Research Article

Certain Concepts in m -Polar Fuzzy Graph Structures

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We apply the concept of m -polar fuzzy sets to graph structures. We introduce certain concepts in m -polar fuzzy graph structures, including strong m -polar fuzzy graph structure, m -polar fuzzy D_i -cycle, m -polar fuzzy D_i -tree, m -polar fuzzy D_i -cut vertex, and m -polar fuzzy D_i -bridge, and we illustrate these concepts by several examples. We present the notions of ϕ -complement of an m -polar fuzzy graph structure and self-complementary, strong self-complementary, totally strong self-complementary m -polar fuzzy graph structures, and we investigate some of their properties.

1. Introduction

Graph theory has applications in many areas of computer science, including data mining, image segmentation, clustering, image capturing, and networking. A graph structure, introduced by Sampathkumar [1], is a generalization of undirected graph which is quite useful in studying some structures including graphs, signed graphs, and graphs in which every edge is labeled or colored. A graph structure helps to study the various relations and the corresponding edges simultaneously.

A fuzzy set [2] is an important mathematical structure to represent a collection of objects whose boundary is vague. Fuzzy models are becoming useful because of their aim in reducing the differences between the traditional models used in engineering and science. Nowadays fuzzy sets are playing a substantial role in chemistry, economics, computer science, engineering, medicine, and decision-making problems. In 1997, Zhang [3] generalized the idea of a fuzzy set and gave the concept of bipolar fuzzy set on a given set X as a map which associates each element of X to a real number in the interval $[-1, 1]$. In 2014, Chen et al. [4] introduced the idea of m -polar fuzzy sets as an extension of bipolar fuzzy sets and showed that bipolar fuzzy sets and 2-polar fuzzy sets are cryptomorphic mathematical notions and that we can obtain concisely one from the corresponding one in [4]. The idea behind this is that “multipolar information”

(not just bipolar information which corresponds to two-valued logic) exists because data for a real world problem are sometimes from n agents ($n \geq 2$). For example, the exact degree of telecommunication safety of mankind is a point in $[0, 1]^n$ ($n \approx 7 \times 10^9$) because different person has been monitored different times. There are many examples such as truth degrees of a logic formula which are based on n logic implication operators ($n \geq 2$), similarity degrees of two logic formula which are based on n logic implication operators ($n \geq 2$), ordering results of a magazine, ordering results of a university, and inclusion degrees (accuracy measures, rough measures, approximation qualities, fuzziness measures, and decision preformation evaluations) of a rough set.

Kauffman [5] gave the definition of a fuzzy graph in 1973 on the basis of Zadeh's fuzzy relations [6]. Rosenfeld [7] discussed the idea of fuzzy graph in 1975. Further remarks on fuzzy graphs were given by Bhattacharya [8]. Several concepts on fuzzy graphs were introduced by Mordeson and Nair [9]. In 2011, Akram introduced the notion of bipolar fuzzy graphs in [10]. Alshehri and Akram [11] introduced the concept of bipolar fuzzy competition graphs. In 2015, Akram and Younas studied certain types of irregular m -polar fuzzy graphs in [12]. Akram and Adeel studied m -polar fuzzy line graphs in [13]. Akram and Waseem introduced certain metrics in m -polar fuzzy graphs in [14]. Dinesh [15] introduced the notion of a fuzzy graph structure and discussed some related properties.

Akram and Akmal [16] introduced the concept of bipolar fuzzy graph structures. In this paper, we apply the concept of m -polar fuzzy sets to graph structures. We introduce certain concepts in m -polar fuzzy graph structures, including strong m -polar fuzzy graph structure, m -polar fuzzy D_i -cycle, m -polar fuzzy D_i -tree, m -polar fuzzy D_i -cut vertex, and m -polar fuzzy D_i -bridge, and we illustrate these concepts by several examples. We present the notions of ϕ -complement of an m -polar fuzzy graph structure and self-complementary, strong self-complementary, totally strong self-complementary m -polar fuzzy graph structures, and we investigate some of their properties.

2. Preliminaries

In this section, we review some basic concepts that are necessary for fully benefit of this paper.

In 1965, Zadeh [2] introduced the notion of a fuzzy set as follows.

Definition 1 (see [2, 6]). A fuzzy set μ in a universe X is a mapping $\mu : X \rightarrow [0, 1]$. A fuzzy relation on X is a fuzzy set ν in $X \times X$. Let μ be a fuzzy set in X and ν fuzzy relation on X . We call ν a fuzzy relation on μ if $\nu(x, y) \leq \min\{\mu(x), \mu(y)\} \forall x, y \in X$.

Definition 2 (see [16]). $\tilde{G}_b = (M, N_1, N_2, \dots, N_n)$ is called a bipolar fuzzy graph structure (BFGS) of a graph structure (GS) $G^* = (U, E_1, E_2, \dots, E_n)$ if $M = (\mu_M^P, \mu_M^N)$ is a bipolar fuzzy set on U and for each $i = 1, 2, \dots, n$, $N_i = (\mu_{N_i}^P, \mu_{N_i}^N)$ is a bipolar fuzzy set on E_i such that

$$\begin{aligned} \mu_{N_i}^P(xy) &\leq \mu_M^P(x) \wedge \mu_M^P(y), \\ \mu_{N_i}^N(xy) &\geq \mu_M^N(x) \vee \mu_M^N(y) \end{aligned} \quad (1)$$

$$\forall xy \in E_i \subset U \times U.$$

Note that $\mu_{N_i}^P(xy) = 0 = \mu_{N_i}^N(xy)$ for all $xy \in U \times U - E_i$ and $0 < \mu_{N_i}^P(xy) \leq 1, -1 \leq \mu_{N_i}^N(xy) < 0 \forall xy \in E_i$, where U and E_i ($i = 1, 2, \dots, n$) are called *underlying vertex set* and *underlying i -edge sets* of $\tilde{G}_b$, respectively.

