)^5 + (A_1 C_2 - A_0 C_3 - A_2 C_1 \\
+ A_4 C_0 - B_1 D_2 + B_2 D_1 - B_3 D_0 + C_1 E_2 - C_2 E_1 \\
+ C_3 E_0 + D_2 F_1) (\omega_1^1)^3 + (A_0 E_1 - A_1 E_0 - B_3 D_1 \\
+ B_1 D_0 - C_0 E_1 + C_1 E_0 - D_0 F_1 + D_1 F_0) \omega_1^1,
\]
\[
f_{43}(\omega_1^1) = -E_2 (\omega_1^1)^5 + (A_1 E_2 + B_3 F_1 + E_0) (\omega_1^1)^3 \\
- (A_0 E_2 + A_2 E_0 + B_1 F_1 - B_2 F_0) (\omega_1^1)^2 + B_0 F_0 \\
- A_0 E_0,
\]
\[
f_{44}(\omega_1^1) = (E_1 - A_3 E_2) (\omega_1^1)^5 + (A_1 E_2 - A_4 E_1 \\
+ A_3 E_0 + B_2 F_1 - B_3 F_0) (\omega_1^1)^3 (A_0 E_1 - A_1 E_0 \\
- B_0 F_1 + B_1 F_0) \omega_1^1.
\]  
(36)

In order to give the main results in this paper, we make the following assumption.

\((H_4)\) (35) has at least one positive root.

If the condition \((H_4)\) holds, then there exists a positive root \(\omega_{10}\) for (35) such that (5) has a pair of purely imaginary root \(\pm i \omega_{10}\). For \(\omega_{10}\),
\[
\tau_{10} = \frac{1}{\omega_{10}} \\
\times \arccos \frac{G_{42}(\omega_{10}) \times G_{43}(\omega_{10}) + G_{43}(\omega_{10}) \times G_{44}(\omega_{10})}{G_{41}(\omega_{10}) + G_{42}(\omega_{10})}.
\]  
(37)

Substituting \(\lambda(\tau_1)\) into (5) and differentiating both sides of it with respect to \(\tau_1\), then
\[
\left[ \frac{d\lambda}{d\tau_1} \right]^{-1} = \frac{P_{40}(\lambda)}{Q_{40}(\lambda)} - \frac{\tau_1}{\lambda},
\]  
(38)

where
\[
P_{40}(\lambda) = 4 \lambda^3 + 3 A_3 \lambda^2 + 2 A_2 \lambda + A_1 + (3 B_3 \lambda^2 \\
+ 2 B_2 \lambda + B_1) e^{-\lambda \tau_1} - (\tau_2 C_3 \lambda^3 + (\tau_2 C_2 - 3 C_3) \lambda^2
\]
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\[ + (\tau_2 C_1 - 2 C_2) \lambda + \tau_2 C_0 - C_1) e^{-\lambda t} - (\tau_2 D_2 \lambda^2 + (\tau_2 D_1 - 2 D_2) \lambda + \tau_2 D_0 - D_1) e^{-\lambda(t + \tau_2)} - (2 \tau_2 E_2 \lambda^2 + 2 (\tau_2 E_1 - E_2) \lambda + 2 \tau_2 E_0 - E_1) e^{-2\lambda t} - (2 \tau_1 F_1 \lambda + 2 \tau_2 F_0 - F_1) e^{-\lambda(t + \tau_2)}, \]

\[ Q_{4\phi} (\lambda) = (B_3 \lambda^4 + B_4 \lambda^3 + B_5 \lambda^2 + B_6 \lambda) e^{-\lambda t} + (D_2 \lambda^3 + D_1 \lambda^2 + D_0) e^{-\lambda(t + \tau_2)} + (F_1 \lambda^2 + F_0 \lambda) e^{-\lambda t}. \]

