Research Article

Comparison Criteria for Nonlinear Functional Dynamic Equations of Higher Order

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We will consider the higher order functional dynamic equations with mixed nonlinearities of the form $x^{[n]}(t) + \sum_{j=0}^N p_j(t) \phi_{\gamma_j}(x(\varphi_j(t))) = 0$, on an above-unbounded time scale $\mathbb{T}$, where $n \geq 2$, $x^{[i]}(t) := r_i(t) \phi_{\alpha_i}[(x^{[i-1]})^\Delta(t)]$, $i = 1, \dots, n-1$, with $x^{[0]} = x$, $\phi_\beta(u) := |u|^\beta \operatorname{sgn} u$, and $\alpha[i, j] := \alpha_i \cdots \alpha_j$. The function $\varphi_i : \mathbb{T} \rightarrow \mathbb{T}$ is a rd-continuous function such that $\lim_{t \rightarrow \infty} \varphi_i(t) = \infty$ for $j = 0, 1, \dots, N$. The results extend and improve some known results in the literature on higher order nonlinear dynamic equations.

1. Introduction

In this paper, we consider comparison criteria for higher order nonlinear dynamic equation with mixed nonlinearities of the form

$$x^{[n]}(t) + \sum_{j=0}^N p_j(t) \phi_{\gamma_j}(x(\varphi_j(t))) = 0, \quad (1)$$

on an above-unbounded time scale $\mathbb{T}$, where

- (i) $n \geq 2$ is an integer, and $x^{[i]}(t) := r_i(t) \phi_{\alpha_i}[(x^{[i-1]})^\Delta(t)]$, $i = 1, 2, \dots, n-1$, $t \in \mathbb{T}$, with $r_n = 1$, $\alpha_n = 1$, and $x^{[0]} = x$;

- (ii) $\phi_\beta(u) := |u|^\beta \operatorname{sgn} u$ for $\beta > 0$.

Without loss of generality we assume $t_0 \in \mathbb{T}$. For $A \subset \mathbb{T}$ and $B \subset \mathbb{R}$, we denote by $C_{\text{rd}}(A, B)$ the space of right-dense continuous functions from A to B and by $C_{\text{rd}}^1(A, B)$ the set of functions in $C_{\text{rd}}(A, B)$ with right-dense continuous

Δ -derivatives. Throughout this paper we make the following assumptions:

- (iii) $\alpha_i, \gamma_j > 0$, $i = 1, 2, \dots, n-1$ and $j = 0, 1, \dots, N$, are constants and $r_i \in C_{\text{rd}}([t_0, \infty)_{\mathbb{T}}, (0, \infty))$ for $i = 1, 2, \dots, n-1$, such that

$$\int_{t_0}^{\infty} r_i^{-1/\alpha_i}(s) \Delta s = \infty, \quad i = 1, 2, \dots, n-1; \quad (2)$$

$$\begin{aligned} \gamma_j &> \gamma_0, & j &= 1, 2, \dots, l; \\ \gamma_j &< \gamma_0, & j &= l+1, l+2, \dots, N. \end{aligned} \quad (3)$$

- (iv) $p_j \in C_{\text{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+)$ such that $p_j \not\equiv 0$, $j = 0, 1, \dots, N$ on $[t_0, \infty)_{\mathbb{T}}$.

- (v) $\varphi_j \in C_{\text{rd}}(\mathbb{T}, \mathbb{T})$ rd-continuous function such that $\lim_{t \rightarrow \infty} \varphi_j(t) = \infty$, $j = 0, 1, \dots, N$, and we let $\varphi(t) := \inf\{\varphi_0(t), \varphi_1(t), \dots, \varphi_N(t)\}$ be a nondecreasing function on $[t_0, \infty)_{\mathbb{T}}$.

Recall that the knowledge and understanding of time scales and time scale notation are assumed. For an excellent introduction to the calculus on time scales, see [1–3]. By a solution of (1) we mean a nontrivial real-valued function

$x \in C_{rd}^1[T_x, \infty)_{\mathbb{T}}$ for some $T_x \geq t_0$ such that $x^{[i]} \in C_{rd}^1[T_x, \infty)_{\mathbb{T}}$, $i = 1, 2, \dots, n-1$, and $x(t)$ satisfies (1) on $[T_x, \infty)_{\mathbb{T}}$, where C_{rd} is the space of right-dense continuous functions. An extendable solution x of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is said to be nonoscillatory.