Definition 3 (see [16]). Let $\tilde{G}_b = (M, N_1, N_2, \dots, N_n)$ be a BFGS of a GS $G^* = (U, E_1, E_2, \dots, E_n)$. Let ϕ be any permutation on the set $\{E_1, E_2, \dots, E_n\}$ and the corresponding permutation on $\{N_1, N_2, \dots, N_n\}$, that is, $\phi(N_i) = N_j$, if and only if $\phi(E_i) = E_j \forall i$.

If $xy \in N_r$ for some r and

$$\begin{aligned} \mu_{N_i}^P(xy) &= \mu_M^P(x) \wedge \mu_M^P(y) - \bigvee_{j \neq i} \mu_{\phi N_j}^P(xy), \\ \mu_{N_i}^N(xy) &= \mu_M^N(x) \vee \mu_M^N(y) - \bigwedge_{j \neq i} \mu_{\phi N_j}^N(xy), \end{aligned} \quad (2)$$

$$i = 1, 2, \dots, n,$$

then $xy \in B_m^\phi$, while m is chosen such that $\mu_{N_m}^P(xy) \geq \mu_{N_i}^P(xy)$ and $\mu_{N_m}^N(xy) \leq \mu_{N_i}^N(xy) \forall i$.

And BFGS $(M, N_1^\phi, N_2^\phi, \dots, N_n^\phi)$, denoted by $\tilde{G}_b^{\phi c}$, is called the ϕ -complement of BFGS $\tilde{G}_b$.

Definition 4 (see [4]). An m -polar fuzzy set (or a $[0, 1]^m$ -set) on X is exactly a mapping $A : X \rightarrow [0, 1]^m$.

Note that a $[0, 1]^m$ -set is an L -set. An L -set on the set X is a synonym of a mapping $A : X \rightarrow L$, where L is a lattice. So, $[0, 1]^m$ is considered to be a partial order set with the pointwise order $\leq$, where m is an arbitrary ordinal number, $\leq$ is defined by $x \leq y \Leftrightarrow p_i(x) \leq p_i(y)$ for each $i \in m$, and $p_i : [0, 1]^m \rightarrow [0, 1]$ is the i th projection mapping ($i \in m$). When $L = [0, 1]$, an L -set on X will be called a fuzzy set on X .

Definition 5 (see [14]). Let C be an m -polar fuzzy subset of a nonempty set V . An m -polar fuzzy relation on C is an m -polar fuzzy subset D of $V \times V$ defined by the mapping $D : V \times V \rightarrow [0, 1]^m$ such that, for all $x, y \in V$, $p_i \circ D(xy) \leq \inf\{p_i \circ C(x), p_i \circ C(y)\}$, $i = 1, 2, \dots, m$, where $p_i \circ C(x)$ denotes the i th degree of membership of the vertex x and $p_i \circ D(xy)$ denotes the i th degree of membership of the edge xy .

Definition 6 (see [4, 14]). An m -polar fuzzy graph is a pair $G = (C, D)$, where $C : V \rightarrow [0, 1]^m$ is an m -polar fuzzy set in V and $D : V \times V \rightarrow [0, 1]^m$ is an m -polar fuzzy relation on V such that

$$p_i \circ D(xy) \leq \inf\{p_i \circ C(x), p_i \circ C(y)\} \quad (3)$$

for all $x, y \in V$.

We note that $p_i \circ D(xy) = 0$ for all $xy \in V \times V - E$ for all $i = 1, 2, 3, \dots, m$. C is called the m -polar fuzzy vertex set of G and D is called the m -polar fuzzy edge set of G , respectively. An m -polar fuzzy relation D on X is called symmetric if $p_i \circ D(xy) = p_i \circ D(yx)$ for all $x, y \in V$.

3. Certain Concepts in m -Polar Fuzzy Graph Structures

Definition 7. Let $G^* = (U, E_1, E_2, \dots, E_n)$ be a graph structure (GS). Let C be an m -polar fuzzy set on U and D_i an m -polar fuzzy set on E_i such that

$$p_j \circ D_i(xy) \leq \inf\{p_j \circ C(x), p_j \circ C(y)\} \quad (4)$$

for all $x, y \in U, i \in n, j \in m$, and $p_j \circ D_i(xy) = 0$ for $xy \in U \times U \setminus E_i, \forall j$. Then $G_{(m)} = (C, D_1, D_2, \dots, D_n)$ is called an m -polar fuzzy graph structure (m -PFGS) on G^* where C is the m -polar fuzzy vertex set of $G_{(m)}$ and D_i is the m -polar fuzzy i -edge set of $G_{(m)}$.