We therefore derive that

\[ \text{Re} \left[ \frac{d\lambda}{dt} \right]_{\lambda=\omega_1}^{-1} = \frac{P_{4\phi} Q_{4\phi} + P_{4\phi} Q_{4t}}{Q_{4\phi} + Q_{4t}}, \]

where

\[ P_{4\phi} = A_1 - 3 A_3 (\omega_1')^2 + \left[ (2C_2 - \tau_2 C_1) \omega_1' \right. \]

\[ + \tau_2 C_3 (\omega_1')^3 \sin \tau_2 \omega_1' + \left( \tau_2 C_2 - 3 C_3 \right) \omega_1' (\omega_1')^2 + C_1 - \tau_2 C_0 \cos \tau_2 \omega_1' + 2 \left( E_2 - \tau_2 E_1 \right) \omega_1' \]

\[ \cdot \sin 2 \tau_2 \omega_1' + \left( 2 \tau_2 E_2 \omega_1' \right)^2 - E_1 - 2 \tau_2 E_0 \right] \cdot \cos 2 \tau_2 \omega_1' + \left( D_2 - \tau_2 D_1 \right) \omega_1' \cos \tau_2 \omega_1' \]

\[ - \left( \tau_2 D_2 \omega_1' + D_1 - \tau_2 D_0 \right) \sin \tau_2 \omega_1' \]

\[ - (F_1 - 2 \tau_2 F_0) \sin 2 \tau_2 \omega_1' + 2 \tau_2 F_1 \omega_1' \cos 2 \tau_2 \omega_1' \]

\[ + 2B_2 \omega_1' \sin \tau_1 \omega_1' \]

\[ + \left( 2D_2 - \tau_2 D_1 \right) \omega_1' \sin \tau_2 \omega_1' \]

\[ + \left( \tau_2 D_2 \omega_1' \right)^2 + D_1 - \tau_2 D_0 \right) \cos \tau_2 \omega_1' \]

\[ + (F_1 - 2 \tau_2 F_0) \cos 2 \tau_2 \omega_1' - 2 \tau_2 F_1 \omega_1' \cos 2 \tau_2 \omega_1' \]

\[ + B_1 - 3 B_3 (\omega_1')^2 \cos \tau_1 \omega_1'. \]

\[ P_{4t} = 2A_2 \omega_1' - 4 (\omega_1')^3 + \left[ (2C_2 - \tau_2 C_1) \omega_1' \right. \]

\[ + \tau_2 C_3 (\omega_1')^3 \cos \tau_2 \omega_1' - \left( \tau_2 C_2 - 3 C_3 \right) \omega_1' (\omega_1')^2 + C_1 - \tau_2 C_0 \sin \tau_2 \omega_1' + 2 \left( E_2 - \tau_2 E_1 \right) \omega_1' \]

\[ \cdot \cos 2 \tau_2 \omega_1' - \left( 2 \tau_2 E_2 \omega_1' \right)^2 - E_1 - 2 \tau_2 E_0 \right] \cdot \sin 2 \tau_2 \omega_1' \]

\[ \cdot \sin 2 \tau_2 \omega_1' \left[ 2D_2 - \tau_2 D_1 \right) \omega_1' \cos \tau_2 \omega_1' \]

\[ - \left( \tau_2 D_2 \omega_1' \right)^2 + D_1 - \tau_2 D_0 \right) \sin \tau_2 \omega_1'. \]

\[ - \left( F_1 - 2 \tau_2 F_0 \right) \sin 2 \tau_2 \omega_1' + 2 \tau_2 F_1 \omega_1' \cos 2 \tau_2 \omega_1' \]

\[ + 2B_2 \omega_1' \cos \tau_1 \omega_1' \]

\[ - \left( 2D_2 - \tau_2 D_1 \right) \omega_1' \sin \tau_2 \omega_1' \]

\[ \cdot \cos \tau_1 \omega_1' + \left( \tau_2 \right) \omega_1' \sin \omega_1' \]

\[ + \left( \tau_2 \right) \omega_1' + D_1 - \tau_2 D_0 \right) \cos \tau_2 \omega_1' \]

\[ + (F_1 - 2 \tau_2 F_0) \cos 2 \tau_2 \omega_1' - 2 \tau_2 F_1 \omega_1' \cos 2 \tau_2 \omega_1' \]

\[ + B_1 - 3 B_3 (\omega_1')^2 \cos \tau_1 \omega_1'. \]

\[ Q_{4\phi} = \left[ D_0 \omega_1' - D_2 (\omega_1')^3 \right] \sin \tau_2 \omega_1' \]

\[ - D_1 (\omega_1')^2 \cos \tau_2 \omega_1' + F_0 \omega_1' \sin 2 \tau_2 \omega_1' \]

\[ - F_1 (\omega_1')^2 \cos 2 \tau_2 \omega_1' + B_3 (\omega_1')^4 - B_1 \omega_1' \cos \tau_2 \omega_1' \]

\[ + D_1 \omega_1' \sin \tau_1 \omega_1' \]

\[ + F_1 \omega_1' \sin \tau_1 \omega_1' \]

\[ + B_1 \omega_1' - B_2 (\omega_1')^3 \sin \tau_1 \omega_1' \]

\[ \sin \tau_1 \omega_1' \]

\[ \cdot \cos \tau_1 \omega_1'. \]

