Research Article

Unilateral Global Bifurcation from Intervals for Fourth-Order Problems and Its Applications

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We establish a unilateral global bifurcation result from interval for a class of fourth-order problems with nondifferentiable nonlinearity. By applying the above result, we firstly establish the spectrum for a class of half-linear fourth-order eigenvalue problems. Moreover, we also investigate the existence of nodal solutions for the following half-linear fourth-order problems:

\[ x^{iv} = \alpha x^+ + \beta x^- + r(t)f(x), \quad 0 < t < 1, \]

\[ x(0) = x(1) = x''(0) = x''(1) = 0, \]

where \( r \neq 0 \) is a parameter, \( a \in C([0,1],(0,\infty)) \), \( x^+ = \max\{x,0\} \), \( x^- = -\min\{x,0\} \), \( \alpha, \beta \in C[0,1] \), and \( f \in C(\mathbb{R}, \mathbb{R}), sf(s) > 0 \), for \( s \neq 0 \). We give the intervals for the parameter \( r \) which ensure the existence of nodal solutions for the above fourth-order half-linear problems if \( f_0 \in [0,\infty) \) or \( f_\infty \in [0,\infty) \), where \( f_0 = \lim_{|s|\to 0} f(s)/s \) and \( f_\infty = \lim_{|s|\to \infty} f(s)/s \). We use the unilateral global bifurcation techniques and the approximation of connected components to prove our main results.

1. Introduction

In the past twenty years, fourth-order BVP have attracted the attention of many specialists in differential equations because of their interesting applications. For example, Bai and Wang [1], Ma and Wang [2], and Chu and O’Regan [3] have investigated the fourth-order BVP by the fixed point theory in cones. Meanwhile, by applying the bifurcation techniques of Rabinowitz [4, 5], Gupta and Mawhin [6], Lazer and McKenna [7], Rynne [8], Liu and O’Regan [9], Ma et. al [10, 11], Shen [12, 13], and Ma [14] studied the existence of nodal solutions for the fourth-order BVP.

Now, consider the following operator equation:

\[ u = \lambda Bu + H(\lambda, u), \quad (1) \]

where \( B \) is a compact linear operator and \( H : \mathbb{R} \times E \to E \) is compact with \( H = O(|u|) \) at \( u = 0 \) uniformly on bounded \( \lambda \) intervals, where \( E \) is a real Banach space with the norm \( \| \cdot \| \).

If the characteristic value \( \mu \) of \( B \) has multiplicity 1 and \( \mathcal{S} = \{ (\lambda, u) : (\lambda, u) \text{ satisfies (1), } u \neq 0 \} \cap \mathbb{R} \times E \), then Dancer [15] has shown that there are two distinct unbounded continua \( C^+_{\mu} \) and \( C^-_{\mu} \), consisting of the bifurcation branch \( C_{\mu} \) of \( \mathcal{S} \) emanating from \((\mu, 0)\), where either \( C^+_{\mu} \) and \( C^-_{\mu} \) are both unbounded or \( C^+_{\mu} \cap C^-_{\mu} \neq \{ (\mu, 0) \} \). This result has been extended to the fourth-order problems by Dai and Han [16]. More specifically, Dai and Han [16] considered the following fourth-order problem:

\[ x^{iv} = \lambda a(t)x + g(t,x,\lambda), \quad 0 < t < 1, \]

\[ x(0) = x(1) = x''(0) = x''(1) = 0, \quad (3) \]

where \( g : [0,1] \times \mathbb{R}^2 \to \mathbb{R} \) is continuous satisfying \( g(t,x,0) \equiv 0 \), and

\[ \lim_{x \to 0} \frac{g(t,x,\lambda)}{|x|} = 0 \quad (4) \]

uniformly on \([0,1]\) and \( \mu \) on bounded sets.

Let \( Y = C[0,1] \) with the norm \( \| x \|_{\infty} = \max_{t \in [0,1]} |x(t)| \). Let \( E = \{ x \in C^3[0,1] : x(0) = x(1) = x''(0) = x''(1) = 0 \} \) with the norm \( \| x \| = \| x \|_{\infty} + \| x' \|_{\infty} + \| x'' \|_{\infty} + \| x''' \|_{\infty} \). Let
E = R x E under the product topology. Let $S_k^+$ denote the set of functions in E which have exactly $k - 1$ generalized simple zeros in $(0, 1)$ and are positive near $t = 0$, set $S_k^- = -S_k^+$, and $S_k = S_k^+ \cup S_k^-$. They are disjoint and open in E. Finally, let $\Phi_k^+ = \mathbb{R} x S_k^+$ and $\Phi_k = \mathbb{R} x S_k$. Let C denote the closure of the set of nontrivial solutions of (3) in $\mathbb{R} x E$, and $C_k^+$ denote the subset of C with $u \in S_k^+$ for $\nu \in \{+,-\}$ and $C_k^- = C_k^+ \cup C_k^-$.

From Dancer [15], Dai and Han [16] obtained that problem (3) has two distinct unbounded subcontinua $C_k^+$ and $C_k^-$, consisting of the bifurcation branch $C_k$ emanating from $(\lambda_k,0)$, which satisfy the following.

**Lemma 1.** Either $C_k^+$ and $C_k^-$ are both unbounded or $C_k^+ \cap C_k^- \neq \{(\lambda_k,0]\}$, and

$$C_k^+ \subset \left( \left( \mathbb{R} \times S_k^+ \right) \cup \{(\lambda_k,0)\} \right),$$

where $\nu \in \{+,-\}$ and $\lambda_k$ were defined as in [10] or [16].

For the abstract unilateral global bifurcation theory, we refer the reader to [4, 5, 15, 17–19] and the references therein.

However, among the above papers, the nonlinearities are linear in the zeros and infinity. The problems involving nondifferentiable nonlinearities have also been investigated by using bifurcation techniques; see Berestycki [20], Schmitt and Smith [19], Rynne [21], Ma and Dai [22], and Dai et al. [23–25] and references therein. Among them, in 1977, Berestycki [20] studied the differential equations involving nondifferentiable nonlinearity. The above Berestycki’s ([20]) result has been improved partially by Schmitt and Smith [19] by applying a set-valued version of Rabinowitz global bifurcation theorem. In 1998, Rynne [21] established the interval bifurcation from $u = 0$ and $u = \infty$ and obtained sets of positive or negative solutions with the approximation technique from Berestycki [20]. Recently, Ma and Dai [22] established the global bifurcation for a Sturm-Liouville problem with a nonsmooth nonlinearity by [15]. Later, Dai et al. [23–25] studied the bifurcation from intervals for Sturm-Liouville problems and its applications and established the unilateral global interval bifurcation for p-Laplacian with non-$p - 1$-linearization nonlinearity, respectively.

On the other hand, half-linear or half-quasilinear problems have attracted the attention of some specialists; see [20, 22, 25]. Among them, Berestycki [20] studied the bifurcation structure for the half-linear equations. Recently, Ma and Dai [22] and Dai and Ma [25] studied the existence of nodal solutions for a class of half-linear or half-quasilinear eigenvalue problems and improved Berestycki’s result, respectively.

Motivated by the above papers, in this paper, we will firstly establish some Dancer-type unilateral global bifurcation results about the continuum of solutions for the fourth-order problems:

$$x^{(4)} = \lambda a(t)x + F(t,x,\lambda), \quad 0 < t < 1,$$
$$x(0) = x(1) = x''(0) = x''(1) = 0,$$

where $\lambda$ is a parameter, the nonlinear term $F$ has the form $F = f + g$, where $f$ and $g$ are continuous functions on $[0, 1] x \mathbb{R}^3$, and $a, f, g$ satisfy the following conditions:

$$\begin{align*}
&(H1) \ a \in C([0, 1], (0, \infty)). \\
&(H2) \ |f(t,x,\lambda)/x| \leq M_1, \text{ for all } t \in [0, 1], 0 < |x| \leq 1, \forall \lambda \in \mathbb{R}, \text{ where } M_1 \text{ is a positive constant.} \\
&(H3) \ g(t,x,\lambda) = o(|x|) \text{ near } x = 0, \text{ uniformly in } t \in [0, 1] \text{ and } \lambda \text{ on bounded sets.}
\end{align*}$$

Let $S$ denote the closure of the set of nontrivial solutions of (6) in $\mathbb{R} x E$, and $S_k^+$ denote the subset of $S$ with $u \in S_k^+$ for $\nu \in \{+,-\}$ and $S_k^- = S_k^+ \cup S_k^-$.

Under assumptions (H1)–(H3), we will show that $I_k = [\lambda_k - d_1, \lambda_k + d_1]$ is a bifurcation interval of problem (6) and there are two distinct unbounded subcontinua, $S_k^+$ and $S_k^-$, consisting of the bifurcation branch $S_k$ from $I_k = [\lambda_k - d_1, \lambda_k + d_1]$, where $\lambda_k$ is given in Lemma 1.

On the basis of the above unilateral global interval bifurcation result, we study the following half-linear eigenvalue problem:

$$x^{(4)} = \lambda a(t)x + \alpha x^+ + \beta x^-, \quad 0 < t < 1,$$
$$x(0) = x(1) = x''(0) = x''(1) = 0,$$

where $x^+ = \max\{x,0\}$, $x^- = \min\{x,0\}$ and $a(t)$ and $\beta(t)$ satisfy the following:

$$\begin{align*}
&(H4) \ a(t), \beta(t) \in C[0, 1].
\end{align*}$$

We will show that there exist two sequences of simple half-eigenvalues for problem (7): $\lambda_1^+ < \lambda_2^+ < \cdots < \lambda_k^+ < \cdots$ and $\lambda_1^- < \lambda_2^- < \cdots < \lambda_k^- < \cdots$. The corresponding half-linear solutions are in $\{\lambda_k^+\} \times S_k^+$ and $\{\lambda_k^-\} \times S_k^-$. Furthermore, aside from these solutions and the trivial ones, there is no other solutions of problem (7).