Definition 8. Let $G_{(m)} = (C, D_1, D_2, \dots, D_n)$ and $G_{(m')} = (C', D'_1, D'_2, \dots, D'_n)$ be two m -PFGSs of a GS $G^* =$

TABLE 1

C	a_1	a_2	a_3	a_4
$p_1 \circ C$	0.1	0.3	0.4	0.2
$p_2 \circ C$	0.0	0.6	0.0	0.0
$p_3 \circ C$	0.0	0.2	0.4	0.3
$p_4 \circ C$	0.1	0.0	0.4	0.4

TABLE 2

D_i	$(a_1 a_2)_1$	$(a_3 a_2)_2$	$(a_2 a_4)_2$
$p_1 \circ D_i$	0.1	0.2	0.2
$p_2 \circ D_i$	0.0	0.0	0.0
$p_3 \circ D_i$	0.0	0.2	0.2
$p_4 \circ D_i$	0.0	0.0	0.0

$(U, E_1, E_2, \dots, E_n)$. Then $G_{(m')}$ is called an m -polar fuzzy subgraph structure of $G_{(m)}$, if

$$\begin{aligned} p_j \circ C'(x) &\leq p_j \circ C(x), \\ p_j \circ D'_i(xy) &\leq p_j \circ D_i(xy), \end{aligned} \quad (5)$$

$$\forall x, y \in U, j \in m, i \in n.$$

$G_{(m')}$ is called an m -polar fuzzy induced subgraph structure of $G_{(m)}$ by a set $W \subseteq U$, if

$$\begin{aligned} p_j \circ C'(x) &= \begin{cases} 0, & x \in U \setminus W \\ p_j \circ C(x), & x \in W, \forall j, \end{cases} \\ p_j \circ D'_i(xy) &= \begin{cases} 0, & x, y \in U \setminus W \\ p_j \circ D_i(xy), & x, y \in W, \forall j, \forall i. \end{cases} \end{aligned} \quad (6)$$

$G_{(m')}$ is called an m -polar fuzzy spanning subgraph structure of $G_{(m)}$, if $C' = C$ and

$$\begin{aligned} p_j \circ D'_i(xy) &\leq p_j \circ D_i(xy) \\ \text{for } j \in m, i \in n, x, y \in U. \end{aligned} \quad (7)$$

Example 9. Consider a graph structure $G^* = (U, E_1, E_2)$ such that $U = \{a_1, a_2, a_3, a_4\}$, $E_1 = \{a_1 a_2\}$, and $E_2 = \{a_3 a_2, a_2 a_4\}$. Let C , D_1 , and D_2 be 4-polar fuzzy sets on U , E_1 , and E_2 , respectively, defined by Tables 1 and 2.

By simple calculations, it is easy to check that $G_{(m)} = (C, D_1, D_2)$ is a 4-polar fuzzy graph structure of G^* as shown in Figure 1. Note that we represent $xy \in D_i$ as

$$(xy)_i = (p_1 \circ D_i(xy), \dots, p_m \circ D_i(xy))_i \quad (8)$$

in all tables and the figures.

A 4-polar fuzzy induced subgraph structure of $G_{(m)}$ by $U \setminus \{a_4\}$ and a 4-polar fuzzy spanning subgraph structure of $G_{(m)}$ (Figure 1) are shown in Figure 2.

Definition 10. Let $G_{(m)} = (C, D_1, D_2, \dots, D_n)$ be an m -PFGS. Then $xy \in E_i$ is called an m -polar fuzzy D_i -edge or simply

FIGURE 1: Four-polar fuzzy graph structure.

a D_i -edge, if $p_j \circ D_i(xy) > 0$ for at least one j and we write that $xy \in D_i$. Consequently, support of D_i is defined by

$$\begin{aligned} \text{supp}(D_i) \\ = \{xy \in E_i : p_j \circ D_i(xy) > 0 \text{ for at least one } j\}. \end{aligned} \quad (9)$$

An m -polar fuzzy D_i -path, in m -PFGS $G_{(m)}$, is a sequence $a_1, a_2, \dots, a_k$ of vertices (distinct except the choice $a_k = a_1$) in U such that each $a_{j-1}a_j$ is an m -polar fuzzy D_i -edge for $1 \leq j \leq k$.

Definition 11. A D_i -strong m -PFGS is an m -polar fuzzy graph structure

$$G_{(m)} = (C, D_1, D_2, \dots, D_n) \quad (10)$$

that satisfies

$$p_j \circ D_i(xy) = \inf \{p_j \circ C(x), p_j \circ C(y)\} \quad (11)$$

for all $xy \in D_i$. $G_{(m)}$ is a strong m -polar fuzzy graph structure, if it is D_i -strong for all i .

Definition 12. An m -PFGS $G_{(m)} = (C, D_1, D_2, \dots, D_n)$ is called complete, if the following three conditions hold:

- (i) m -polar fuzzy graph structure $G_{(m)}$ is strong.
- (ii) $\text{supp}(D_i) \neq \emptyset, \forall i$.
- (iii) For each pair of vertices $x, y \in U$, xy is a D_i -edge for some i .

Example 13. Consider a graph structure $G^* = (U, E_1, E_2)$ such that $U = \{b_1, b_2, b_3, b_4\}$, $E_1 = \{b_1 b_2, b_1 b_4, b_3 b_4\}$, and $E_2 = \{b_3 b_2, b_2 b_4, b_1 b_3\}$. Let C , D_1 , and D_2 be 4-polar fuzzy sets on U , E_1 , and E_2 , respectively, defined by Tables 3 and 4.

By simple calculations, it is easy to check that $G_{(m)} = (C, D_1, D_2)$ is a strong and complete 4-polar fuzzy graph structure of G^* , as shown in Figure 3. $b_2 b_1 b_4 b_3$ is a 4-polar fuzzy D_1 -path consisting of all D_1 -edges constituting $\text{supp}(D_1) = \{b_2 b_1, b_1 b_4, b_4 b_3\}$.