Therefore, if condition (H_{42}) \ P_{4\phi} Q_{4\phi} + P_{4\phi} Q_{4t} \neq 0 holds, then \text{Re}[d\lambda/dt]_{\lambda=\omega_1'} \neq 0.\ Thus, we have the following results according to the Hopf bifurcation theorem in [15].

**Theorem 3.** For system (2), if conditions (H_{41}) - (H_{42}) hold and \tau_2 \in (0, \tau_2^0), then the positive equilibrium \( D_\ast (S_\ast, I_\ast, Q_\ast, R_\ast) \) is asymptotically stable for \tau_1 \in (0, \tau_1^*), and a family of periodic solutions bifurcate from \( D_\ast (S_\ast, I_\ast, Q_\ast, R_\ast) \) near \tau_1 = \tau_1^*. 

Then, we get
\[
\cos \tau_2 \omega'_2 = \frac{G_{52} (\omega'_2) \times G_{56} (\omega'_2) + G_{53} (\omega'_2) \times G_{55} (\omega'_2)}{G_{51} (\omega'_2) \times G_{53} (\omega'_2) + G_{52} (\omega'_2) \times G_{54} (\omega'_2)},
\]
\[
\sin \tau_2 \omega'_2 = \frac{G_{51} (\omega'_2) \times G_{56} (\omega'_2) - G_{54} (\omega'_2) \times G_{55} (\omega'_2)}{G_{51} (\omega'_2) \times G_{53} (\omega'_2) + G_{52} (\omega'_2) \times G_{54} (\omega'_2)}.
\]
Thus, we can obtain the following function with respect to \( \omega'_2 \):
\[
\cos^2 \tau_2 \omega'_2 + \sin^2 \tau_2 \omega'_2 = 1.
\]
Suppose that \((H_{52})\), \((46)\) has at least one positive root.
If condition \((H_{52})\) holds, there exists a positive root \(\omega'_2 > 0\) of \((46)\) such that \((46)\) has a pair of purely imaginary roots \(\pm i\omega'_2\). For \(\omega'_2\),
\[
\tau'_2 = \frac{1}{\omega'_2},
\]
\[
\times \arccos \frac{G_{52} (\omega'_2) \times G_{56} (\omega'_2) + G_{53} (\omega'_2) \times G_{55} (\omega'_2)}{G_{51} (\omega'_2) \times G_{53} (\omega'_2) + G_{52} (\omega'_2) \times G_{54} (\omega'_2)}.
\]
Taking the derivative with respect to \(\tau'_2\) in \((42)\), we can obtain
\[
\left[ \frac{d \lambda}{d \tau'_2} \right]^{-1} = \frac{P_{50} (\lambda)}{Q_{50} (\lambda)} - \frac{\tau'_2}{\lambda},
\]
where
\[
P_{50} (\lambda) = 3C_3 \lambda^2 + 2C_2 \lambda + C_1 + \left( 4A^3 + 3A_3 \lambda^2 + 2A_2 + A_1 \lambda + A_0 \right) e^{\lambda \tau_2} + \frac{2A_2}{\lambda} + A_1 e^{-\lambda \tau_2} - \left( \tau_1 D_2 \lambda^2 + (\tau_1 D_1 - 2D_2 \lambda + \tau_1 D_0 - D_1) e^{-\lambda \tau_1} - \tau_1 F_0 \lambda e^{-\lambda (\tau_2 + \tau_1)} - \left( \tau_1 B_3 \lambda^3 + (\tau_1 B_2 - 3B_3) \lambda^2 + (\tau_1 B_1 - 2B_2) \lambda + \tau_1 B_0 \right) e^{\lambda (\tau_2 - \tau_1)}.\right.
\]
\[
Q_{50} (\lambda) = (E_2 \lambda^3 + E_1 \lambda^2 + E_0 \lambda) e^{\lambda \tau_2} + (F_1 \lambda^2 + F_0 \lambda) \cdot e^{-\lambda (\tau_2 + \tau_1)} - \left( \lambda^5 + A_3 \lambda^4 + A_2 \lambda^3 + A_1 \lambda^2 + A_0 \right) \cdot e^{\lambda \tau_2} - (B_3 \lambda^4 + B_2 \lambda^3 + B_1 \lambda^2 + B_0 \lambda) e^{\lambda (\tau_2 - \tau_1)}.
\]
Define
\[
\text{Re} \left[ \frac{d \lambda}{d \tau'_2} \right]_{\lambda = i \omega'_2} = \frac{P_{50} Q_{50} + P_{50} Q_{50}'}{Q_{50}^2 + Q_{50}'}.
\]
Obviously, if the condition \((H_{52})\), \(P_{50} Q_{50} + P_{50} Q_{50}' \neq 0\), then \(\text{Re} \left[ d \lambda/d \tau'_2 \right]_{\lambda = i \omega'_2} \neq 0\). Thus, we have the following results according to the Hopf bifurcation theorem in [15].
Theorem 4. For system (2), if the conditions \((H_{51})-(H_{52})\) hold and \(\tau_1 \in (0, \tau_{10})\), then the positive equilibrium \(D_+(S_*, I_*, Q_*, R_*)\) is asymptotically stable for \(\tau_2 \in [0, \tau'_2)\) and system (2) undergoes a Hopf bifurcation at the positive equilibrium \(D_+(S_*, I_*, Q_*, R_*)\) near \(\tau_2 = \tau'_2\).