Following the above eigenvalue theory (see Theorem 19), we will investigate the existence of nodal solutions for the following fourth-order problem:

$$x^{(4)} = \alpha x^+ + \beta x^- + \lambda a(t) f (x), \quad 0 < t < 1,$$
$$x(0) = x(1) = x''(0) = x''(1) = 0.$$

Let $\Sigma$ denote the closure of the set of nontrivial solutions of (8) in $\mathbb{R} x E$, with $\Sigma_k^+$ denoting the subset of $\Sigma$ with $u \in S_k^+$ for $\nu \in \{+,-\}$ and $\Sigma_k^- = \Sigma_k^+ \cup \Sigma_k^-$. For $\alpha = \beta = 0$, Dai and Han [16] have established the existence of nodal solutions for problem (8) with crossing nonlinearity. In this paper, we assume that $f$ satisfies the following assumptions:

$$\begin{align*}
&(H5) \ sf(s) > 0 \text{ for } s \neq 0. \\
&(H6) \ f_0 \in (0, \infty) \text{ and } f_{\infty} \in (0, \infty). \\
&(H7) \ f_0 \in (0, \infty) \text{ and } f_{\infty} = \infty. \\
&(H8) \ f_0 \in (0, \infty) \text{ and } f_{\infty} = 0. \\
&(H9) \ f_0 = 0 \text{ and } f_{\infty} = \infty. \\
&(H10) \ f_0 = 0 \text{ and } f_{\infty} = 0. \\
&(H11) \ f_0 = 0 \text{ and } f_{\infty} = 0.
\end{align*}$$
where
\[
\lim_{|x| \to 0} \frac{f(x)}{x} = f_0, \quad \forall t \in [0, 1],
\]
\[
\lim_{|x| \to \infty} \frac{f(x)}{x} = f_\infty, \quad \forall t \in [0, 1].
\]

The rest of this paper is arranged as follows. In Section 2, we have given some preliminaries. In Section 3, we establish the unilateral global bifurcation result from the interval for problem (6). In Section 4, on the basis of the unilateral global interval bifurcation result, we will establish the spectrum for a class of the half-linear fourth-order eigenvalue problem (7) (see Theorem 19). In Section 5, following the above eigenvalue theory (see Theorem 19), we will investigate the existence of nodal solutions for a class of the half-linear fourth-order problem (8).

### 2. Hypotheses and Lemmas

We define the linear operator \( L : D(L) \subset E \to Y \):
\[
Lx = x'''(t), \quad x \in D(L),
\]
with \( D(L) = \{ x \in C^4[0, 1] \mid x(0) = x(1) = x''(0) = x''(1) = 0 \} \).

From [26, p. 439-440], we consider the following auxiliary problem:
\[
x'''(t) = h(t), \quad t \in (0, 1),
\]
\[
x(0) = x(1) = x''(0) = x''(1) = 0,
\]
for a given \( h \in C[0, 1] \). We can get that problem (11) can be equivalently written as
\[
x(t) = L^{-1}(h)(t) = \int_0^1 k(t, s) h(s) \, ds,
\]
where
\[
k(t, s) = \begin{cases} 
\frac{1}{6} s (1-t) \left(2t-s^2-t^2\right), & s \leq t, \\
\frac{1}{6} t (1-s) \left(2s-t^2-s^2\right), & s > t.
\end{cases}
\]

Then \( L \) is a closed operator and \( L^{-1} : Y \to E \) is completely continuous.

Define the Nemytskii operator \( H : \mathbb{R} \times E \to E \) by
\[
H(\lambda, x)(t) = \lambda L^{-1} [ax] + L^{-1} [g(t, x, \lambda)].
\]

Then it is clear that \( H \) is continuous operator and problem (3) can be equivalently written as
\[
x = H(\lambda, x).
\]

Clearly, \( H : \mathbb{R} \times E \to E \) is completely continuous and \( H(\mu, 0) = 0, \forall \mu \in \mathbb{R} \).

Let
\[
\overline{g}(t, x, \lambda) = \max_{0 \leq s \leq 1} |g(t, s, \lambda)|
\]
\[
\forall t \in [0, 1], \lambda \text{ on bounded sets},
\]
and then \( \overline{g} \) is nondecreasing with respect to \( x \), and
\[
\lim_{x \to 0} \frac{\overline{g}(t, x, \lambda)}{x} = 0
\]
uniformly for \( t \in [0, 1] \) and \( \lambda \) on bounded sets. Further it follows from (17) that
\[
\frac{g(t, x, \lambda)}{\|x\|} \leq \frac{\overline{g}(t, |x|, \lambda)}{\|x\|} \leq \frac{\overline{g}(t, \|x\|, \lambda)}{\|x\|} \to 0 \quad \text{as} \quad \|x\| \to 0
\]
uniformly for \( t \in [0, 1] \) and \( \lambda \) on bounded sets.

In the following, we summarize some preliminary results from [10, 16].

**Definition 2** (see [16]). Let \( u \in E \) and \( t_* \in I \) such that \( u(t_*) = u''(t_*) = 0 \). We call that \( t_* \) a generalized simple zero if \( u'(t_*) \neq 0 \) or \( u'''(t_*) \neq 0 \). Otherwise, we call that \( t_* \) a generalized double zero. If there is no generalized double zero of \( u \), we call that \( u \) a nodal solution.

**Lemma 3** (see [10] or [16]). Let (H1) hold. The linear eigenvalue problem
\[
x'''(t) = \lambda a(t) x, \quad 0 < t < 1,
\]
\[
x(0) = x(1) = x''(0) = x''(1) = 0
\]
has an infinite sequence of positive eigenvalues:
\[
0 < \lambda_1(a) < \lambda_2(a) < \cdots < \lambda_k(a) \to +\infty \quad (k \to +\infty).
\]

Moreover, each eigenvalue is simple. To each eigenvalue \( \lambda_k(a) \) there corresponds an essential unique eigenfunction \( \psi_k \) which has exactly \( k-1 \) generalized simple zeros in \((0, 1)\) and is positive near 0.

**Lemma 4** (see [16]). If \((\lambda, x)\) is a nontrivial solution of (6) under assumptions (H2) and (H3) and \( x \) has a generalized double zero, then \( x \equiv 0 \).

**Remark 5.** By Lemma 4, we can see that if \((\lambda, x)\) is a nontrivial solution of (6) under assumptions (H2) and (H3), then \( x \in \bigcup_{k=1}^{\infty} \mathcal{S}_k \).

**Lemma 6** (see [20, Lemma 2]). Let \( j \) and \( k \) be two integers such that \( j \geq k \geq 2 \). Suppose that there exist two families of real numbers:
\[
\xi_0 = 0 < \xi_1 < \xi_2 < \cdots < \xi_{k-1} < \xi_k = 1,
\]
\[
\eta_0 = 0 < \eta_1 < \eta_2 < \cdots < \eta_{j-1} < \eta_j = 1.
\]
Then, if \( \xi_1 \leq \eta_1 \), there exist two integers \( p \) and \( q \) having the same parity: \( 1 \leq p \leq k - 1 \) and \( 1 \leq q \leq j - 1 \), such that
\[
\xi_p \leq \eta_q \leq \eta_{q+1} \leq \xi_{p+1}.
\]

Definition 7 (see [27]). Let \( X \) be a Banach space and let \( |C_n| \) be a family of subsets of \( X \). Then the superior limit \( D \) of \( \{C_n\} \) is defined by
\[
D := \limsup_{n \to \infty} C_n = \{ x \in X \mid \exists \{n_i\} \subseteq \mathbb{N}, \ x_{n_i} \in C_{n_i} \text{ such that } x_{n_i} \to x \}.
\]

Lemma 8 (see [27]). Each connected subset of metric space \( X \) is contained in a component, and each connected component of \( X \) is closed.

Lemma 9 (see [28]). Let \( X \) be a Banach space and let \( |C_n| \) be a family of closed connected subsets of \( X \). Assume that
\begin{enumerate}[(i)]
  \item there exist \( z_n \in C_n, n = 1, 2, \ldots, \) and \( z^* \in X \), such that \( z_n \to z^* \);
  \item \( r_n = \sup \{\|x\| \mid x \in C_n\} = \infty \);
  \item for all \( R > 0 \), \( \bigcup_{n \in \mathbb{N}} C_n \cap B_R \) is a relative compact set of \( X \), where
\end{enumerate}
\[
B_R = \{ x \in X \mid \|x\| \leq R \}.
\]

Then there exists an unbounded component \( C \) in \( D \) and \( z^* \in C \).

3. Unilateral Global Bifurcation

The main result for problem (6) is the following theorem.

Theorem 10. Let \( (H1), (H2), \) and \( (H3) \) hold. Let \( d_1 = M_1/\alpha_0 \), where \( \alpha_0 = \min_{k \in \mathbb{N}} \alpha(k) \), and let \( k_k = [\lambda_k - d_1, \lambda_k + d_1] \) for every \( k \in \mathbb{N} \). The component \( C_{k}^+ \) of \( \mathcal{H}_k \cup (I_k \times \{0\}) \), containing \( I_k \times \{0\} \), is unbounded and lies in \( \Phi_k \cup (I_k \times \{0\}) \) and the component \( C_{k}^- \) of \( \mathcal{H}_k \cup (I_k \times \{0\}) \), containing \( I_k \times \{0\} \), is unbounded and lies in \( \Phi_k \cup (I_k \times \{0\}) \).

In order to prove Theorem 10, we need the following results.

Lemma 11. If \( \mathcal{H}_k^+ \) is bounded, we can find a neighborhood \( \Theta \) of \( C_k^+ \) such that \( \partial \Theta \cap C_k^+ = \emptyset \), where \( C_k^+(v = +, -) \) is given by Theorem 10.

Proof. We only prove the case of \( C_k^+ \) since the case of \( C_k^- \) is similar.

Let \( U \) be a uniform neighborhood of \( C_k^+ \) in \( \mathbb{R} \times E \).
We discuss two cases.

Case 1 (if \( \partial U \cap C_k^+ \neq \emptyset \)). Since the solutions of problem (6) are bounded in \( \mathbb{R} \times E \), then \( C_k^+ \) is compact in \( \mathbb{R} \times E \). It follows that \( \mathcal{H} \cap S_k^+ \) is compact metric space. Obviously, \( C_k^+ \) and \( \partial U \cap C_k^+ \) are two disjoint closed subsets of \( \mathcal{H} \cap S_k^+ \). Because of the maximal connectedness of \( C_k^+ \), there does not exist a component \( S^+ \) of \( \mathcal{H} \cap S_k^+ \) such that \( C_k^+ \cap (\partial U \cap S_k^+) \neq \emptyset \). By [27] or [4, Lemma 1.1.1], there exist two disjoint compact subsets \( K_1, K_2 \) of \( \mathcal{H} \cap S_k^+ \), such that \( \mathcal{H} \cap S_k^+ = K_1 \cup K_2 \), \( C_k^+ \subset K_1 \), \( \partial U \cap S_k^+ \subset K_2 \). Evidently, \( d(K_1, K_2) > 0 \). Let \( r = \min(d(K_1, K_2), d(K_1, \partial U)) \), and let \( U' \) be the r/2-neighborhood of \( K_1 \).

By Lemma 1, there are two unbounded continua \( C_k^+ \) and \( C_k^- \) of \( S_k^+ \) bifurcating from \( (\lambda_k, 0) \), consisting of the bifurcation branch \( C_k^+ \), which satisfy the following result.