FIGURE 2: Four-polar fuzzy subgraph structures.

TABLE 3

C	b_1	b_2	b_3	b_4
$p_1 \circ C$	0.3	0.2	0.1	0.4
$p_2 \circ C$	0.4	0.2	0.1	0.2
$p_3 \circ C$	0.3	0.0	0.2	0.3
$p_4 \circ C$	0.5	0.1	0.1	0.1

TABLE 4

D_i	$(b_1b_2)_1$	$(b_1b_4)_1$	$(b_3b_4)_1$	$(b_1b_3)_2$	$(b_2b_4)_2$	$(b_3b_2)_2$
$p_1 \circ D_i$	0.2	0.3	0.1	0.1	0.2	0.1
$p_2 \circ D_i$	0.2	0.2	0.1	0.1	0.2	0.1
$p_3 \circ D_i$	0.0	0.3	0.2	0.2	0.0	0.0
$p_4 \circ D_i$	0.1	0.1	0.1	0.1	0.1	0.1

Definition 14. Let $G_{(m)} = (C, D_1, D_2, \dots, D_n)$ be an m -PFGS and let $j \in m$. The j -strength or j -gain of a D_i -path $p = b_1b_2 \cdots b_l$, denoted by $G \cdot P_{i,j}$, is defined by

$$G \cdot P_{D_{i,j}} = \bigwedge_{k=2}^l [p_j \circ D_i(b_{k-1}b_k)]. \quad (12)$$

Example 15. Consider the 4-PFGS $G_{(m)} = (C, D_1, D_2)$, shown in Figure 3, in which $P = b_2b_1b_4b_3$ is a 4-polar fuzzy D_1 -path. Then j -gain, for $j = 1, 2, 3, 4$, of this path P is given by $G \cdot P_{D_{1,1}} = 0.1$, $G \cdot P_{D_{1,2}} = 0.1$, $G \cdot P_{D_{1,3}} = 0.0$, and $G \cdot P_{D_{1,4}} = 0.1$.

Definition 16. Let $G_{(m)} = (C, D_1, D_2, \dots, D_n)$ be an m -polar fuzzy graph structure and let $j \in m$. Then $D_{i,j}$ -strength of connectedness between two vertices x and y is defined by $p_{j,i}^\infty(xy) = \bigvee_{k \geq 1} \{p_{j,i}^k(xy)\}$, where for $k \geq 2$ and $p_{j,i}^1(xy) = p_j \circ D_i(xy)$

$$\begin{aligned} p_{j,i}^k(xy) &= (p_{j,i}^{k-1} \circ p_{j,i}^1)(xy) \\ &= \bigvee_z \{p_{j,i}^{k-1}(xz) \wedge p_{j,i}^1(zy)\}, \end{aligned} \quad (13)$$

for $k \geq 2$ and $p_{j,i}^1(xy) = p_j \circ D_i(xy)$.

TABLE 5: Connectedness between vertices.

	$(b_1b_2)_1$	$(b_1b_3)_2$	$(b_1b_4)_1$	$(b_2b_4)_2$	$(b_3b_3)_2$	$(b_3b_2)_1$
$p_{1,1}^1$	0.2	0.0	0.3	0.0	0.0	0.1
$p_{1,1}^2$	0.2	0.1	0.0	0.2	0.0	0.0
$p_{1,1}^3$	0.0	0.0	0.1	0.2	0.1	0.1
$p_{1,1}^4$	0.1	0.1	0.0	0.1	0.1	0.0
$p_{1,1}^5$	0.0	0.0	0.1	0.1	0.1	0.1
$p_{1,1}^6$	0.1	0.1	0.0	0.1	0.1	0.0
$p_{1,1}^\infty$	0.2	0.1	0.3	0.2	0.1	0.1

FIGURE 3: A strong 4-polar fuzzy graph structure.

Example 17. Consider the 4-PFGS $G_{(m)} = (C, D_1, D_2)$ as shown in Figure 3. All the calculated $D_{i,j}$ -strengths of connectedness between any two vertices of $G_{(m)}$, for $i = 1$, $j = 1$, are given in Table 5.

Definition 18. An m -polar fuzzy D_i -cycle is an m -PFGS $G_{(m)} = (C, D_1, D_2, \dots, D_n)$ such that

- (i) $(\text{supp}(C), \text{supp}(D_1), \text{supp}(D_2), \dots, \text{supp}(D_n))$ is an E_i -cycle (it is called a D_i -cycle),
- (ii) There is no unique D_i -edge uv in $G_{(m)}$ such that for any $j \in m$

$$p_j \circ D_i(uv) = \min \{p_j \circ D_i(xy) : xy \in \text{supp}(D_i)\}. \quad (14)$$

FIGURE 4: A 4-polar fuzzy D_2 -cycle.

Example 19. Let $G_{(m)} = (C, D_1, D_2)$ be the 4-PFGS, shown in Figure 4; then $G_{(m)}$ is a 4-polar fuzzy D_2 -cycle.