3. Properties of the Hopf Bifurcation

In this section, we investigate the direction of the Hopf bifurcation and the stability of the Hopf bifurcation of system (2) when \(\tau_2 > 0\) and \(\tau_1 = \tau_{1*} \in (0, \tau_{10})\) by using the normal form theory and the center manifold theorem in [15]. Throughout this section, we assume that \(\tau_{1*} < \tau'_2\).

Let \(u_1(t) = S(t) - S_*, u_2(t) = I(t) - I_*, u_3(t) = Q(t) - Q_*, u_4(t) = R(t) - R_*\), and \(\tau_2 = \tau'_2 + \mu, \mu \in R\) and normalize the time delay by \(t \to (t/\tau_2)\). Then system (1) can be transformed into

\[
\begin{align*}
\dot{u}(t) &= L_\mu u + F(\mu, u),
\end{align*}
\]

where

\[
\begin{align*}
L_\mu &= \left(\tau'_{20} + \mu\right)
\end{align*}
\]

By the Riesz representation theorem, there is a \(4 \times 4\) matrix function with bounded variation components \(\eta(\theta, \mu), \theta \in [-1, 0]\) such that

\[
L_\mu \phi = \int_{-1}^{0} d\eta(\theta, \mu) \phi(\theta), \quad \phi \in C([-1, 0], R^4). \tag{53}
\]

In fact, we choose

\[
\eta(\theta, \mu) = \begin{cases}
(\tau'_{20} + \mu) (A_{\text{trix}} + B_{\text{trix}} + C_{\text{trix}}), & \theta = 0,
(\tau'_{20} + \mu) (B_{\text{trix}} + C_{\text{trix}}), & \theta \in \left(-\tau_{1*}, 0\right),
(\tau'_{20} + \mu) C_{\text{trix}}, & \theta \in \left(-1, -\tau_{1*}/\tau_{20}\right),
0, & \theta = -1.
\end{cases} \tag{54}
\]

For \(\phi \in C([-1, 0], R^4),\) we define

\[
A(\mu) = \begin{cases}
d\phi(\theta)/d\theta, & -1 \leq \theta < 0,
\int_{-1}^{0} d\eta(\theta, \mu) \phi(\theta), & \theta = 0,
\end{cases} \tag{55}
\]

\[
R(\mu) = \begin{cases}
0, & -1 \leq \theta < 0,
F(\mu, \phi), & \theta = 0.
\end{cases}
\]

Then system (51) is equivalent to the following operator equation:

\[
\dot{u}(t) = A(\mu) u + R(\mu) u. \tag{56}
\]