Lemma 12. \( C_k^+ \) and \( C_k^- \) are both unbounded and
\[
\mathcal{H}_k^+ \subset \Phi_k \cup \{(\lambda_k, 0)\} \quad (v = +, -).
\]

Proof. By Lemma 1.24 of [4], there exists a bounded open neighborhood \( O_k \) of \( (\lambda_k, 0) \) such that
\[
\left( C_k^+ \cap \{(\lambda_k, 0)\} \right) \cap O_k \subset \Phi_k^+ \quad (v = +, -).
\]

It follows that
\[
\mathcal{H}_k^+ \cap O_k \subset \Phi_k^+ \cup \{(\lambda_k, 0)\} \quad (v = +, -).
\]

By Lemma 4, we show that \( C_{k, \epsilon} \), \( \lambda_k \), \( (\lambda_k, 0) \) cannot leave \( \Phi_k^+ \) outside of a neighborhood of \( (\lambda_k, 0) \). Thus, we have
\[
\mathcal{H}_k^+ \subset \Phi_k^+ \cup \{(\lambda_k, 0)\} \quad (v = +, -).
\]
Next, we will prove that both \( C_{k, \epsilon} \) and \( C_{k, \epsilon}^- \) are unbounded.

Without loss of generality, we may suppose that \( C_{k, \epsilon} \) is bounded. Therefore, in view of (32) and Lemma 1, there exists \( (\lambda_*, u_*) \in C_{k, \epsilon} \cap C_{k, \epsilon}^- \) such that \( (\lambda_*, u_*) \neq (\lambda_k, 0) \) and \( u_* \in S_k^+ \cap S_k^- \). This contradicts the definitions of \( S_k^+ \) and \( S_k^- \).

To prove Theorem 10, the next lemma will play a key role.
Lemma 13. Let $\varepsilon_n, 0 < \varepsilon_n < 1$, be a sequence converging to 0. If there exists a sequence $(\lambda_n, x_n) \in \mathbb{R} \times S_k^n$ such that $(\lambda_n, x_n)$ is a nontrivial solution of problem (27) corresponding to $\varepsilon = \varepsilon_n$, and $(\lambda_n, x_n)$ converges to $(\lambda, 0)$ in $\mathbb{R} \times E$, then $\lambda \in I_k$.

Proof. Without loss of generality, we may assume that $\|x_n\| \leq 1$. Let $y_n = x_n/\|x_n\|$; then $y_n$ satisfies the problem

$$
\begin{align*}
y'''_n &= \lambda_n a(t) y_n + f_n(t) + g_n(t), & 0 < t < 1, \\
y(0) &= y'(0) = y''(0) = 0,
\end{align*}
$$

where

$$
\begin{align*}
f_n(t) &= \frac{f(t, x_n, \|x_n\|, \lambda_n)}{\|x_n\|}, \\
g_n(t) &= \frac{g(t, x_n, \lambda_n)}{\|x_n\|}.
\end{align*}
$$

By (18), it follows that

$$
\lim_{n \to \infty} g_n(t) = 0,
$$

uniformly for $t \in [0, 1]$ and $\lambda$ on bounded sets. Furthermore, (H2) implies that

$$
\left| f_n(t) \right| = \left| \frac{f(t, x_n, \|x_n\|, \lambda_n)}{\|x_n\|} \right| \leq \frac{M_1}{\|x_n\|} \leq M_1
$$

for all $t \in [0, 1]$, $\lambda \in I_k$.

Note that $\|y_n\| = 1$ implies $\|y_n\|_{\infty} \leq 1$. Using this fact with (35) and (36), we have that $\lambda_n a(t) y_n + f_n(t) + g_n(t)$ is bounded in $C^4[0, 1]$ for $n$ large enough. The compactness of $L^1$ implies that $y_n$ is convergence in $C^4[0, 1]$. Without loss of generality, we may assume that $y_n \to y$ in $E$ with $\|y\| = 1$. Clearly, we have $y \in S_k^n$.

We claim that $y \in S_k^n$. On the contrary, supposing that $y \in S_k^r$, by Lemma 4, then $y \equiv 0$, which is a contradiction with $\|y\| = 1$.

Now, we deduce the boundedness of $\lambda$. Let $\varphi_k^r \in S_k^r$ be an eigenfunction of problem (19) corresponding to $\lambda_k$.

Let

$$
\begin{align*}
t_0 &= 0 < t_1 < t_2 < \cdots < t_k = 1, \\
s_0 &= 0 < s_1 < s_2 < \cdots < s_{k-1} < s_k = 1
\end{align*}
$$

be, respectively, the sequences of generalized simple zeros of $y_n$ and $\varphi_k^r$.

Suppose $t_1 \leq s_1$; then we deduce from Lemma 6 existence of integers $p$ and $q$ having the same parity such that

$$
t_p \leq s_q \leq t_{p+1}.
$$

Therefore, without loss of generality, we choose $\xi_1 = t_0$, $\eta_1 = t_1$, $\xi_2 = s_q$, and $\eta_2 = s_{q+1}$ since $p$ and $q$ have the same parity $y_n$ and $\varphi_k^r$ do not vanish and have the same sign in both the intervals $[\xi_1, \eta_1]$ and $[\xi_2, \eta_2]$. It follows that $y_n(\xi_1) = y_n(\eta_1) = y''(\xi_1) = y''(\eta_1) = 0$, and $\varphi_k^r(\xi_2) = \varphi_k^r(\eta_2) = (\varphi_k^r)'(\xi_2) = (\varphi_k^r)'(\eta_2) = 0$.

We can assume without loss of generality that $y_n > 0$ and $\varphi_k^r > 0$ in $(\xi_1, \eta_1)$. By the Picone identity in [29, Theorem 4], we have that

$$
\begin{align*}
\int_{\xi_1}^{\eta_1} \left[ \frac{y_n}{\varphi_k^r} \left( y_n'(\varphi_k^r)' - \varphi_k^r y_n'' \right) + \frac{y_n'}{\varphi_k} \left( \varphi_k y_n' - y_n(\varphi_k)' \right) - y_n(\varphi_k)'' \left( \frac{y_n}{\varphi_k} \right)' \right] dt &= A_1 + B_1, \\
A_1 &= \int_{\xi_1}^{\eta_1} \left[ y_n - y_n(\varphi_k)' \left( \frac{y_n}{\varphi_k} \right)' \right] dt, \\
B_1 &= \int_{\xi_1}^{\eta_1} \left[ y_n'' - 2\varphi_k''(\varphi_k)'^2 \left( \frac{y_n}{\varphi_k} \right)'^2 \right] dt.
\end{align*}
$$

The left-hand side of (39) equals

$$
\begin{align*}
&- \lim_{t \to -\xi_1} \left[ \frac{y_n}{\varphi_k} \left( y_n(\varphi_k)' - \varphi_k y_n' \right) \right] \\
&+ \frac{y_n'}{\varphi_k} \left[ \varphi_k y_n' - y_n(\varphi_k)' \right] - y_n(\varphi_k)'' \left( \frac{y_n}{\varphi_k} \right)' \\
&+ \lim_{t \to -\eta_1} \left[ \frac{y_n}{\varphi_k} \left( y_n(\varphi_k)' - \varphi_k y_n' \right) \right]
\end{align*}
$$

and $H_{\xi_1} + H_{\eta_1}$.

We prove that $H_{\xi_1} \leq 0$. If $\varphi_k^r(\xi_1) \neq 0$, then $H_{\xi_1} = 0$. If $\varphi_k^r(\xi_1) = 0$, noting the conclusion of Lemma 3, then $(\varphi_k^r)'(\xi_1) > 0$. By L'Hospital rule, we have that
\[ H_{\xi_i} = -\lim_{t \to \xi_i^-} y'_n \left( y_n (\varphi_k^\nu)''' - \varphi_k^\nu y''' ight) + y_n \left( y_n (\varphi_k^\nu)' - \varphi_k^\nu y' \right)'
\]
\[ + \lim_{t \to \xi_i^-} y''_n \left( \varphi_k^\nu y'' - y_n (\varphi_k^\nu)'' \right) + \lim_{t \to \xi_i^-} y''_n \left( \varphi_k^\nu y'' + \varphi_k^\nu y''' - y_n (\varphi_k^\nu)'' - y_n (\varphi_k^\nu)''' \right)
\]
\[ + (\varphi_k^\nu)''(\xi_i) \lim_{t \to \xi_i^-} y''_n \left[ y_n (\varphi_k^\nu)' - y_n (\varphi_k^\nu)' \right] + y_n \left[ y''_n (\varphi_k^\nu) - y_n (\varphi_k^\nu)'' \right]
\]
\[ = \frac{\left[ y''_n(\xi_i) \right]^2 (\varphi_k^\nu)''(\xi_i)}{(\varphi_k^\nu)'(\xi_i)} + \frac{(\varphi_k^\nu)'(\xi_i)}{(2\varphi_k^\nu)'(\xi_i)} \lim_{t \to \xi_i^-} y''_n \left[ y_n (\varphi_k^\nu)' - y_n (\varphi_k^\nu)' \right] + 2 y''_n \left[ y''_n (\varphi_k^\nu) - y_n (\varphi_k^\nu)'' \right] + y_n \left[ y''_n (\varphi_k^\nu) - y_n (\varphi_k^\nu)'' \right]
\]
\[ = \frac{\left[ y''_n(\xi_i) \right]^2 (\varphi_k^\nu)''(\xi_i)}{(\varphi_k^\nu)'(\xi_i)}
\]

In the following, we will prove that
\[ (\varphi_k^\nu)''(t) \leq 0, \quad \forall t \in [\xi_1, \eta_1]. \quad \text{(42)} \]

Let \( u = \varphi_k^\nu(t) + 1 \). Then \( u''' = \lambda_k a \varphi_k^\nu \) and \( u \geq 1 \) in \( (\xi_1, \eta_1) \). For some \( \varepsilon > 0 \) small enough, let \( v \in C^1([-\varepsilon, 1 + \varepsilon]) \) and \( b \geq 0 \) be such that \( v(-\varepsilon) = v(1 + \varepsilon) = v''(-\varepsilon) = v''(1 + \varepsilon) = 0 \), \( v|_{[t]} = u \), and \( v''' = \lambda_k b v \). Then we have
\[ v'' = \lambda_k b v, \quad t \in (-\varepsilon, 1 + \varepsilon), \]
\[ v(-\varepsilon) = v(1 + \varepsilon) = v''(-\varepsilon) = v''(1 + \varepsilon) = 0. \quad \text{(43)} \]

Set \( m = (2\xi_1 - \varepsilon)/2 \) and \( n = (2\eta_1 + \varepsilon)/2 \). Let \( w \in C^1([m, n]) \) and \( c \geq 0 \) be such that \( w|_{[t]} = v, \) \( w \equiv 0 \) in \( (m, n) \) and \( w(m) = w(n) = w''(m) = w''(n) = 0 \) and \( w''' = \lambda_k c w \). Set \( z = u'' \); then \( z \) should be a solution of the problem
\[ \frac{d^2}{d t^2} z = \lambda_k c w, \quad t \in (m, n), \]
\[ z(m) = z(n) = 0. \quad \text{(44)} \]

The Strong Maximum Principle implies that \( z < 0 \) in \( (m, n) \). This follows that \( (\varphi_k^\nu)''(t) \leq 0 \) in \( [\xi_1, \eta_1] \). It follows that \( H_{\xi_i} \leq 0 \).