Definition 20. Let $G_{(m)} = (C, D_1, D_2, \dots, D_n)$ be an m -polar fuzzy graph structure and let x be a vertex in $G_{(m)}$. Let $(C', D'_1, D'_2, \dots, D'_n)$ be an m -polar fuzzy subgraph structure of $G_{(m)}$ induced by $U \setminus \{x\}$, such that for $i \in n, j \in m$

$$p_j \circ C'(x) = 0,$$

$$p_j \circ D'_i(xv) = 0, \quad (15)$$

for all edges xv in $G_{(m)}$,

$$p_j \circ C'(v) = p_j \circ C(v),$$

$$p_j \circ D'_i(uv) = p_j \circ D_i(uv), \quad (16)$$

for $v \neq x, u \neq x$.

Then x is called an m -polar fuzzy $D_{i,j}$ -cut vertex for some i , if for some $u, v \in U \setminus \{x\}$

$$p_{j,i}^{\infty}(uv) > p_{j,i}^{\infty'}(uv) \quad (17)$$

and x is called an m -polar fuzzy D_i -cut vertex for some i , if it is $D_{i,j}$ -cut vertex for all j .

Definition 21. Let $G_{(m)} = (C, D_1, D_2, \dots, D_n)$ be an m -polar fuzzy graph structure and xy an edge in $G_{(m)}$. Let $(C, D'_1, D'_2, \dots, D'_n)$ be an m -polar fuzzy spanning subgraph structure of $G_{(m)}$ such that

$$p_j \circ D'_i(xy) = 0,$$

$$p_j \circ D'_i(uv) = p_j \circ D_i(uv), \quad (18)$$

for edges $uv \neq xy, i \in n, j \in m$.

Then xy is an m -polar fuzzy $D_{i,j}$ -bridge for some i , if for some $u, v \in U$

$$p_{j,i}^{\infty}(uv) > p_{j,i}^{\infty'}(uv). \quad (19)$$

And xy is called an m -polar fuzzy D_i -bridge for some i , if it is $D_{i,j}$ -bridge for all j .

Example 22. Consider the 4-PFGS $G_{(m)} = (C, D_1, D_2)$ as shown in Figure 3. The vertex b_1 is a 4-polar fuzzy $D_{1,1}$ -cut vertex since $p_{1,1}^{\infty}(b_2b_3) = 0.1 > 0.0 = p_{1,1}^{\infty'}(b_2b_3)$ and $p_{1,1}^{\infty}(b_2b_4) = 0.2 > 0.0 = p_{1,1}^{\infty'}(b_2b_4)$, where (W, D'_1, D'_2) is the 4-polar fuzzy subgraph structure of $G_{(m)}$ induced by $W = C \setminus \{b_1\}$.

The edge b_1b_4 is a 4-polar fuzzy $D_{1,1}$ -bridge since $p_{1,1}^{\infty}(b_1b_4) = 0.3 > 0.0 = p_{1,1}^{\infty'}(b_1b_4)$, $p_{1,1}^{\infty}(b_1b_3) = 0.1 > 0.0 = p_{1,1}^{\infty'}(b_1b_3)$, $p_{1,1}^{\infty}(b_2b_3) = 0.1 > 0.0 = p_{1,1}^{\infty'}(b_2b_3)$, and $p_{1,1}^{\infty}(b_2b_4) = 0.2 > 0.0 = p_{1,1}^{\infty'}(b_2b_4)$, where (C, D'_1, D'_2) is the 4-polar fuzzy spanning subgraph structure of $G_{(m)}$ that excludes the D_1 -edge b_1b_4 .

Definition 23. An m -PFGS $G_{(m)} = (C, D_1, D_2, \dots, D_n)$ is a D_i -tree if $(\text{supp}(C), \text{supp}(D_1), \text{supp}(D_2), \dots, \text{supp}(D_n))$ is an E_i -tree.

Definition 24. An m -PFGS $G_{(m)} = (C, D_1, D_2, \dots, D_n)$ is an m -polar fuzzy D_i -tree if there is an m -polar fuzzy spanning subgraph structure $G'_{(m)} = (C', D'_1, D'_2, \dots, D'_n)$ such that $G'_{(m)}$ is a D'_i -tree and for every edge xy not belonging to $G'_{(m)}$

$$p_j \circ D_i(xy) < p_{j,i}^{\infty'}(xy) \quad \forall j. \quad (20)$$

If the above condition holds for $j \in m$, $G_{(m)}$ is called a $D_{i,j}$ -tree.

Example 25. Consider the 4-PFGS $G_{(m)} = (C, D_1, D_2)$ as shown in Figure 5.

Then $G_{(m)}$ is a 4-polar fuzzy $D_{1,1}$ -tree since $p_1 \circ D_1(b_2b_5) = 0.2 < 0.3 = p_{1,1}^{\infty'}(b_2b_5)$, where (C, D'_1, D'_2) is the 4-polar fuzzy spanning subgraph structure of $G_{(m)}$ that excludes the D_1 -edge b_2b_5 . Also $G_{(m)}$ is a D_2 -tree.