Next, we define the adjoint operator \(A^*\) of \(A\)

\[
A^*(\varphi) = \begin{cases}
d\varphi(s)/ds, & 0 < s \leq 1,
\int_{-1}^{0} d\eta(\theta, 0) \varphi(-s), & s = 0,
\end{cases} \tag{57}
\]

and a bilinear inner product

\[
\langle \varphi(s), \phi(\theta) \rangle = \overline{\varphi(0)}(0) \phi(0) - \int_{\theta - 1}^{\theta} \int_{\xi = 0}^{\theta} \overline{\varphi(\xi - \theta)} d\eta(\theta) \phi(\xi) d\xi, \tag{58}
\]

where \(\eta(\theta) = \eta(\theta, 0).\)
Let \( \rho(\theta) = (1, \rho_2, \rho_3, \rho_4) e^{i\omega_2 \tau_0 - i\omega_3 \tau_0^2} \) be the eigenvectors of \( A(0) \) corresponding to \( +i\omega_2 \tau_0 \) and \( \rho'(s) = (1/K)(1, \rho_2', \rho_3', \rho_4') e^{i\omega_2 \tau_0 - i\omega_3 \tau_0^2} \) be the eigenvectors of \( A^* \) corresponding to \( -i\omega_2 \tau_0 \). Then, we obtain

\[
\begin{align*}
\rho_2 &= \frac{i\omega_2 - a_1 - b_1 e^{-i\tau_1 \omega_2}}{b_2 e^{-i\tau_1 \omega_2}}, \\
\rho_3 &= \frac{a_{32} (i\omega_2 - a_1 - b_1 e^{-i\tau_1 \omega_2})}{(i\omega_2 - a_{33} - c_{33} e^{-i\tau_2 \omega_2}) b_2 e^{-i\tau_1 \omega_2}}, \\
\rho_4 &= \frac{c_{42} \rho_2 + c_{43} \rho_3}{(i\omega_2 - a_{44}) e^{i\tau_2 \omega_2}}, \\
\rho_2' &= -\frac{i\omega_2 e^{i\tau_1 \omega_2}}{b_{21} e^{i\tau_1 \omega_2}}, \\
\rho_3' &= \frac{c_{42} \rho_2 + c_{43} \rho_3}{c_{33} e^{i\tau_2 \omega_2}}, \\
\rho_4' &= \frac{1}{(i\omega_2 + a_{33} + c_{33} e^{i\tau_2 \omega_2}) c_{43} e^{i\tau_2 \omega_2}}, \\
\rho_3' &= \left[ (i\omega_2 + a_{22} + b_{22} e^{i\tau_1 \omega_2}) \rho_2 + b_{12} e^{i\tau_1 \omega_2} \right] \\
&\quad \times \left( \frac{c_{43} e^{i\tau_2 \omega_2} - a_{33} e^{i\tau_2 \omega_2}}{i\omega_2 + c_{33} e^{i\tau_2 \omega_2}} \right)^{-1}.
\end{align*}
\]

From (58), we can obtain

\[
\mathcal{K} = 1 + \rho_2 \overline{\rho_2} + \rho_3 \overline{\rho_3} + \rho_4 \overline{\rho_4} + \tau_1 e^{-2i\omega_2 \tau_1} (b_{11} + b_{21} \overline{\rho_2} + \rho_2 (b_{12} + b_{22} \overline{\rho_2}')) + \tau_2 e^{-2i\omega_2 \tau_2} (\rho_2 (c_{22} \overline{\rho_2} + c_{42} \overline{\rho_4}) + \rho_3 (c_{33} \overline{\rho_3} + c_{43} \overline{\rho_4})).
\]

such that \( \langle \rho^*, \rho \rangle = 1, \langle \rho^*, \overline{\rho} \rangle = 0 \).

Following the algorithms given in [15] and using similar computation process in [16], we can get the coefficients which determine the properties of the Hopf bifurcation:

\[
\begin{align*}
g_{20} &= 2\beta \tau_0 e^{i\tau_1 \omega_2} \left( \rho_2 - 1 \right) \left( -\frac{\tau_1}{\tau_2} \right) \left( -\frac{1}{\tau_2} \right), \\
g_{11} &= \beta \tau_0 e^{i\tau_1 \omega_2} \left( \rho_2 - 1 \right) \left( -\frac{\tau_1}{\tau_2} \right) \left( -\frac{1}{\tau_2} \right) + \left( -\frac{\tau_1}{\tau_2} \right) \left( -\frac{1}{\tau_2} \right), \\
g_{12} &= 2\beta \tau_0 e^{i\tau_1 \omega_2} \left( \rho_2 - 1 \right) \left( \rho_2 - 1 \right) \left( \frac{1}{\tau_2} \right) \left( \frac{1}{\tau_2} \right), \\
g_{21} &= 2\beta \tau_0 e^{i\tau_1 \omega_2} \left( \rho_2 - 1 \right) \left( \frac{1}{\tau_2} \right) \left( \frac{1}{\tau_2} \right) \left( -\frac{\tau_1}{\tau_2} \right) + \left( -\frac{\tau_1}{\tau_2} \right) \left( -\frac{1}{\tau_2} \right),
\end{align*}
\]