Next, we will prove that \( H_{\xi_i} \leq 0 \). If \( \varphi_k^\nu(\eta_i) \neq 0 \), then
\[ H_{\xi_i} = 0. \quad \text{If } \varphi_k^\nu(\eta_i) = 0, \text{ noting the conclusion of Lemma 3, then } (\varphi_k^\nu)'(\eta_i) < 0. \]

By L'Hospital rule, we have that
\[ H_{\xi_i} = -\frac{\left[ y_n'(\eta_i) \right]^2 (\varphi_k^\nu)''(\eta_i)}{(\varphi_k^\nu)'(\eta_i)}. \quad \text{(45)} \]

By (42), we can show that \( H_{\xi_i} \leq 0 \). Therefore, the left-hand side of (39) \( \leq 0 \).

By (42), we have \( B_1 \geq 0 \).
So, by (48), we have that

\[
\int_{t_1}^{t_2} (\lambda_k - \lambda) a(t) \phi_k^2 dt - M_1 \int_{t_1}^{t_2} \phi_k^2 dt \leq 0,
\]

and

\[
\int_{t_2}^{t_1} (\lambda_k - \lambda) a(t) (\phi_k^2) dt - M_1 \int_{t_2}^{t_1} (\phi_k^2) dt \leq 0.
\]

Furthermore, it follows that

\[
\text{if } \lambda \leq \lambda_k, \quad \int_{t_1}^{t_2} [(\lambda_k - \lambda) a_0 (t) - M_1] \phi_k^2 dt \leq 0,
\]

hence \( \lambda \geq \lambda_k - d_1 \),

\[
\text{if } \lambda \geq \lambda_k, \quad \int_{t_1}^{t_2} [(\lambda - \lambda_k) a_0 dt - M_1] (\phi_k^2) dt \leq 0,
\]

hence \( \lambda \leq \lambda_k + d_1 \).

Therefore, we have that \( \lambda \in I_k \).

**Lemma 14.** For \( \nu \in \{+, -\} \), the component \( C_k^\nu \) of \( S_k^* \cup (I_k \times \{0\}) \) satisfies \( C_k^\nu \subset \Phi_k^* \cup (I_k \times \{0\}) \), where \( C_k^\nu (\nu = +, -) \) is given by Theorem 10.

**Proof.** We only prove the case of \( C_k^+ \) since the case of \( C_k^- \) is similar. For any \((\lambda, x) \in C_k^+ \), there are two possibilities: (i) \( x \in S_k^+ \) or (ii) \( x \in \partial S_k^+ \). It is obvious that \((\lambda, x) \in \Phi_k^+ \) in the case of (i). While case (ii) implies that \( x \) has at least one double zero in \([0, 1]\), Lemma 4 follows that \( x = 0 \). Hence, there exists a sequence \((\lambda_n, x_n) \in \Phi_k^+ \) such that \((\lambda_n, x) \) is a solution of problem (27) corresponding to \( \epsilon = 0 \), and \((\lambda_n, x_n) \) converges to \((\lambda, 0) \) in \( \mathbb{R} \times \mathbb{R} \). By Lemma 13, we have \( \lambda \in I_k \); that is, \((\lambda, x) \) is \( I_k \times \{0\} \) in the case of (ii). Hence, \( C_k^+ \subset \Phi_k^* \cup (I_k \times \{0\}) \).

**Proof of Theorem 10.** We only prove the case of \( C_k^+ \) since the case of \( C_k^- \) is similar. Let \( C_k^+ \) be the component of \( S_k^* \cup (I_k \times \{0\}) \), containing \( I_k \times \{0\} \). By Lemma 14, we can show that \( C_k^+ \subset \Phi_k^* \cup (I_k \times \{0\}) \).

Suppose on the contrary that \( C_k^+ \) is bounded. By Lemma 11, we can find a neighborhood \( \mathcal{O} \) of \( \Phi_k^* \) such that \( \partial \mathcal{O} \cap S_k^* = \emptyset \).

In order to complete the proof of this theorem, we consider problem (27). By Lemma 1, there are two continua \( C_{k,e} \) and \( C_{k,e}^* \) consisting of the bifurcation branch \( \Lambda_k \times \mathbb{R} \). By Lemma 12, we have that \( C_{k,e}^+ \) and \( C_{k,e}^- \) are unbounded and \( C_{k,e}^+ \cap \partial \mathcal{O} \subset \Phi_k^* \cup \{(\lambda_k, 0)\} \).

So there exists \((\lambda, x) \in C_{k,e}^+ \cap \partial \mathcal{O} \) for all \( \epsilon > 0 \). Since \( \partial \mathcal{O} \) is bounded in \( \mathbb{R} \times \mathbb{R} \), (27) shows that \((\lambda, x) \) is bounded in \( \mathbb{R} \times \mathbb{C}^4 \) independently of \( \epsilon \). By the compactness of \( L^{-1} \), one can find a sequence \( \epsilon_n \to 0 \) such that \((\lambda_n, x) \) converges to a solution \((\lambda, x) \) of (6). So \( x \in S_k^O \). If \( x \in \partial S_k^O \), then from Lemma 4 or Remark 5 it follows that \( x \equiv 0 \). Note that \((\lambda, x) \in C_{k,e} \cap \partial \mathcal{O} \) since \( S_k^O \cap \partial \mathcal{O} \) is closed subset of \( \mathbb{R} \times \mathbb{R} \). By Lemma 13, \( \lambda \in I_k \), which contradicts the definition of \( \mathcal{O} \). On the other hand, if \( x \in S_k^O \), then \((\lambda, x) \in \Phi_k^+ \) which contradicts \( \Phi_k^+ \cap \partial \mathcal{O} = \emptyset \).

From Theorem 10 and its proof, we can easily get the following two corollaries.

**Corollary 15.** There exist two unbounded subcontinua \( X_k^0 \) and \( X_k^0 \) of solutions of (6) in \( \mathbb{R} \times \mathbb{R} \), bifurcating from \( I_k \times \{0\} \), and \( X_k^0 \subset (\mathbb{R} \times \mathbb{R}^S_k) \cup (I_k \times \{0\}) \) for \( \nu = + \) and \( \nu = - \).

We relax the assumption of \( \alpha \) as the following:

\[(A1) \quad \alpha(t) \in \mathbb{C} \] is a sign-changing weight.

**Corollary 16.** Let (A1) hold and \( f \equiv 0 \). There exist two unbounded subcontinua \( X_k^c \) and \( X_k^0 \) of solutions of (6) in \( \mathbb{R} \times \mathbb{R} \), bifurcating from \( (\lambda_k, 0) \), and \( X_k^c \subset (\mathbb{R} \times \mathbb{R}) \cup (I_k \times \{0\}) \) for \( \nu = + \) and \( \nu = - \).

4. Spectrum of Half-Quasi-Linear Problems

In this section, we consider the half-linear problem (7). Problem (7) is called half-linear because it is positive homogeneous in the cones \( u > 0 \) and \( u < 0 \). Similar to that of [20], we say that \( \lambda \) is a half-eigenvalue of problem (7) if there exists a nontrivial solution \((\lambda, u) \). \( \lambda \) is said to be simple if \( \nu = + \) for all solutions \((\lambda, u) \) of problem (7).

In order to prove our main results, we need the following Sturm type comparison result.

**Lemma 17.** Let \( b_i(t) \geq \max \{b_1(t), b_2(t) + \alpha(t) + \beta(t), b_2(t) - \alpha(t) - \beta(t)\} \) for \( t \in (0, 1) \) and \( b_i(t) \in \mathbb{C}(0, 1) \), \( i = 1, 2 \). Also let \( u_1, u_2 \) be solutions of the following differential equations:

\[
u^{(i)} = b_i(t) u + \alpha u + \beta u, \quad t \in (0, 1), \quad i = 1, 2,
\]

respectively. If \( (c, d) \subset (0, 1), u_1(c) = u_1'(b) = 0, \) and \( u_1'(c) = 0 \) or \( u_1'(d) = 0 \) and \( u_1(d) = 0, \) and \( u_1'(d) = 0, \) and \( u_1(c) \neq 0 \), then there exists \( \tau \in (c, d) \) such that \( u_2(t) = 0 \) or \( b_1(t) = b_2(t) = b_1 + b_2 = b_1 + b_2 + \alpha + \beta \) or \( b_1 = b_1 - \beta \) and \( u_2(t) = \mu u_1(t) \) for some constant \( \mu \neq 0 \).

**Proof.** We discuss four cases.

**Case 1** \( u_1(t) > 0, u_2(t) > 0 (in (c, d)), \) Then an easy calculation shows that

\[
\int_c^d (u_1^{(i)} u_2 - u_2^{(i)} u_1) dt = \int_c^d (b_1 - b_2) u_1 u_2 dt < 0.
\]

Similar to the step of the proof of Lemma 3.1 of [16], we can obtain the left-hand side of (55) \( \geq 0 \). This is a contradiction.
Case 2 \((u_1(t) > 0, u_2(t) < 0, t \in (c, d))\). Similar to (55), we can get
\[
\int_c^d \left( u_1''' u_2 - u_2''' u_1 \right) dt = \int_c^d \left( b_1 - b_2 + \alpha + \beta \right) u_1 u_2 dt > 0. \tag{56}
\]
Similar to the step of the proof of Lemma 3.1 of [16], we can obtain that the left-hand side of (56) \(\leq 0\). This is a contradiction.

Case 3 \((u_1(t) < 0, u_2(t) < 0, t \in (c, d))\). Similar to Case 1, we can get the result.

Case 4 \((u_1(t) < 0, u_2(t) > 0, t \in (c, d))\). We can get
\[
\int_c^d \left( u_1''' u_2 - u_2''' u_1 \right) dt = \int_c^d \left( b_1 - b_2 - \alpha - \beta \right) u_1 u_2 dt > 0. \tag{57}
\]
Similar to the proof of Lemma 3.1 of [16], we can obtain the left-hand side of (57) \(\leq 0\). This is a contradiction. \(\square\)

By Lemma 17, we obtain the following result that will be used later.

**Lemma 18.** Let \(I = [a, b]\) be such that \(I \subset [0, 1]\) and \(\text{meas}(I) > 0\). Let \(g_n \in C([0, 1], \mathbb{R})\) be such that
\[
\lim_{n \to +\infty} g_n(t) = +\infty \quad \text{uniformly on } I. \tag{58}
\]
Let \(y_n \in E\) be a solution of the equation
\[
y_n'''(t) = g_n(t) y_n + a y_n + \beta y_n, \quad t \in I. \tag{59}
\]
Then the number of zeros of \(y_n\) in \(I\) goes to infinity as \(n \to +\infty\).