Definition 26. An m -PFGS $G_{(m)}^1 = (C_1, D_{11}, D_{12}, \dots, D_{1n})$ of a graph structure $G^{*1} = (U_1, E_{11}, E_{12}, \dots, E_{1n})$ is said to be isomorphic to an m -PFGS $G_{(m)}^2 = (C_2, D_{21}, D_{22}, \dots, D_{2n})$ of a graph structure $G^{*2} = (U_2, E_{21}, E_{22}, \dots, E_{2n})$, if there exists a bijection $h : U_1 \rightarrow U_2$ and a permutation ϕ on the set $\{1, 2, \dots, n\}$ such that for $k \in m, i \in n$

$$p_k \circ C_1(u_1) = p_k \circ C_2(h(u_1)), \quad \forall u_1 \in U_1,$$

$$p_k \circ D_{1i}(u_1u_2) = p_k \circ D_{2\phi(i)}(h(u_1)h(u_2)), \quad (21)$$

$$\forall u_1u_2 \in E_{1i}.$$

Example 27. Two isomorphic 4-PFGSs $G_{(m_1)} = (C_1, D_{11}, D_{12})$ and $G_{(m_2)} = (C_2, D_{21}, D_{22})$ are shown in Figure 6. This isomorphism holds under the permutation $\phi = (12)$ and the mapping $h : U_1 \rightarrow U_2$, defined by $h(a_1) = b_3, h(a_2) = b_4, h(a_3) = b_1$, and $h(a_4) = b_2$.

Definition 28. An m -PFGS $G_{(m)}^1 = (C_1, D_{11}, D_{12}, \dots, D_{1n})$ is said to be identical to an m -PFGS $G_{(m)}^2 = (C_2, D_{21}, D_{22}, \dots,$

FIGURE 5: A 4-polar fuzzy $D_{1,1}$ -tree.

FIGURE 6: Two isomorphic 4-polar fuzzy graph structures.

D_{2n}) if there exists a bijection $h : U \rightarrow U$, such that for $j \in m$, $i \in n$

$$\begin{aligned} p_j \circ C_1(u) &= p_j \circ C_2(h(u)), \quad \forall u \in U, \\ p_j \circ D_{1i}(u_1 u_2) &= p_j \circ D_{2i}(h(u_1) h(u_2)), \quad (22) \\ &\forall u_1 u_2 \in E_{1i}. \end{aligned}$$

Example 29. Two 4-PFGSs $G_{(m_1)} = (C_1, D_{11}, D_{12})$ and $G_{(m_2)} = (C_2, D_{21}, D_{22})$, identical under the mapping $h : U_1 \rightarrow U_2$, defined by $h(a_1) = b_3$, $h(a_2) = b_4$, $h(a_3) = b_1$, and $h(a_4) = b_2$, are shown in Figure 7.

Definition 30. Let $G_{(m)} = (C, D_1, D_2, \dots, D_n)$ be an m -polar fuzzy graph structure. Let ϕ be any permutation on the set $\{1, 2, \dots, n\}$ and $\phi(D_i) = D_j \Leftrightarrow \phi(i) = j$. If $xy \in D_r$ for some $r \in n$ and

$$\begin{aligned} p_j \circ D_i^\phi(xy) &= \inf(p_j \circ C(x), p_j \circ C(y)) - \bigvee_{k \neq i} p_j \\ &\quad \circ D_{\phi(k)}(xy) \end{aligned} \quad (23)$$

for $j \in m$, $i \in n$ then $xy \in D_s^\phi$, where s is chosen such that, for each j ,

$$p_j \circ D_s^\phi(xy) = \max \{p_j \circ D_i^\phi(xy) : i \in n\}. \quad (24)$$

Then an m -PFGS constructed with C and n m -polar fuzzy relations $(D_s^\phi, s = 1, 2, \dots, n)$ on C , denoted by $G_{(m)}^{\phi c} = (C, D_1^\phi, D_2^\phi, \dots, D_n^\phi)$, is called the ϕ -complement of m -polar fuzzy graph structure $G_{(m)}$.

Proposition 31. A ϕ -complement of an m -polar fuzzy graph structure is always a strong m -PFGS. Moreover, if $\phi^{-1}(r) = i$ for $r, i \in n$, then all D_r -edges in m -PFGS $G_{(m)} = (C, D_1, D_2, \dots, D_n)$ become D_i^ϕ -edges in $G_{(m)}^{\phi c} = (C, D_1^\phi, D_2^\phi, \dots, D_n^\phi)$.

Proof. From the definition of ϕ -complement $G_{(m)}^{\phi c}$ of an m -PFGS $G_{(m)}$,

$$\begin{aligned} p_j \circ D_i^\phi(xy) &= \inf(p_j \circ C(x), p_j \circ C(y)) - \bigvee_{k \neq i} p_j \\ &\quad \circ D_{\phi(k)}(xy), \quad j \in m, i \in n. \end{aligned} \quad (25)$$

FIGURE 7: Identical 4-polar fuzzy graph structures.

We can see that $\inf(p_j \circ C(x), p_j \circ C(y)) \geq 0$, $\bigvee_{k \neq i} p_j \circ D_{\phi(k)}(xy) \geq 0$, and $p_j \circ D_k(xy) \leq \inf(p_j \circ C(x), p_j \circ C(y))$, $\forall D_k$. So

$$\bigvee_{k \neq i} p_j \circ D_{\phi(k)}(xy) \leq \inf(p_j \circ C(x), p_j \circ C(y)) \implies p_j \circ D_i^\phi(xy) = \inf(p_j \circ C(x), p_j \circ C(y)) \quad (26)$$

$$-\bigvee_{k \neq i} p_j \circ D_{\phi(k)}(xy) \geq 0 \quad \forall i, j.$$

Moreover, the maximum value of $p_j \circ D_i^\phi(xy)$ occurs when its negative part $(-\bigvee_{k \neq i} p_j \circ D_{\phi(k)}(xy))$ becomes zero, and it is zero when $\phi(i) = r$ and xy is a D_r -edge in $G_{(m)}$. Therefore, $p_j \circ D_i^\phi(xy) = \inf(p_j \circ C(x), p_j \circ C(y))$ for $xy \in D_r$, $\phi(i) = r$.