with

\[
\begin{align*}
W_{20}(\theta) &= \frac{i\beta \tau_0}{\tau_2 \tau_0} e^{-i\tau_1 \omega_2} \left( -\frac{\tau_1}{\tau_2} \right) \left( -\frac{1}{\tau_2} \right) + \left( -\frac{\tau_1}{\tau_2} \right) \left( -\frac{1}{\tau_2} \right), \\
W_{11}(\theta) &= \frac{i\beta \tau_0}{\tau_2 \tau_0} e^{-i\tau_1 \omega_2} \left( \frac{1}{\tau_2} \right) \left( \frac{1}{\tau_2} \right) \left( -\frac{\tau_1}{\tau_2} \right) + \left( -\frac{\tau_1}{\tau_2} \right) \left( -\frac{1}{\tau_2} \right),
\end{align*}
\]

where \( E_1 \) and \( E_2 \) can be determined by the following equations, respectively,

\[
\begin{align*}
E_1 &= \text{2} \left( \begin{array}{cccc}
\alpha_{11} & b_1 & 0 & 0 \\
b_1 & \alpha_{22} & 0 & 0 \\
0 & -\alpha_{32} & \alpha_{33} & 0 \\
0 & -\alpha_{42} & -\alpha_{43} & \alpha_{44}
\end{array} \right), \\
E_2 &= \text{2} \left( \begin{array}{cccc}
\alpha_{11} & b_1 & 0 & 0 \\
b_1 & \alpha_{22} + b_{22} + c_2 & 0 & 0 \\
0 & \alpha_{32} & \alpha_{33} + a_{44} & 0 \\
0 & \alpha_{42} & \alpha_{43} + a_{44}
\end{array} \right),
\end{align*}
\]
with
\[ \begin{align*}
a'_{11} &= 2i\omega'_{10} - a_{11} - b_{11} e^{-2i\tau_1}, \\
a'_{22} &= 2i\omega'_{10} - a_{22} - b_{22} e^{-2i\tau_1} - c_{22} e^{-2i\tau_2}, \\
a'_{33} &= 2i\omega'_{10} - a_{33} - c_{33} e^{-2i\tau_2}, \\
E_{11} &= -\beta_1 \left( \frac{\tau_1}{\tau_{10}} \right) \rho(1) - \left( \frac{\tau_1}{\tau_{10}} \right) \rho(2), \\
E_{12} &= \beta_1 \left( \frac{\tau_1}{\tau_{10}} \right) \rho(1) - \left( \frac{\tau_1}{\tau_{10}} \right) \rho(2), \\
E_{21} &= -\beta_2 \left( \frac{\tau_1}{\tau_{10}} \right) \rho(1) - \left( \frac{\tau_1}{\tau_{10}} \right) \rho(2) + \bar{\rho}(1) \left( \frac{\tau_1}{\tau_{10}} \right) \rho(1) - \left( \frac{\tau_1}{\tau_{10}} \right) \rho(2), \\
E_{22} &= \beta_2 \left( \frac{\tau_1}{\tau_{10}} \right) \rho(1) - \left( \frac{\tau_1}{\tau_{10}} \right) \rho(2) + \bar{\rho}(1) \left( \frac{\tau_1}{\tau_{10}} \right) \rho(1) - \left( \frac{\tau_1}{\tau_{10}} \right) \rho(2),
\end{align*} \]

by Figures 1–3. However, if we choose \( \tau_1 = 0.9755 > 0 \), we obtain a Hopf bifurcation occurs, which can be illustrated by Figures 4–6. When a Hopf bifurcation occurs, the state of computer viruses propagation changes from an equilibrium point to a limit cycle and it means that the propagation of computer viruses is out of control. Similarly, we have \( \omega_{10} = 1.6022, \tau_{10} = 1.7852 \) when \( \tau_1 = 0, \tau_2 > 0 \). The corresponding waveforms and phase plots are shown in Figures 7–12.