**Proof.** Set \(a^0 := \max_{t \in [0, 1]} |a(t)|\) and \(\beta^0 := \max_{t \in [0, 1]} |\beta(t)|\). By simple computation, we can show that
\[
g_n(t) + a\frac{y_n'}{y_n} + \beta\frac{y_n'}{y_n} \geq g_n(t) - a^0 - \beta^0 \quad \forall t \in [0, 1]. \tag{60}
\]
After taking a subsequence if necessary, we may assume that
\[
g_{n_j}(t) - a^0 - \beta^0 \geq \lambda_j \quad \forall t \in I \tag{61}
\]
as \(j \to +\infty\), where \(\lambda_j\) is the \(j\)th eigenvalue of the following problem:
\[
u''' = \lambda u, \quad 0 < t < 1,
\]
\[
u(a) = u(b) = u''(a) = u''(b) = 0. \tag{62}
\]
Let \(\varphi_j\) be the corresponding eigenfunction of \(\lambda_j\). It is easy to check that the distance between any two consecutive zeros of \(\varphi_j\) is \((a - b) / j\) (also see [30]).

Hence, the number of zeros of \(\varphi_j\) goes to infinity as \(j \to +\infty\). Note that the conclusion of Lemma 17 also is valid if \(\alpha = \beta = 0\). Using these facts and Lemma 17, we can obtain the desired results. \(\square\)

On the basis of the unilateral global interval bifurcation result, we establish the spectrum of the half-linear problem (8). More precisely, we will use Theorem 10 to prove the following result.

**Theorem 19.** There exist two sequences of simple half-
eigenvalues for problem (7), \(\lambda_1^< < \lambda_2^< < \cdots\) and \(\lambda_1^> < \lambda_2^> < \cdots\). The corresponding half-linear solutions are in \([\lambda_1^< \times \mathbb{S}_{\lambda_1^<}^\pm] \times S_{\lambda_1^<}^\pm\). Furthermore, aside from these solutions and the trivial ones, there is no other solutions of problem (7).

**Proof of Theorem 19.** By Theorem 10, we know that there exists at least one solution of problem (7), \((\lambda_j^<, x_j^<) \in \mathbb{R} \times S_{\lambda_1^<}^\pm\), where \(j = 1, 2, \ldots\). We claim that, for any solution \((\lambda, x)\) of problem (7) lies in some \(S_{\lambda_1^<}^\pm\). Next, we only prove the case of \(\nu = +\) since the case of \(\nu = -\) is similar.

We divide the proof into three steps.

**Step 1 (we show that \(\lambda = \lambda_j^<\)).** We may assume without loss of generality that the first generalized simple zero of \(x_j^<\) to occur in \((0, 1)\) is a generalized simple zero of \(x\). That is, there exists \(c \in [0, 1]\) such that \(x(c) = x''(c) = 0\) and \(x'(c) \neq 0\) or \(x'''(c) \neq 0\), and \(x\) and \(x_j^<\) do not vanish and have the same sign in \((0, c)\). It follows that
\[
\int_0^c \left( x''' x_j^< - (x_j^<)''' x \right) dt = \int_0^c (\lambda - \lambda_j^<) xx_j^< dt. \tag{64}
\]
Similar to the proof of Lemma 3.1 of [16], we can obtain the left-hand side of (64) \(\geq 0\). So, one has that \(\lambda_j^< \leq \lambda\).

On the other hand, similar to the proof of Lemma 13, by Lemma 6, there must exist an interval \((d, e) \subset (0, 1)\) such that \(x\) and \(x_j^<\) do not vanish and have the same sign in \((d, e)\), and \(x_j^<(d) = x_j^< (e) = (x_j^<)'(d) = (x_j^<)'(e) = 0\). We have
\[
\int_d^e \left( (x_j^<)''' x - x''' x_j^< \right) dt = \int_d^e (\lambda_j^< - \lambda) x_j^< x dt. \tag{65}
\]
Similar to the proof of Lemma 3.1 of [16], we can obtain the left-hand side of (65) \(\geq 0\). So, one has that \(\lambda \leq \lambda_j^<\). Hence \(\lambda = \lambda_j^<\).

**Step 2 (we will prove that \(x = hx_j^<\) for some positive constant \(h\)).** Without loss of generality, we may assume that \(x\) and \(x_j^<\)
are positive in \((0, c)\). By the Picone identity in [29, Theorem 4], noting \(\lambda = \lambda_k^+\), we have that

\[
\int_0^c \left\{ x \left( x_k' \right)' - x_k \right\}' dt = C_1,
\]

where \(C_1 = \int_0^c \left\{ x'' - \frac{x}{x_k'} \right\}' \left\{ x_k' \right\}' \right\}' dt.

Using similar methods of the proof of Lemma 13, we can show that the left-hand side of (66) \(\leq 0\). Hence, \(C_1 \leq 0\). It follows that

\[
\int_0^c \left\{ \left( x'' - \frac{x}{x_k'} \left( x_k' \right)'' \right) \right\}' \left\{ \left( x_k' \right)' \right\}' dt \leq 0.
\]

Thus, \(\left\{ x/x_k' \right\}'' = 0\). Furthermore, we obtain that \(x = h_1 x_k'\) on \((0, c)\) for some positive constant \(h_1\). We may assume without loss of generality that the first zero of \(x_k\) to occur in \((c, 1]\) is a generalized simple zero of \(x\). That is, there exists \(c_1 \in (c, 1]\) such that \(x(c_1) = x'(c_1) = 0\) and \(x''(c_1) \neq 0\). Using similar methods similar to the above, we can show that \(x = h_2 x_k''\) on \((c_1, 1]\) for some positive constant \(h_2\). Clearly, \(x''(c) = h_1 (x_k')''(c) = h_2 (x_k'')''(c)\) and Lemma 3 imply \(h_1 = h_2\). Repeating the above process \(k\) times, we can show that \(x = h k x_k^n\) for some positive constant \(h_k\).

Step 3 (we prove that \(\lambda_k^+ (v = +) or v = -\) are increasing). In fact, if \((\lambda_k^+, u)\) and \((\lambda_j^+, v)\) are the solutions of problem (7) with \(u \in S_k^n, v \in S_j^n\) and \(k < j\), the first generalized simple zero of \(uv\) to occur in \((0, 1)\) is a generalized simple zero of \(v\). Indeed, if this were not so, by Lemma 6, using method similar to the proof in Step 1, we could obtain \(\lambda_k^+ < \lambda_j^+\), which is impossible, since the half-eigenvalues were shown to be simple. Therefore, by Lemma 17, we can get \(\lambda_k^+ < \lambda_j^+\).

Naturally, we can consider the bifurcation structure of the perturbation of problem (7) of the form

\[
x''' = \lambda a(t)x + ax^+ + \beta x^- + g(t, x, \lambda),
\]

0 \(t < 1\), (69)

\[
x(0) = x(1) = x''(0) = x''(1) = 0,
\]

where \(g(t, x, \lambda)\) satisfies (4).

\[\text{Theorem 20. For } v = +, - (\lambda_k^+ 0) \text{ is a bifurcation point for problem (69). Moreover, there exists an unbounded continua } \mathcal{D}_{k,0}^+ \text{ of solutions of problem (69) emanating from } (\lambda_k^+, 0), \text{ such that } \mathcal{D}_{k,0}^+ \subset ((\mathbb{R} \times S_k^n) \cup \{(\lambda_k^+, 0)\}).
\]

\[\text{Proof. Let } \alpha^0 = \max_{t \in [0,1]} |a(t)| \text{ and } \beta^0 = \max_{t \in [0,1]} |\beta(t)|:
\]

\[
I_k = \left[ \lambda_k - \frac{\alpha^0 + \beta^0}{\alpha_k}, \lambda_k + \frac{\alpha^0 + \beta^0}{\alpha_k} \right].
\]

Corollary 15 shows that there exist two unbounded subcontinua \(\mathcal{D}_{k,0}^+ \text{ and } \mathcal{D}_{k,0}^-\) of solutions of (69) in \(\mathbb{R} \times E\), bifurcating from \(I_k \times \{0\}\), and \(\mathcal{D}_{k,0}^+ \subset (\mathbb{R} \times S_k^n) \cup (I_k \times \{0\})\) for \(v = +\) and \(v = -\). Let us show that \(\mathcal{D}_{k,0}^+ \cap (\mathbb{R} \times \{0\}) = \{(\lambda_k^+, 0)\}\); that is, \((\lambda_k^+, 0)\) is a bifurcation point for problem (69). Indeed, if there exists \((\lambda_n, x_n)\) being a sequence of solutions of problem (69) converging to \((\lambda, 0)\), let \(y_n = x_n/\|x_n\|\), and then \(y_n\) should be a solution of problem

\[
y_n = L^{-1} \left( \lambda_n a y_n + \alpha y_n^+ + \beta y_n^- + \frac{g(t, x_n, \lambda)}{\|x_n\|} \right).
\]

By (18) and the compactness of \(L^{-1}\), we obtain that for some convenient subsequence \(y_n \to y_0\) as \(n \to +\infty\). Now \(y_0\) verifies the equation

\[
Ly_0 = \lambda a(t) y_0 + \alpha y_0^+ + \beta y_0^- (72)
\]

and \(\|y_0\| = 1\). This implies that \(\lambda = \lambda_k^+\) for some \(k \in N\) and \(v \in \{+, -\}\).

\[\text{5. Nodal Solutions for Half-Linear Eigenvalue Problems}\]

We start this section by studying the following eigenvalue problem:

\[
x''' = \lambda a(t)x + ax^+ + \lambda ra(t)f(x), \quad 0 < t < 1,
\]

\[
x(0) = x(1) = x''(0) = x''(1) = 0,
\]

where \(\lambda > 0\) is a parameter.

Let \(\xi(x), \zeta(x) \in C([R, R])\) be such that

\[
f(x) = f_0 x + \zeta(x),
\]

(74)

\[
f(x) = f_c x + \xi(x)
\]
with
\[
\lim_{|x| \to 0} \frac{\xi(x)}{x} = 0, \quad (75)
\]
\[
\lim_{|x| \to \infty} \frac{\xi(x)}{x} = 0. \quad (76)
\]

Let us consider
\[
x'''(t) = \alpha + \beta x - \lambda a(t) \xi(x), \quad 0 < t < 1,
\]
\[
x(0) = x(1) = x''(0) = x''(1) = 0
\]
as a bifurcation problem from the trivial solution \(x \equiv 0\), and
\[
x'''(t) = \alpha x + \beta x - \lambda a(t) \xi(x), \quad 0 < t < 1,
\]
\[
x(0) = x(1) = x''(0) = x''(1) = 0
\]
as a bifurcation problem from infinity.

Applying Theorem 20 to problem (76), we have the following result.

**Lemma 21.** Let (H1), (H4), (H5), and (H6) hold. For \(v = +, -, (\lambda_k^v/r_{f0}, 0)\) is a bifurcation point for problem (76). Moreover, there exists an unbounded continuum \(\mathcal{D}_k^v\) of solutions of problem (76) that joins \((\lambda_k^v/r_{f0}, 0)\) to infinity, such that \(\mathcal{D}_k^v \subset ((\mathbb{R} \times S_k) \cup \{(\lambda_k^v/r_{f0}, 0)\})\).