In the last few years, there has been an increasing interest in obtaining sufficient conditions for the oscillation/nonoscillation of solutions of different classes of dynamic equations; we refer the reader to [4–13] and the references cited therein. Special cases of (1) have been studied by many authors. When $\alpha_i = r_i = N = 1$ and $\varphi_i(t) = t$ for $i = 1, 2, \dots, n$, $p(t) = q_0(t) = 0$, and $q_1(t) \geq 0$, Grace et al. [14] established some oscillation criteria for higher order nonlinear dynamic equation of the form

$$x^{\Delta^n}(t) + p(t)(x^\sigma(t))^\gamma = 0, \quad (4)$$

where γ is the ratio of positive odd integers. In paper by Grace [15], some new criteria for the oscillation of the even order dynamic equation

$$\left[r(t)(x^{\Delta^{n-1}}(t))^{\alpha} \right]^{\Delta} + p(t)(x^\sigma(t))^\gamma = 0, \quad (5)$$

where α and γ are the ratios of positive odd integers, were given. Recently, Hassan and Kong [16] obtained asymptotics and oscillation criteria for the n th-order half-linear dynamic equation with deviating argument

$$(x^{[n-1]})^{\Delta}(t) + p(t)\phi_{\alpha[1, n-1]}(x(g(t))) = 0, \quad (6)$$

and Grace and Hassan [17] establish oscillation criteria for more general higher order dynamic equation

$$x^{[n]}(t) + p(t)\phi_\gamma(x^\sigma(\varphi(t))) = 0. \quad (7)$$

The purpose of this paper is to derive comparison criteria for higher order nonlinear dynamic equation with mixed nonlinearities (1).

2. Preliminaries

We will employ the following lemmas. Consider the inequality

$$(r(t)\phi_\alpha(x^\Delta(t)))^\Delta + Q(t)\phi_\gamma(x(\varphi(t))) \leq 0, \quad (8)$$

where r and Q are positive real-valued, rd-continuous functions on $\mathbb{T}$ and r satisfies condition (2), $\varphi : \mathbb{T} \rightarrow \mathbb{T}$ is a rd-continuous function and $\varphi(t) \rightarrow \infty$ as $t \rightarrow \infty$, and α and γ are positive real numbers.

Now, we present the following lemma.

Lemma 1. *If inequality (8) has an eventually positive solution, then the equation*

$$(r(t)\phi_\alpha(x^\Delta(t)))^\Delta + Q(t)\phi_\gamma(x(\varphi(t))) = 0 \quad (9)$$

has also an eventually positive solution.

Proof. Let $x(t)$ be an eventually positive solution of inequality (8). It is easy to see that $x^\Delta > 0$ eventually. Let t_0 be sufficiently large so that $x(t) > 0$, $x(\varphi(t)) > 0$, and $y(t) := r(t)\phi_\alpha(x^\Delta(t))$ for $t \in [t_0, \infty)_{\mathbb{T}}$. Then, in view of

$$x(t) = x(t_0) + \int_{t_0}^t \phi_\alpha^{-1}\left(\frac{y(s)}{r(s)}\right) \Delta s, \quad (10)$$

there is $t_1 \geq t_0$ such that $\varphi(t) \geq t_0$, for $t \geq t_1$. Inequality (8) becomes

$$y^\Delta(t) + Q(t)\phi_\gamma\left(x(t_0) + \int_{t_0}^{\varphi(t)} \phi_\alpha^{-1}\left(\frac{y(s)}{r(s)}\right) \Delta s\right) \leq 0. \quad (11)$$

Integrating (11) from t to $v \geq t \geq t_1$ and letting $v \rightarrow \infty$, we have

$$y(t) \geq G(t, y(t)), \quad \text{for } t \in [t_1, \infty)_{\mathbb{T}}, \quad (12)$$

where

$$G(t, y(t)) := \int_t^\infty Q(v)\phi_\gamma\left(x(t_0) + \int_{t_0}^{\varphi(v)} \phi_\alpha^{-1}\left(\frac{y(s)}{r(s)}\right) \Delta s\right) \Delta v. \quad (13)$$

Now, we define a sequence of successive approximations $\{w_j(t)\}$ as follows:

$$\begin{aligned} w_0(t) &:= y(t), \\ w_{j+1}(t) &:= G(t, w_j(t)), \quad j = 0, 1, 2, \dots \end{aligned} \quad (14)$$

It is easy to show that

$$\begin{aligned} 0 &< w_j(t) \leq y(t), \\ w_{j+1}(t) &\leq w_j(t), \\ j &= 0, 1, 2, \dots \end{aligned} \quad (15)$$