For \( \tau_1 > 0, \tau_2 > 0 \) and \( \tau_2 = 1.05 \in (0, \tau_{20}) \), we can find \( \omega'_1 = 1.9403 \) and then we obtain \( \tau'_{10} = 0.8903 \). Figures 13–15 show that the positive equilibrium \( D_1 \) of system (66) is asymptotically stable when \( \tau_1 = 0.785 < 0.8903 = \tau'_{10} \) and Figures 16–18 show that there is a Hopf bifurcation occurs at the positive equilibrium \( D_1 \) of system (66) and a family of periodic solutions bifurcate from \( D_1 \) when \( \tau_1 = 0.9755 > \tau_{10} = 0.8903 \). Similarly, we have \( \omega'_1 = 2.0411 \) and \( \tau'_{20} = 1.3485 \) when \( \tau_2 > 0, \tau_1 = 0.65 \in (0, \tau_{10}) \). The corresponding waveforms and plots are shown in Figures 19–24. In addition, we can obtain \( \mu_2 = 1.3897, \beta_2 = -2.2318 < 0, T_2 = 8.4012 > 0 \). Thus, we can conclude that the Hopf bifurcation is supercritical, the bifurcating periodic solutions are stable, and the period of the periodic solutions increases according to Theorem 5.

### 5. Critical Analysis and Research Perspectives

In this paper, an SIQR computer virus model with two delays is investigated. Compared with the system considered in [14],
Figure 1: The trajectory of $S, I, Q,$ and $R$ when $\tau_1 = 1.3025 < 1.3265 = \tau_{10}$.

Figure 2: The phase plot of $S, I,$ and $Q$ when $\tau_1 = 1.3025 < 1.3265 = \tau_{10}$.

Figure 3: The phase plot of $S, Q,$ and $R$ when $\tau_1 = 1.3025 < 1.3265 = \tau_{10}$.
Figure 4: The trajectory of $S$, $I$, $Q$, and $R$ when $\tau_1 = 1.3625 > 1.3265 = \tau_{10}$.

Figure 5: The phase plot of $S$, $I$, and $Q$ when $\tau_1 = 1.3625 > 1.3265 = \tau_{10}$.

Figure 6: The phase plot of $S$, $Q$, and $R$ when $\tau_1 = 1.3625 > 1.3265 = \tau_{10}$. 
Figure 7: The trajectory of $S(t)$, $I(t)$, $Q(t)$, and $R(t)$ when $\tau_2 = 1.65 < 1.7852 = \tau_{20}$.

Figure 8: The phase plot of $S(t)$, $I(t)$, and $Q(t)$ when $\tau_2 = 1.65 < 1.7852 = \tau_{20}$.

Figure 9: The phase plot of $S(t)$, $Q(t)$, and $R(t)$ when $\tau_2 = 1.65 < 1.7852 = \tau_{20}$.
Figure 10: The trajectory of $S$, $I$, $Q$, and $R$ when $\tau_2 = 1.8625 > 1.7852 = \tau_{20}$.

Figure 11: The phase plot of $S$, $I$, and $Q$ when $\tau_2 = 1.8625 > 1.7852 = \tau_{20}$.

Figure 12: The phase plot of $S$, $Q$, and $R$ when $\tau_2 = 1.8625 > 1.7852 = \tau_{20}$. 
Figure 13: The trajectory of $S(t)$, $I(t)$, $Q(t)$, and $R(t)$ when $\tau_1 = 0.785 < 0.8903 = \tau_1^{\prime}$ and $\tau_2 = 1.05$.

Figure 14: The phase plot of $S$, $I$, and $Q$ when $\tau_1 = 0.785 < 0.8903 = \tau_1^{\prime}$ and $\tau_2 = 1.05$.

Figure 15: The phase plot of $S$, $Q$, and $R$ when $\tau_1 = 0.785 < 0.8903 = \tau_1^{\prime}$ and $\tau_2 = 1.05$. 
Figure 16: The trajectory of $S(t)$, $I(t)$, $Q(t)$, and $R(t)$ when $\tau_1 = 0.9755 > 0.8903 = \tau_{10}'$ and $\tau_2 = 1.05$.