We add the points \(\{(\lambda, \infty) \mid \lambda \in \mathbb{R}\}\) to space \(\mathbb{R} \times E\); by the results of Rabinowitz [31], we have the following lemma.

**Lemma 22.** Let (H1), (H4), (H5), and (H6) hold. For \(v = +, -, (\lambda_k^v/r_{f0}, \infty)\) is a bifurcation point for problem (77). Moreover, there exists an unbounded continuum \(\mathcal{D}_k^v\) of solutions of problem (77) meeting \((\lambda_k^v/r_{f0}, \infty)\), such that \(\mathcal{D}_k^v \subset ((\mathbb{R} \times S_k) \cup \{(\lambda_k^v/r_{f0}, \infty)\})\).

We note that problem (76) and problem (77) are the same, and each of them is equivalent to problem (73). By Lemmas 21 and 22, we obtain the following lemma.

**Lemma 23.** Let (H1), (H4), (H5), and (H6) hold. There exists an unbounded continuum \(\mathcal{D}_k^v = \mathcal{D}_k^v \cap \mathcal{D}_k^v\) of solutions of problem (73) emanating from \((\lambda_k^v/r_{f0}, 0)\) or \((\lambda_k^v/r_{f0}, \infty)\), such that \(\mathcal{D}_k^v \subset ((\mathbb{R} \times S_k) \cup \{(\lambda_k^v/r_{f0}, 0)\})\) and \(\mathcal{D}_k^v \subset ((\mathbb{R} \times S_k) \cup \{(\lambda_k^v/r_{f0}, \infty)\})\), and \(\mathcal{D}_k^v\) joins \((\lambda_k^v/r_{f0}, 0)\) to \((\lambda_k^v/r_{f0}, \infty)\).

**Remark 24.** Any solution of (73) of the form \((1, x)\) yields a solution \(x\) of (8). In order to prove our main results, one will only show that \(\mathcal{D}_k^v\) crosses the hyperplane \([1] \times E\) in \(\mathbb{R} \times E\).

**Remark 25.** From (H5) and (H6), we can see that there exists a positive constant \(Q\) such that \(f(s)/s \geq Q\) for all \(s \neq 0\).

**Theorem 26.** Let (H1), (H4), (H5), and (H6) hold. For \(v = +, -, \) either \(\lambda_k^v/r_{f0} < r < \lambda_k^v/r_{f0}\) or \(\lambda_k^v/r_{f0} < r < \lambda_k^v/r_{f0}\). Then problem (8) possesses a solution \(x_k^n\) such that \(x_k^n\) has exactly \(k - 1\) zeros in \((0, 1)\) and \(v x_k^n\) is positive near 0.

**Proof.** By Remark 24, in order to prove Theorem 26, we only show that \(\mathcal{D}_k^v\) joins \((\lambda_k^v/r_{f0}, 0)\) to \((\lambda_k^v/r_{f0}, \infty)\). Let \((\mu_n, x_n) \in \mathcal{D}_k^v\) satisfy
\[
\mu_n + \|x_n\| \to \infty. \quad (78)
\]
We note that \(\mu_n > 0\) for all \(n \in \mathbb{N}\) since \((0, 0)\) is the only solution of (73) for \(\lambda = 0\) and \(\mathcal{D}_k^v \cap ((0) \times E) = 0\).

**Case 1.** \(\lambda_k^v/r_{f0} < r < \lambda_k^v/r_{f0}\). In this case, we show that
\[
\left(\frac{\lambda_k^v/r_{f0}}{r_{f0}}, \frac{\lambda_k^v/r_{f0}}{r_{f0}}\right) \subseteq \{\lambda \in \mathbb{R} \mid (\lambda, x) \in \mathcal{D}_k^v\}. \quad (79)
\]

We divide the proof into two steps.

**Step 1.** We show that if there exists a constant number \(M > 0\) such that
\[
\mu_n \in (0, M), \quad (80)
\]
for \(n\) large enough, then \(\mathcal{D}_k^v\) joins \((\lambda_k^v/r_{f0}, 0)\) to \((\lambda_k^v/r_{f0}, \infty)\).

In this case
\[
\|x_n\| \to \infty. \quad (81)
\]
Let \(\tilde{\xi}(\cdot) = \max\{|\tilde{\xi}(s)| : 0 \leq |s| \leq x\}\), then \(\tilde{\xi}\) is nondecreasing, and
\[
\lim_{x \to \infty} \frac{\tilde{\xi}(x)}{x} = 0. \quad (82)
\]
Moreover, from (84) and the fact that \(\tilde{\xi}\) is nondecreasing, we have that
\[
\lim_{n \to \infty} \frac{\tilde{\xi}(x_n(t))/\|x_n\|}{\|x_n\|} = 0, \quad \forall t \in [0, 1], \quad (85)
\]
since
\[
\frac{\tilde{\xi}(x_n(t))}{\|x_n\|} \leq \tilde{\xi}(\|x_n(t)\|/\|x_n\|) \leq \frac{\tilde{\xi}(\|x_n(t)\|)}{\|x_n\|} \to 0, \quad (86)
\]
\[
n \to \infty \forall t \in [0, 1].
\]
We divide the equation
\[
\begin{align*}
x_n''' &= \mu_n r a(t) f_{\infty} x_n + \alpha x_n^+ + \beta x_n^- \\
&\quad + \mu_n r a(t) \xi(x_n), \quad 0 < t < 1, \\
x_n(0) &= x_n(1) = x_n''(0) = x_n''(1) = 0
\end{align*}
\]
by \(\|x_n\|\) and set \(y_n = x_n/\|x_n\|\). Since \(y_n\) is bounded in \(E\), choosing a subsequence and relabeling if necessary, we have that \(y_n \to y\) for some \(y \in E\) with \(\|y\| = 1\).

By the compactness of \(L^{-1}\), we obtain that
\[
\begin{align*}
y''' &= \mu r a(t) f_{\infty} y + \alpha y^+ + \beta y^-, \quad 0 < t < 1, \\
y(0) &= y(1) = y''(0) = y''(1) = 0,
\end{align*}
\]
where \(\mu = \lim_{n \to \infty} \mu_n\), again choosing a subsequence and relabeling if necessary.

It is clear that \(\|y\| = 1\) and \(y \in D_k^\gamma \subseteq D_k\) since \(D_k\) is closed in \(R \times E\). Moreover, by Theorem 27, \(\mu f_{\infty} = \lambda_k^\gamma\), so that
\[
\mu = \frac{\lambda_k^\gamma}{r f_{\infty}}.
\]
Thus, \(D_k\) joins \((\lambda_k^\gamma/r f_{\infty}, 0)\) to \((\lambda_k^\gamma/r f_{\infty}, \infty)\).

Step 2. We show that there exists a constant number \(M > 0\) such that \(\mu_n \in (0, M]\), for \(n\) large enough.

On the contrary, we suppose that \(\lim_{n \to \infty} |\mu_n| = +\infty\).

Since \((\mu_n, x_n) \in D_k^\gamma\), it follows from the compactness of \(L^{-1}\) that
\[
\begin{align*}
x_n''' &= \mu_n r a(t) \bar{f}_n(t) x_n + \alpha x_n^+ + \beta x_n^-, \quad 0 < t < 1, \\
x_n(0) &= x_n(1) = x_n''(0) = x_n''(1) = 0,
\end{align*}
\]
where \(\bar{f}_n : [0, 1] \times [0, \infty) \to R\) by
\[
\bar{f}_n(t) = \begin{cases} 
\frac{f(x_n)}{x_n}, & x_n \neq 0, \ t \in [0, 1], \\
\frac{f_0}{x_n}, & x_n = 0, \ t \in [0, 1].
\end{cases}
\]

By Remark 25, we have
\[
\lim_{n \to \infty} \mu_n r \bar{f}_n(t) = \pm \infty.
\]

Let \(\varphi_k^\nu\) be an eigenfunction corresponding to \(\lambda_k^\nu\). But if \(\lim_{n \to \infty} \mu_n r \bar{f}_n(t) = -\infty\), applying Lemma 17 to \(x_n\) and \(\varphi_k^\nu\), we have that \(\varphi_k^\nu\) must change sign for \(n\) large enough, which is impossible. So \(\lim_{n \to \infty} \mu_n r \bar{f}_n(t) = +\infty\). By Lemma 18, we get that \(x_n\) must change sign for \(n\) large enough, and this contradicts the fact that \(x_n \in S_k^\nu\).

Case 2 \((\lambda_k^\nu/f_0 < r < \lambda_k^\nu/f_{\infty})\). In this case, if \((\mu_n, x_n) \in D_k^\nu\) is such that
\[
\begin{align*}
\lim_{n \to \infty} (\mu_n + \|x_n\|) &= \infty, \\
\lim_{n \to \infty} \mu_n &= \infty,
\end{align*}
\]
then
\[
\left(\frac{\lambda_k^\nu}{r f_0}, \frac{\lambda_k^\nu}{r f_{\infty}}\right) \subseteq \{\lambda \in (0, \infty) \mid (\lambda, x) \in D_k^\nu\}
\]
and, moreover, \((\{1\} \times E) \cap D_k^\nu \neq \emptyset\).

Assume that there exists \(M > 0\) such that, for all \(n \in N\),
\[
\mu_n \in (0, M].
\]

Applying a similar argument to that used in Step 1 of Case 1, after taking a subsequence and relabeling if necessary, it follows that
\[
(\mu_n, x_n) \longrightarrow \left(\frac{\lambda_k^\nu}{r f_{\infty}}, \infty\right), \quad n \to \infty.
\]
Again \(D_k^\nu\) joins \((\lambda_k^\nu/r f_0, 0)\) to \((\lambda_k^\nu/r f_{\infty}, \infty)\) and the result follows.

\[\square\]

Theorem 27. Let \((H1), (H4), (H5), \) and \((H7)\) hold. For \(\nu \in \{+, -\}\), assume that one of the following conditions holds.

(i) \(r \in (0, \lambda_k^\nu/f_0)\) for \(\lambda_k^\nu > 0\).

(ii) \(r \in (0, \lambda_k^\nu/f_0) \cup \lambda_k^\nu/f_0, 0\) for \(\nu \lambda_k^\nu > 0\).

(iii) \(r \in (0, \lambda_k^\nu/f_0) \cup \lambda_k^\nu/f_0, 0\) for \(\nu \lambda_k^\nu < 0\).

(iv) \(r \in (\lambda_k^\nu/f_0, 0)\) for \(\lambda_k^\nu < 0\).

Then problem (8) possesses two solutions \(x_k^\nu\) and \(\bar{x}_k^\nu\) such that \(x_k^\nu\) has exactly \(k - 1\) zeros in \((0, 1)\) and is positive near 0 and \(x_k^\nu\) has exactly \(k - 1\) zeros in \((0, 1)\) and is negative near 0.