Then, the sequence $\{w_j(t)\}$ is nonincreasing and bounded for each $t \geq t_1$. This means that we may define $w(t) := \lim_{j \rightarrow \infty} w_j(t) \geq 0$. Since

$$0 \leq w(t) \leq w_j(t) \leq y(t), \quad \forall j \geq 0, \quad (16)$$

we find that

$$\int_{t_1}^t w_j(s) \Delta s \leq \int_{t_1}^t y(s) \Delta s. \quad (17)$$

By Lebesgue's dominated convergence theorem on time scale, one can easily find

$$w(t) = G(t, w(t)). \quad (18)$$

Therefore

$$\begin{aligned} w^\Delta(t) &= -Q(t)\phi_\gamma\left(x(t_0) + \int_{t_0}^{\varphi(t)} \phi_\alpha^{-1}\left(\frac{w(s)}{r(s)}\right) \Delta s\right) \\ &= -Q(t)\phi_\gamma(w(\varphi(t))), \end{aligned} \quad (19)$$

where

$$m(t) := x(t_0) + \int_{t_0}^t \phi_\alpha^{-1} \left(\frac{w(s)}{r(s)} \right) \Delta s. \quad (20)$$

Then

$$\begin{aligned} m(t) &> 0, \\ r(t) \phi_\alpha(m^\Delta(t)) &= w(t), \end{aligned} \quad (21)$$

for $t \geq t_1$.

Equation (19) then gives

$$(r(t) \phi_\alpha(m^\Delta(t)))^\Delta + Q(t) \phi_\gamma(m(\varphi(t))) = 0. \quad (22)$$

Hence (9) has a positive solution $m(t)$. This completes the proof. $\square$

The second one is cited from [18, 19].

Lemma 2. Assume that (3) holds. Then there exists an N -tuple $(\eta_1, \eta_2, \dots, \eta_N)$ with $\eta_j > 0$ satisfying

$$\begin{aligned} \sum_{j=1}^N \gamma_j \eta_j &= \gamma_0, \\ \sum_{j=1}^N \eta_j &= 1. \end{aligned} \quad (23)$$

The next lemma is cited from [17] and improves the well-known lemma of Kiguradze.

Lemma 3. Assume that (2) holds. If (1) has an eventually positive solution x , then there exists an integer $m \in [0, n]$ with $m + n$ being odd such that

$$\begin{aligned} m &\geq 1 \\ \text{implies } x^{[k]} &> 0 \quad \text{for } k = 0, 1, \dots, m-1, \end{aligned} \quad (24)$$

eventually, and

$$\begin{aligned} m &\leq n \\ \text{implies } (-1)^{m+k} x^{[k]} &> 0 \quad \text{for } k = m, \dots, n, \end{aligned} \quad (25)$$

eventually.

3. Main Results

In the following main theorem, we will use the following notations: $\alpha[h, k] := \alpha_h \cdots \alpha_k$ for $1 \leq h \leq k \leq n-1$ and $\alpha[h, k] = 1$ for $h > k$ and, for any $u, v \in \mathbb{T}$, define $R_j(v, u)$,

$P_j(t)$, and $\bar{P}_j(t)$, $j = 0, \dots, n-1$, by the following recurrence formulas:

$$\begin{aligned} R_j(v, u) &:= \begin{cases} \left[\int_u^v \frac{R_{j-1}(s, u)}{r_{m-j+1}(s)} \Delta s \right]^{1/\alpha_{m-j+1}}, & j = 1, \dots, m, \\ 1, & j = 0; \end{cases} \\ P_j(t) &:= \begin{cases} \left[\frac{1}{r_{n-j}(t)} \int_t^\infty P_{j-1}(s) \Delta s \right]^{1/\alpha_{n-j}}, & j = 1, \dots, n-1, \\ p(t), & j = 0; \end{cases} \\ \bar{P}_j(t) &:= \begin{cases} \left[\frac{1}{r_{n-j}(t)} \int_t^\infty \bar{P}_{j-1}(s) \Delta s \right]^{1/\alpha_{n-j}}, & j = 1, \dots, n-1, \\ \bar{p}(t), & j = 0, \end{cases} \end{aligned} \quad (26)$$

with $p(t) := p_0(t) + \prod_{j=1}^N [p_j(t)/\eta_j]^{\eta_j}$ and $\bar{p}(t) := \sum_{j=0}^N p_j(t)$, provided the improper integrals involved are convergent.