Hence $G_{(m)}^{\phi c} = (C, D_1^\phi, D_2^\phi, \dots, D_n^\phi)$ is a strong m -PFGS and every D_r -edge in $G_{(m)}$ becomes D_i^ϕ -edge for $\phi^{-1}(r) = i$. $\square$

Remark 32. If $G_{(m)}$ is an m -polar fuzzy graph structure of a GS G^* then ϕ -complement of $G_{(m)}$ is a strong m -polar fuzzy graph structure of ϕ^{-1} -complement of G^* .

Definition 33. Let $G_{(m)} = (C, D_1, D_2, \dots, D_n)$ be an m -polar fuzzy graph structure and ϕ be any permutation on the set $\{1, 2, \dots, n\}$. Then

- (i) $G_{(m)}$ is called *self-complementary* if $G_{(m)}$ is isomorphic to $G_{(m)}^{\phi c}$,
- (ii) $G_{(m)}$ is called *strong self-complementary* if $G_{(m)}$ is identical to $G_{(m)}^{\phi c}$.

Definition 34. Let $G_{(m)} = (C, D_1, D_2, \dots, D_n)$ be an m -polar fuzzy graph structure. Then

- (i) $G_{(m)}$ is called *totally self-complementary* if $G_{(m)}$ is isomorphic to $G_{(m)}^{\phi c}$ for every permutation ϕ on the set $\{1, 2, \dots, n\}$,

- (ii) $G_{(m)}$ is called *totally strong self-complementary* if $G_{(m)}$ is identical to $G_{(m)}^{\phi c}$ for every permutation ϕ on the set $\{1, 2, \dots, n\}$.

Theorem 35. An m -PFGS $G_{(m)}$ is strong if and only if it is totally self-complementary.

Proof. We suppose that $G_{(m)}$ is a strong m -PFGS of a GS $G^* = (U, E_1, E_2, \dots, E_n)$ and ϕ is any permutation on the set $\{1, 2, \dots, n\}$.

By Proposition 31, $G_{(m)}^{\phi c}$ is strong and all D_i -edges in $G_{(m)}$ become D_j^ϕ -edges for $\phi^{-1}(i) = j$. So

$$p_k \circ D_i(a_1 a_2) = \inf(p_k \circ C(a_1), p_k \circ C(a_2)) = p_k \circ D_j^\phi(a_1 a_2), \quad \forall k. \quad (27)$$

Using the identity mapping $h : U \rightarrow U$, we get $p_k \circ C(a) = p_k \circ C(h(a))$, $\forall k \in m$, $a \in U$, and

$$p_k \circ D_i(a_1 a_2) = \inf(p_k \circ C(a_1), p_k \circ C(a_2)) = \inf(p_k \circ C(h(a_1)), p_k \circ C(h(a_2))) = p_k \circ D_j^\phi(h(a_1) h(a_2)), \quad \forall k. \quad (28)$$

That is,

$$p_k \circ D_i(a_1 a_2) = p_k \circ D_j^\phi(h(a_1) h(a_2)), \quad \forall k. \quad (29)$$

Hence $G_{(m)}$ is isomorphic to $G_{(m)}^{\phi c}$ under the identity mapping $h : U \rightarrow U$ and a permutation ϕ . Since ϕ was an arbitrary permutation on $\{1, 2, \dots, n\}$, $G_{(m)}$ is totally self-complementary.

Conversely suppose that $G_{(m)}$ is a totally self-complementary m -PFGS and we have to prove that $G_{(m)}$ is strong m -PFGS.

Since $G_{(m)}$ and $G_{(m)}^{\phi c}$ are isomorphic for all permutations on $\{1, 2, \dots, n\}$, therefore by definition of isomorphism $p_k \circ D_i(a_1 a_2) = p_k \circ D_j^\phi(h(a_1) h(a_2)) = \inf(p_k \circ C(h(a_1)), p_k \circ C(h(a_2)))$.

$C(h(a_2))) = \inf(p_k \circ C(a_1), p_k \circ C(a_2))$, for all $a_1 a_2 \in E_i$, $i \in n$, $k \in m$. Hence $G_{(m)}$ is a strong m -PFGS.

This completes the proof. $\square$

Theorem 36. Let $G_{(m)}$ be a strong m -PFGS on a GS $G^* = (U, E_{11}, E_{12}, \dots, E_{1n})$. If G^* is totally strong self-complementary and C is an m -polar fuzzy set on U such that for each $j \in m$, $p_j \circ C$ assigns a constant value $c_j \in [0, 1]$ to every $u \in U$, then $G_{(m)}$ is totally strong self-complementary.

Proof. We assume that G^* is totally strong self-complementary and $p_j \circ C(u) = c_j$ for all $u \in U$, where $c_j \in [0, 1]$ is a constant for each $j \in m$.