Figure 17: The phase plot of $S(t)$, $I(t)$, and $Q(t)$ when $\tau_1 = 0.9755 > 0.8903 = \tau_{10}'$ and $\tau_2 = 1.05$.

Figure 18: The phase plot of $S(t)$, $Q(t)$, and $R(t)$ when $\tau_2 = 0.9755 > 0.8903 = \tau_{20}'$ and $\tau_2 = 1.05$. 
Figure 19: The trajectory of $S(t)$, $I(t)$, $Q(t)$, and $R(t)$ when $\tau_2 = 1.235 < 1.3485 = \tau_1^{20}$ and $r_1 = 0.65$.

Figure 20: The phase plot of $S(t)$, $I(t)$, and $Q(t)$ when $\tau_2 = 1.235 < 1.3485 = \tau_1^{20}$ and $r_1 = 0.65$.

Figure 21: The phase plot of $S(t)$, $Q(t)$, and $R(t)$ when $\tau_2 = 1.235 < 1.3485 = \tau_1^{20}$ and $r_1 = 0.65$. 
Figure 22: The trajectory of $S$, $I$, $Q$, and $R$ when $\tau_2 = 1.52 > 1.3485 = \tau_{20}$ and $\tau_1 = 0.65$.

Figure 23: The phase plot of $S$, $I$, and $Q$ when $\tau_2 = 1.52 > 1.3485 = \tau_{20}$ and $\tau_1 = 0.65$.

Figure 24: The phase plot of $S$, $Q$, and $R$ when $\tau_2 = 1.52 > 1.3485 = \tau_{20}$ and $\tau_1 = 0.65$. 
the system in this paper is more general because it accounts for not only the time delay due to the latent period of the computer virus but also the time delay due to the period that the antivirus software uses to clean the computer viruses in the infectious and the quarantined nodes. The sufficient conditions for the stability of the positive equilibrium and existence of the Hopf bifurcation for the possible combinations of two delays are obtained. When the conditions are satisfied, then there exists a critical value of the delay below which system (2) is locally asymptotically stable and above which system (2) is unstable. The direction and the stability of the bifurcating periodic solutions are determined by applying the normal theory and the center manifold theorem. Numerical simulations show that the computer viruses may be controlled by shortening the time delay due to the latent period of the computer viruses and the time delay due to the period that the antivirus software uses to clean the computer viruses in the infectious and the quarantined nodes.

However, it should be pointed out that we suppose that the recovered computers have a permanent immunization period and they can no longer be infected. This is not consistent with real situation. In order to overcome this limitation and considering that the recovered computers may be infected again after a temporary immunity period, it is definitely an interesting work to investigate the following more general SIQRS model with multiple delays:

\[
\begin{align*}
\frac{dS(t)}{dt} = &\ (1 - p)b - \beta S(t - \tau_1)I(t - \tau_1) - dS(t) \\
+ &\ \sigma R(t - \tau_3), \\
\frac{dI(t)}{dt} = &\ \beta S(t - \tau_1)I(t - \tau_1) - (\delta + d + \alpha_1)I(t) \\
- &\ \gamma I(t - \tau_2), \\
\frac{dQ(t)}{dt} = &\ \delta I(t) - (d + \alpha_2)Q(t) - \epsilon Q(t - \tau_2), \\
\frac{dR(t)}{dt} = &\ \gamma I(t - \tau_2) + pb + \epsilon Q(t - \tau_2) - dR(t) \\
- &\ \sigma R(t - \tau_3),
\end{align*}
\]  

where \( \sigma \) is the transition rate from the recovered computers to the susceptible computers. \( \tau_3 \) is the temporary immunity period after which a recovered computer may be infected again. We leave the analysis of the more complicated bifurcations of system (67) as the future work.

Further research directions include the possibility of linking the results obtained with the model proposed in the present paper with the results coming from the networks theory; see the recent review paper [17]. Specifically, the interest focuses on the possibility to gain a deep understanding of the impact of the network topology on the viral prevalence; see [18, 19]. In this context, the model proposed in this paper, which is a coarse-grained compartment-level model, can be considered as a network where all nodes are classified into a few compartments according to their states and whose degree distribution takes into account the rule of interactions proposed in the present paper.

### Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

### Acknowledgments

This work was supported by Natural Science Foundation of the Higher Education Institutions of Anhui Province (KJ2015A144).

### References