Theorem 4. Assume that for sufficiently large $T \in [t_0, \infty)_{\mathbb{T}}$ the first-order dynamic equation

$$\begin{aligned} z^\Delta(t) + K_m(t) \phi_{\gamma_0/\alpha[1, n]}(z(\varphi(t))) &= 0, \\ \text{for } \varphi(t) &\in [T, \infty)_{\mathbb{T}}, \end{aligned} \quad (27)$$

is oscillatory, where

$$K_m(t) := P_{n-m-1}(t) [R_m(\varphi(t), T)]^{\gamma_0/\alpha[m+1, n]}, \quad (28)$$

for every number $m \in \{1, \dots, n-1\}$ with $m+n$ being odd. Then

- (1) if n is even, every solution of (1) is oscillatory,
- (2) if n is odd and

$$\int_T^\infty \bar{P}_{n-1}(t) \Delta t = \infty, \quad (29)$$

then every solution of (1) either is oscillatory or tends to zero eventually.

Proof. Assume that (1) has a nonoscillatory solution x on $[t_0, \infty)_{\mathbb{T}}$. Then, without loss of generality, $x(t) > 0$ and $x(\varphi_j(t)) > 0$, for $t \in [t_0, \infty)_{\mathbb{T}}$ and $j = 0, 1, \dots, N$. By Lemma 3, there exists an integer m , $0 \leq m < n$, with $m+n$ being odd such that (24) and (25) hold for $t \geq T_1 \in [t_0, \infty)_{\mathbb{T}}$.

(I) When $m \geq 1$, from (1), we get

$$\begin{aligned} -x^{[n]}(t) &= \sum_{j=0}^N p_j(t) \phi_{\gamma_j}(x(\varphi_j(t))) \\ &\geq \sum_{j=0}^N p_j(t) \phi_{\gamma_j}(x(\varphi(t))) \\ &= \phi_{\gamma_0}(x(\varphi(t))) \sum_{j=0}^N p_j(t) [x(\varphi(t))]^{\gamma_j - \gamma_0}. \end{aligned} \quad (30)$$

From (23), we have

$$\sum_{j=1}^N \gamma_j \eta_j - \gamma_0 \sum_{j=1}^N \eta_j = 0. \quad (31)$$

Using the arithmetic-geometric mean inequality (see [20, Page 17]), we have

$$\sum_{j=1}^N \eta_j v_j \geq \prod_{j=1}^N v_j^{\eta_j}, \quad \text{for any } v_j \geq 0, \quad j = 1, \dots, N. \quad (32)$$

Then for $t \geq T_1$,

$$\begin{aligned} & \sum_{j=0}^N p_j(t) [x(\varphi(t))]^{\gamma_j - \gamma_0} \\ &= p_0(t) + \sum_{j=1}^N \eta_j \frac{p_j(t)}{\eta_j} [x(\varphi(t))]^{\gamma_j - \gamma_0} \\ &\geq p_0(t) + \prod_{j=1}^N \left[\frac{p_j(t)}{\eta_j} \right]^{\eta_j} [x(\varphi(t))]^{\eta_j(\gamma_j - \gamma_0)} \\ &= p_0(t) + \prod_{j=1}^N \left[\frac{p_j(t)}{\eta_j} \right]^{\eta_j} = p(t). \end{aligned} \quad (33)$$

This together with (30) shows that

$$-x^{[n]}(t) \geq p(t) \phi_{\gamma_0}(x(\varphi(t))), \quad \text{for } t \geq T_1. \quad (34)$$

Integrating above inequality from $t \geq T_1$ to $v \in [t, \infty)_{\mathbb{T}}$ and then using the fact that x is strictly increasing and φ is nondecreasing, we get

$$\begin{aligned} -x^{[n-1]}(v) + x^{[n-1]}(t) &\geq \int_t^v p(s) \phi_{\gamma_0}(x(\varphi(s))) \Delta s \\ &\geq \phi_{\gamma_0}(x(\varphi(t))) \int_t^v p(s) \Delta s, \end{aligned} \quad (35)$$

and by (25) we see that $x^{[n-1]}(v) > 0$. Hence by taking limits as $v \rightarrow \infty$ we have

$$x^{[n-1]}(t) \geq \phi_{\gamma_0}(x(\varphi(t))) \int_t^\infty p(s) \Delta s, \quad (36)$$

which implies

$$\begin{aligned} & [x^{[n-2]}(t)]^\Delta \\ &\geq \phi_{\alpha_{n-1}}^{-1} [\phi_{\gamma_0}(x(\varphi(t)))] \left[\frac{1}{r_{n-1}(t)} \int_t^\infty p(s) \Delta s \right]^{1/\alpha_{n-1}} \\ &= \phi_{\alpha_{[n-1,n]}}^{-1} [\phi_{\gamma_0}(x(\varphi(t)))] P_1(t). \end{aligned} \quad (37)$$