Proof. We will only prove the case of (i) since the proofs of the cases for (ii), (iii), and (iv) are completely analogous.

Inspired by the idea of [32], we define the cut-off function of \(f\) as the following:
\[
f^{[n]}(s) = \begin{cases} 
f(s), & s \in [-n, n], \\
\frac{2n^2 - f(n)}{n} (s - n) + f(n), & s \in (n, 2n), \\
\frac{2n^2 + f(-n)}{n} (s + n) + f(-n), & s \in (-2n, -n), \\
ns, & s \in (-\infty, -2n] \cup [2n, +\infty).
\end{cases}
\]
We consider the following problem:

\[ x''' = ax' + b \beta x - \lambda r a(t) f^{[n]}(x), \quad 0 < t < 1, \]  
\[ x(0) = x(1) = x''(0) = x''(1) = 0. \]  

(98)

Clearly, we can see that \( \lim_{n \to \infty} f^{[n]}(s) = f(s), (f^{[n]})_0 = f_0 \), and \((f^{[n]})(\infty) = n \).

Similar to the proof of Theorem 26, by Lemma 23 and Remark 24, there exists an unbounded continuum \( \mathcal{D}_k^{[n]} \) of solutions of problem (98) emanating from \((\lambda_k^r / rf_0, 0)\), such that \( \mathcal{D}_k^{[n]} \subset (\Phi_k^r \cup \{(\lambda_k^r / rf_0, 0)\}) \), and \( \mathcal{D}_k^{[n]} \) joins \((\lambda_k^r / rf_0, 0)\) to \((\lambda_k^r / rn, \infty)\).

Taking \( z_n = (\lambda_k^r / rn, \infty) \) and \( z^* = (0, \infty) \), we have that \( z_n \to z^* \).

So condition (i) in Lemma 9 is satisfied with \( z^* = (0, \infty) \).

Obviously, \( r_n = \sup \| \lambda + \| x \| \in (\lambda, x) \in \mathcal{D}_k^{[n]} \| = \infty \), \( (99) \)

and accordingly, (ii) in Lemma 9 holds and (iii) in Lemma 9 can be deduced directly from the Arezela-Ascoli Theorem and the definition of \( f^{[n]} \).

Therefore, by Lemma 9, \( \limsup_{n \to \infty} \mathcal{D}_k^{[n]} \) contains an unbounded connected component \( \mathcal{D}_k^r \) with \((0, 0) \in \mathcal{D}_k^r \). Form \( \lim_{n \to \infty} f^{[n]}(s) = f(s), (f^{[n]})_0 = f_0 \), (98) can be converted to the equivalent equation (73). Since \( \mathcal{D}_k^{[n]} \subset \Phi_k^r \), we conclude that \( \mathcal{D}_k^r \subset \Phi_k^r \). Moreover, \( \mathcal{D}_k^r \subset \Sigma_k^r \) by (8).

In the following, we show that \((\lambda_k^r / rf_0, 0) \in \mathcal{D}_k^r \). Let \((\mu_n, x_n) \in \mathcal{D}_k^r \setminus \{(\lambda_k^r / rf_0, 0)\} \) satisfy \( \lim_{n \to \infty} \| x_n \| = 0 \). Let \( \xi \in C(\mathbb{R}, \mathbb{R}) \) be such that

\[ f(s) = f_0 + \xi(s) \]  

with \( \lim_{s \to 0} \xi(s) = 0 \), let \( \xi(x) = \max_{0 \leq |x| \leq \xi(s)} \xi(s) \), then \( \xi \) is nondecreasing, and

\[ \lim_{x \to 0} \frac{\xi(x)}{x} = 0. \]  

(101)

We divide the equation

\[ x''' = \mu_n ra(t) f_0 x_n + ax_n + b \beta x_n - \lambda r a(t) \xi(x_n), \quad 0 < t < 1, \]  
\[ x_n(0) = x_n(1) = x''(0) = x''(1) = 0. \]  

(102)

Let \( y_n = x_n / \| x_n \| \); \( y_n \) should be the solutions of problem

\[ y''' = \mu_n ra(t) f_0 y_n + ay_n + b \beta y_n \]  
\[ + \mu_n ra(t) \xi(x_n) / \| x_n \| \]  
\[ t \in (0, 1), \]  
\[ y_n(0) = y_n(1) = y''(0) = y''(1) = 0. \]  

(103)

Since \( y_n \) is bounded in \( E \), choosing a subsequence and relabeling if necessary, we have that \( y_n \to y \) for some \( y \in E \).

Furthermore, from (101) and the fact that \( \xi \) is nondecreasing, we have that

\[ \lim_{n \to \infty} \| x_n \| = 0, \]  

(104)

since

\[ \frac{\xi(\| x_n \|)}{\| x_n \|} \leq \frac{\xi(\| x_n \|_\infty)}{\| x_n \|_\infty} \leq \frac{\xi(\| x_n \|)}{\| x_n \|} \to 0, \quad n \to +\infty. \]  

(105)

By (104) and the compactness of \( L^{-1} \), we obtain that

\[ Ly = \mu r a(t) f_0 y + \alpha y + \beta \beta y. \]  

(106)

It is clear that \( \| y \| = 1 \) and \( y \in \mathcal{D}_k^r \subseteq \mathcal{D}_k^r \) since \( \mathcal{D}_k^r \) is closed in \( \mathbb{R} \times E \). Moreover, by Theorem 19, \( \mu r f_0 = \lambda_k^r \), so that

\[ \mu = \lambda_k^r / rf_0. \]  

(107)

Thus, \( \mathcal{D}_k^r \) joins \((\lambda_k^r / rf_0, 0) \) to \((0, 0) \).

\[ \square \]

**Theorem 28.** Let (H1), (H4), (H5), and (H8) hold. For \( \nu \in \{+,-\} \), assume that one of the following conditions holds.

(i) \( r \in (\lambda_k^r / rf_0, +\infty) \) for \( \lambda_k^r > 0 \).

(ii) \( r \in (-\infty, \lambda_k^r / rf_0) \cup (\lambda_k^r / rf_0, +\infty) \) for \( \lambda_k^r > 0 \).

(iii) \( r \in (-\infty, \lambda_k^r / rf_0) \cup (\lambda_k^r / rf_0, +\infty) \) for \( \lambda_k^r < 0 \).

(iv) \( r \in (-\infty, \lambda_k^r / rf_0) \) for \( \lambda_k^r < 0 \).

Then problem (8) possesses two solutions \( x^+_k \) and \( x^-_k \) such that \( x^+_k \) has exactly \( k \) - 1 zeros in \((0, 1)\) and is positive near 0 and \( x^-_k \) has exactly \( k \) - 1 zeros in \((0, 1)\) and is negative near 0.

**Proof.** Define

\[ f^{[n]}(s) = \begin{cases} 1, & s \in (-\infty, 0) \\ \frac{n}{2 + f(n)} (s + n) + f(-n), & s \in (-2n, 0), \\ \frac{n}{2 - f(n)} (s - n) - f(n), & s \in (0, 2n), \\ f(s), & s \in [n, 2n]. \end{cases} \]  

(108)

We consider the following problem:

\[ x''' = ax' + b \beta x - \lambda r a(t) f^{[n]}(x), \quad 0 < t < 1, \]  
\[ x(0) = x(1) = x''(0) = x''(1) = 0. \]  

(109)

Clearly, we can see that \( \lim_{n \to \infty} f^{[n]}(s) = f(s), (f^{[n]})_0 = f_0 \), and \( (f^{[n]})_\infty = 1/n \).

Similar to the proof of Theorem 26, by Lemma 23 and Remark 24, there exists an unbounded continuum \( \mathcal{D}_k^{[n]} \) of solutions of problem (109) emanating from \((\lambda_k^r / rf_0, 0)\), such
that $D_k[n] \subset (\Phi_k \cup \{0\})$, and $D_k[n]$ joins $(\lambda_k/r f_0, 0)$ to $(n \lambda_k/r, \infty)$.

Taking $z_n = (n \lambda_k/r, \infty)$ and $z^* = (0, 0)$, we have that $z_n \to z^*$.

So condition (i) in Lemma 9 is satisfied with $z^* = (0, 0)$.

Obviously,

$$r_n = \sup \{ \lambda + \|x\| : (\lambda, x) \in D_k(n) \} = \infty,$$

and accordingly, (ii) in Lemma 9 holds and (iii) in Lemma 9 can be deduced directly from the Arzela-Ascoli Theorem and the definition of $f[n]$.

Therefore, by Lemma 9, $\limsup_n \sup \{ f[n](s) : s \in [0, 1] \} = \infty$.

From $\lim_{n \to \infty} f[n](s) = f(s)$, (109) can be converted to the equivalent equation (73). Since $D_k[n] \subset \Phi_k$, we conclude $D_k \subset \Phi_k$. Moreover, $D_k \subset \Sigma_k$ by (8).

Similar to the proof of Theorem 27, we can obtain that $(\lambda_k/r f_0, 0) \in D_k$.

**Theorem 29.** Let (H1), (H4), (H5), and (H9) hold. For $\nu \in \{+, -, \}$, assume that one of the following conditions holds.

(i) $r \in (\lambda_k/r f_0, +\infty)$ for $\lambda_k^+ > 0$.

(ii) $r \in (\lambda_k/r f_0, +\infty)$ for $\lambda_k^- > 0$.

(iii) $r \in (-\infty, 0)$ for $\lambda_k^+ > 0$.

(iv) $r \in (-\infty, 0)$ for $\lambda_k^- < 0$.

Then problem (8) possesses two solutions $x_k^+$ and $x_k^-$ such that $x_k^+$ has exactly $k - 1$ zeros in $[0, 1)$ and is positive near 0 and $x_k^-$ has exactly $k - 1$ zeros in $[0, 1)$ and is negative near 0.

**Proof.** We will only prove the case of (i) since the proofs of the cases for (ii), (iii), and (iv) are completely analogous.

If $(\lambda, x)$ is any nontrivial solution of problem (73), dividing problem (73) by $\|x\|^2$ and setting $y = x/\|x\|^2$ yields

$$y''' = a y' + \beta y^\nu + \lambda r(a)(t) \frac{f(x)}{\|x\|}, \quad 0 < t < 1,$$

$$y(0) = y(1) = y''(0) = y''(1) = 0.$$  

Define

$$\tilde{f}(y) = \left\{ \begin{array}{ll} \|y\|^2 f \left( \frac{y}{\|y\|} \right), & \text{if } y \neq 0, \\ 0, & \text{if } y = 0. \end{array} \right.$$  

(112)

Evidently, problem (111) is equivalent to

$$y''' = ay' + \beta y^\nu + \lambda r(a) \tilde{f}(y), \quad 0 < t < 1, y(0) = y(1) = y''(0) = y''(1) = 0.$$  

(113)

It is obvious that $(\lambda, 0)$ is always the solution of problem (113). By simple computation, we can show that $f_0 = f_{\infty} \in (0, \infty)$ and $f_{\infty} = f_0 = 0$. Similar to the proof of Theorem 28, there exists an unbounded continuum $\mathcal{C}_k$ of solutions of problem (113) emanating from $(\lambda_k/r f_0, 0)$, such that $\mathcal{C}_k \subset \{ (\lambda_k/r f_0, 0) \}$, and $\mathcal{C}_k$ joins $(\lambda_k/r f_0, 0)$ to $(0, \infty)$.