Since G^* is totally strong self-complementary, for any permutation ϕ^{-1} with $\phi^{-1}(k) = i$, there exists a bijection $h : U \rightarrow U$, such that, for every E_i -edge $a_1 a_2$ in G^* , $h(a_1)h(a_2)$ (an E_k -edge in G^*) is an E_i -edge in $(G^*)^{\phi^{-1}}$. Consequently, for every D_i -edge $a_1 a_2$ in $G_{(m)}$, $h(a_1)h(a_2)$ (a D_k -edge in $G_{(m)}$) is a D_i^{ϕ} -edge in $G_{(m)}^{\phi c}$. Moreover,

$$p_j \circ C(h(a)) = c_j = p_j \circ C(a), \quad \forall a, h(a) \in U, j \in m \quad (30)$$

and $G_{(m)}$ is a strong m -PFGS, so we get

$$\begin{aligned} p_j \circ D_i(a_1 a_2) &= \inf(p_j \circ C(a_1), p_j \circ C(a_2)) \\ &= \inf(p_j \circ C(h(a_1)), p_j \circ C(h(a_2))) \\ &= p_j \circ D_i^{\phi}(h(a_1)h(a_2)), \\ &\quad \forall a_1 a_2 \in E_i, i \in n, j \in m. \end{aligned} \quad (31)$$

This shows that $G_{(m)}$ is identical to $G_{(m)}^{\phi c}$. Hence $G_{(m)}$ is totally strong self-complementary, since ϕ was an arbitrary permutation. This completes the proof. $\square$

Converse of Theorem 36 does not hold because every strong and totally strong self-complementary m -PFGS does not necessarily include an m -polar fuzzy vertex set C such that for each $j \in m$, $p_j \circ C$ assigns a constant value $c_j \in [0, 1]$ to every $u \in U$. This can be observed in m -PFGS shown in Figure 9.

Example 37. There are no other totally self-complementary m -PFGSs than strong m -polar fuzzy graph structures. So all strong m -PFGSs are the examples of totally self-complementary m -PFGSs.

Example 38. An m -PFGS, shown in Figure 8, is strong self-complementary but not totally strong self-complementary because this m -PFGS is identical to its ϕ -complement only when $\phi = (12)(34)$.

An m -PFGS, shown in Figure 9, is totally strong self-complementary.

FIGURE 8: A strong self-complementary 4-PFGS.

FIGURE 9: A totally strong self-complementary 4-PFGS.

4. Conclusions

A graph structure is a useful tool in solving the combinatorial problems in different areas of computer science and computational intelligence systems. It helps to study various relations and the corresponding edges simultaneously. Sometimes information in a network model is based on multiagent, multiattribute, multiobject, multipolar information or uncertainty rather than a single bit. An m -polar fuzzy model is useful for such network models which gives more and more precision, flexibility, and comparability to the system as compared to the classical, fuzzy, and bipolar fuzzy models. We have introduced certain concepts in m -polar fuzzy graph structures. We are extending our work to (1) neutrosophic graph structures, (2) intuitionistic neutrosophic soft graph

structures, (3) roughness in graph structures, and (4) intuitionistic fuzzy soft graph structures.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

References

- [1] E. Sampathkumar, "Generalized graph structures," *Bulletin of Kerala Mathematics Association*, vol. 3, no. 2, pp. 65–123, 2006.
- [2] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, pp. 338–353, 1965.
- [3] W.-R. Zhang, "Bipolar fuzzy sets," in *Proceedings of the IEEE International Conference on Fuzzy Systems*, pp. 835–840, 1997.
- [4] J. Chen, S. Li, S. Ma, and X. Wang, "M-Polar fuzzy sets: an extension of bipolar fuzzy sets," *The Scientific World Journal*, vol. 2014, Article ID 416530, 8 pages, 2014.
- [5] A. Kauffman, *Introduction a la Theorie des Sous-Ensembles Flous*, vol. 1, Masson et Cie S.A., Renens, Switzerland, 1973.
- [6] L. A. Zadeh, "Similarity relations and fuzzy orderings," *Information Sciences*, vol. 3, no. 2, pp. 177–200, 1971.
- [7] A. Rosenfeld, *Fuzzy Graphs, Fuzzy Sets and their Applications*, Edited by L. A. Zadeh, K. S. Fu, M. Shimura, Academic Press, New York, NY, USA, 1975.
- [8] P. Bhattacharya, "Some remarks on fuzzy graphs," *Pattern Recognition Letters*, vol. 6, no. 5, pp. 297–302, 1987.
- [9] J. N. Mordeson and P. S. Nair, *Fuzzy Graphs and Fuzzy Hypergraphs*, Physica, Heidelberg, Germany, 1998, 2nd Edition, 2001.
- [10] M. Akram, "Bipolar fuzzy graphs," *Information Sciences*, vol. 181, no. 24, pp. 5548–5564, 2011.
- [11] N. O. Alshehri and M. Akram, "Bipolar fuzzy competition graphs," *Ars Combinatoria*, vol. 121, pp. 385–402, 2015.
- [12] M. Akram and H. R. Younas, "Certain types of irregular m-polar fuzzy graphs," *Journal of Applied Mathematics and Computing*, 2015.
- [13] M. Akram and A. Adeel, "m-polar fuzzy graphs and m-polar fuzzy line graphs," *Journal of Discrete Mathematical Sciences & Cryptography*, pp. 1–21, 2015.
- [14] M. Akram and N. Waseem, "Certain metrics in m-polar fuzzy graphs," *New Mathematics and Natural Computation*, vol. 12, no. 2, pp. 135–155, 2016.
- [15] T. Dinesh, *A study on graph structures, incidence algebras and their fuzzy analogues [Ph.D. thesis]*, Kannur University, Kannur, India, 2011.
- [16] M. Akram and R. Akmal, "Application of bipolar fuzzy sets in graph structures," *Applied Computational Intelligence and Soft Computing*, vol. 2016, Article ID 5859080, 13 pages, 2016.

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