Under the inversion $y \to y/\|y\|^2 = x$, we obtain $\mathcal{C}_k \to D_k$ being an unbounded component of solutions of problem (73) emanating from $(0, \infty)$, such that $D_k \subset ((R \times \mathcal{C}_k) \cup \{ (\lambda_k/r f_0, 0) \})$, and $\mathcal{C}_k$ joins $(0, \infty)$ to $(\lambda_k/r f_0, \infty)$.

Moreover, by (8), we can obtain that $D_k \subset \Sigma_k$. Thus, $D_k$ is an unbounded component of solutions of problem (8) such that $D_k$ joins $(0, \infty)$ to $(\lambda_k/r f_0, \infty)$.

**Theorem 30.** Let (H1), (H4), (H5), and (H10) hold. For $\nu \in \{+, -, \}$, assume that one of the following conditions holds.

(i) $r \in (0, +\infty)$ for $\lambda_k^+ > 0$.

(ii) $r \in (0, +\infty)$ for $\lambda_k^- > 0$, or $\nu \lambda_k^- < 0$.

(iii) $r \in (-\infty, 0)$ for $\lambda_k^- < 0$.

Then problem (8) possesses two solutions $x_k^+$ and $x_k^-$ such that $x_k^+$ has exactly $k - 1$ zeros in $[0, 1)$ and is positive near 0 and $x_k^-$ has exactly $k - 1$ zeros in $[0, 1)$ and is negative near 0.

**Proof.** Define

$$f[n](s) = \left\{ \begin{array}{ll} ns, & s \in (-\infty, -2n) \cup [2n, +\infty), \\ \frac{2n^2 + f(-n)}{n} (s + n) + f(-n), & s \in (-2n, -n), \\ \frac{2n^2 - f(n)}{n} (s - n) + f(n), & s \in (n, 2n), \\ f(s), & s \in \left[ -n, -\frac{2}{n} \right], \\ -\left[ f\left( -\frac{2}{n} \right) + \frac{1}{n^2} \right] (ns + 2) + f\left( -\frac{2}{n} \right), & s \in \left[ -\frac{2}{n}, -\frac{n}{2} \right], \\ \left[ f\left( \frac{2}{n} \right) - \frac{1}{n^2} \right] (ns - 2) + f\left( \frac{2}{n} \right), & s \in \left[ \frac{2}{n}, \frac{n}{2} \right], \\ 1/n, & s \in \left[ 1/n, 1 \right]. \end{array} \right.$$  

(114)
We consider the following problem:

\[ x^{n+1} = \alpha x^n + \beta x^- + \lambda r(a(t) f[n](x), \quad 0 < t < 1, \]
\[ x(0) = x(1) = x''(0) = x''(1) = 0. \]

(115)

Clearly, we can see that \( \lim_{n \to \infty} f[n](s) = f(s) \), \( (f[n])_0 = 1/n \), and \( (f[n])_\infty = n \).

Applying the similar method used in the proof of Theorem 26, by Lemma 23 and Remark 24, there exists an unbounded continuum \( \mathcal{D}_k^{(n)} \) of solutions of problem (115) emanating from \( (n \lambda^+/r, 0) \) or \((\lambda^+/rn, \infty)\), such that \( \mathcal{D}_k^{(n)} \subset ((\mathbb{R} \times \mathbb{S}_k^+) \cup \{0, \infty\}) \) and \( \mathcal{D}_k^{(n)} \subset ((\mathbb{R} \times \mathbb{S}_k^+) \cup \{0, \infty\}) \), and \( \mathcal{D}_k^{(n)} \) joins \( (n \lambda^+/r, 0) \) to \( (\lambda^+/rn, \infty) \).

Taking \( z_n = (n \lambda^+/r, 0) \) and \( z^* = (0, \infty) \) or \( z_n = (\lambda^+/rn, \infty) \) and \( z^* = (0, \infty) \), we have that \( z_n \to z^* \). By Lemma 9, we obtain an unbounded component \( \mathcal{D}_k^\infty \) with \( (0, 0) \in \mathcal{D}_k^\infty \).

From \( \lim_{n \to \infty} f[n](s) = f(s) \), (115) can be converted to the equivalent equation (73). Thus, \( \mathcal{D}_k^\infty \) is an unbounded component of solutions of problem (73) emanating from \( (0, 0) \) or \( (0, 0) \), such that \( \mathcal{D}_k^\infty \subset (\mathbb{R} \times \mathbb{S}_k^+) \cup \{0, \infty\} \) and \( \mathcal{D}_k^\infty \subset (\mathbb{R} \times \mathbb{S}_k^+) \cup \{0, \infty\} \) and \( \mathcal{D}_k^\infty \) joins \( (0, 0) \) to \( (0, 0) \).

\[ f[n](s) = \begin{cases} \frac{1}{n}, & s \in (-\infty, -2n] \cup [2n, +\infty). \\ \frac{2 + f(-n)}{n} (s + n) + f(-n), & s \in (-2n, -n), \\ \frac{2 - f(n)}{n} (s - n) + f(n), & s \in (n, 2n), \\ f(s), & s \in [n - \frac{2}{n}, \frac{2}{n}], \\ \frac{-f(\frac{n-2}{n}) + \frac{1}{n^2}}{n} (ns + 2) + f(\frac{2}{n}), & s \in (\frac{2}{n}, \frac{2}{n}), \\ \frac{f(\frac{n-2}{n}) - \frac{1}{n^2}}{n} (ns - 2) + f(\frac{2}{n}), & s \in (\frac{1}{n}, \frac{1}{n}), \\ \frac{1}{n}, & s \in (-\frac{1}{n}, \frac{1}{n}). \end{cases} \]

(116)

Since \( \mathcal{D}_k^{(n)} \subset (\mathbb{R} \times \mathbb{S}_k^+) \), we conclude \( \mathcal{D}_k^\infty \subset (\mathbb{R} \times \mathbb{S}_k^+) \). Moreover, by Remark 24 and (8), we can obtain that \( \mathcal{D}_k^\infty \subset \mathcal{S}_k^+ \).

Thus, \( \mathcal{D}_k^\infty \) is an unbounded component of solutions of problem (8) emanating from \( (\infty, 0) \) or \( (0, \infty) \), such that \( \mathcal{D}_k^\infty \subset (\mathbb{R} \times \mathbb{S}_k^+) \cup \{0, \infty\} \) and \( \mathcal{D}_k^\infty \subset (\mathbb{R} \times \mathbb{S}_k^+) \cup \{0, \infty\} \), and \( \mathcal{D}_k^\infty \) joins \( (0, 0) \) to \( (0, 0) \).

Applying the similar method used in the proof of Theorem 27, we obtain an unbounded connected component \( \mathcal{D}_k^\infty \) with \( (0, 0) \in \mathcal{D}_k^\infty \).

**Theorem 31.** Let \((H1), (H4), (H5), \) and \((H11)\) hold. For \( \nu \in \{+, -\} \), assume that one of the following conditions holds.

(i) There exists a \( \lambda_{ik}^+ > 0 \) for \( \lambda_{ik}^+ > 0 \), such that \( r \in (0, \lambda_{ik}^+) \).

(ii) There exists a \( \lambda_{ik}^+ > 0 \) for \( \lambda_{ik}^+ > 0 \), such that \( r \in (\infty, \lambda_{ik}^+) \cup (\lambda_{ik}^+, +\infty) \).

(iii) There exists a \( \lambda_{ik}^- < 0 \) for \( \lambda_{ik}^- < 0 \), such that \( r \in (\lambda_{ik}^-, 0) \cup (0, \lambda_{ik}^-) \).

(iv) There exists a \( \lambda_{ik}^- < 0 \) for \( \lambda_{ik}^- < 0 \), such that \( r \in (\lambda_{ik}^-, 0) \).

Then problem (8) possesses two solutions \( x_k^+ \) and \( x_k^- \), such that \( x_k^+ \) has exactly \( k - 1 \) zeros in \( (0, 1) \) and is positive near 0 and \( x_k^- \) has exactly \( k - 1 \) zeros in \( (0, 1) \) and is negative near 0.

**Proof.** Define

Since \( \mathcal{D}_k^{(n)} \subset (\mathbb{R} \times \mathbb{S}_k^+) \), we conclude \( \mathcal{D}_k^\infty \subset (\mathbb{R} \times \mathbb{S}_k^+) \). Moreover, by Remark 24 and (8), we can obtain that \( \mathcal{D}_k^\infty \subset \mathcal{S}_k^+ \).

Thus, \( \mathcal{D}_k^\infty \) is an unbounded component of solutions of problem (8) emanating from \( (\infty, 0) \) or \( (0, \infty) \), such that \( \mathcal{D}_k^\infty \subset (\mathbb{R} \times \mathbb{S}_k^+) \cup \{0, \infty\} \) and \( \mathcal{D}_k^\infty \subset (\mathbb{R} \times \mathbb{S}_k^+) \cup \{0, \infty\} \), and \( \mathcal{D}_k^\infty \) joins \( (0, 0) \) to \( (0, 0) \).

Applying the similar method used in the proof of Theorem 27, we obtain an unbounded connected component \( \mathcal{D}_k^\infty \) with \( (0, 0) \in \mathcal{D}_k^\infty \).

Moreover, we can obtain the desired results.

**Remark 32.** If \( \alpha = \beta = 0 \) and \( f_0, f_\infty \in (0, \infty) \), Ma et al. [10, 11] studied problem (8) under the conditions \((H5) \) and \((H6) \); in the situation, the results of Theorem 26 improve on Theorem 3.1 of [10] and Theorem 1.1 of [11], respectively.

**Remark 33.** When \( \alpha = \beta = 0 \) and \( f_0, f_\infty \in (0, \infty) \), Dai and Han [16] investigated the existence of nodal solutions for problem (8). Thus, our results partially extend and improve the corresponding Theorem 4.1 of [16].

**Remark 34.** When \( \alpha = \beta = 0 \) and \( f_0, f_\infty \in (0, \infty) \), Lazer and McKenna [7] investigated the existence of nodal solutions for problem (8). Thus, our results partially extend and improve the corresponding Theorem 1 of [7].
Remark 35. The nonlinear term of (8) is not necessarily homogeneous linearizable at the origin and infinity because of the influence of the term $ax^r + bx^{-r}$. Clearly, so the bifurcation results of [7, 8, 10–13, 16] cannot be applied directly to obtain our results.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References