Integrating above inequality (37) from $t \geq T_1$ to $v \in [t, \infty)_{\mathbb{T}}$ and letting $v \rightarrow \infty$, we get

$$\begin{aligned} -x^{[n-2]}(t) &\geq \phi_{\alpha_{[n-1,n]}}^{-1} [\phi_{\gamma_0}(x(\varphi(t)))] \int_t^\infty P_1(s) \Delta s, \\ -[x^{[n-3]}(t)]^\Delta &\geq \phi_{\alpha_{[n-2,n]}}^{-1} [\phi_{\gamma_0}(x(\varphi(t)))] \\ &\quad \cdot \left[\frac{1}{r_{n-2}(t)} \int_t^\infty P_1(s) \Delta s \right]^{1/\alpha_{n-2}} \\ &= \phi_{\alpha_{[n-2,n]}}^{-1} [\phi_{\gamma_0}(x(\varphi(t)))] P_2(t). \end{aligned} \quad (38)$$

Continuing this process $(n - m - 3)$ times, we find

$$-x^{[m]}(t)^\Delta \geq \phi_{\alpha_{[m+1,n]}}^{-1} [\phi_{\gamma_0}(x(\varphi(t)))] P_{n-m-1}(t), \quad (39)$$

for $t \geq T_1$.

Also, from (24) and (25), we get

$$\begin{aligned} x^{[m-1]}(t) &= x^{[m-1]}(T_1) \\ &\quad + \int_{T_1}^t \phi_{\alpha_m}^{-1}(x^{[m]}(s)) \left[\frac{1}{r_m(s)} \right]^{1/\alpha_m} \Delta s \\ &\geq \phi_{\alpha_m}^{-1}(x^{[m]}(t)) \int_{T_1}^t \left[\frac{1}{r_m(s)} \right]^{1/\alpha_m} \Delta s \\ &= \phi_{\alpha_m}^{-1}(x^{[m]}(t)) R_1(t, T_1). \end{aligned} \quad (40)$$

It follows that

$$\begin{aligned} & (x^{[m-2]}(t))^\Delta \\ &\geq \phi_{\alpha_{[m-1,m]}}^{-1}(x^{[m]}(t)) \left[\frac{R_1(t, T_1)}{r_{m-1}(t)} \right]^{1/\alpha_{m-1}}. \end{aligned} \quad (41)$$

Then for $t \in [T_1, \infty)_{\mathbb{T}}$,

$$\begin{aligned} x^{[m-2]}(t) &\geq x^{[m-2]}(t) - x^{[m-2]}(T_1) \\ &\geq \int_{T_1}^t \phi_{\alpha_{[m-1,m]}}^{-1}(x^{[m]}(s)) \left[\frac{R_1(s, T_1)}{r_{m-1}(s)} \right]^{1/\alpha_{m-1}} \Delta s \\ &\geq \phi_{\alpha_{[m-1,m]}}^{-1}(x^{[m]}(t)) \int_{T_1}^t \left[\frac{R_1(s, T_1)}{r_{m-1}(s)} \right]^{1/\alpha_{m-1}} \Delta s \\ &= \phi_{\alpha_{[m-1,m]}}^{-1}(x^{[m]}(t)) R_2(t, T_1). \end{aligned} \quad (42)$$

Analogously, we have

$$x(t) \geq \phi_{\alpha_{[1,m]}}^{-1}(x^{[m]}(t)) R_m(t, T_1). \quad (43)$$

Then for $\varphi(t) \in [T_1, \infty)_{\mathbb{T}}$

$$x(\varphi(t)) \geq \phi_{\alpha_{[1,m]}}^{-1}(x^{[m]}(\varphi(t))) R_m(\varphi(t), T_1). \quad (44)$$

From (39) and (44), we get

$$\begin{aligned} -[x^{[m]}(t)]^\Delta &\geq P_{n-m-1}(t) [R_m(\varphi(t), T_1)]^{\gamma_0/\alpha[m+1,n]} \\ &\cdot \phi_{\gamma_0/\alpha[1,n]}(x^{[m]}(\varphi(t))) = K_m(t) \\ &\cdot \phi_{\gamma_0/\alpha[1,n]}(x^{[m]}(\varphi(t))). \end{aligned} \quad (45)$$

Let $z(t) := x^{[m]}(t) > 0$; we get

$$-z^\Delta(t) \geq K_m(t) \phi_{\gamma_0/\alpha[1,n]}(z(\varphi(t))), \quad (46)$$

or

$$z^\Delta(t) + K_m(t) \phi_{\gamma_0/\alpha[1,n]}(z(\varphi(t))) \leq 0. \quad (47)$$

In view of Corollary 2.3.5 in [21], there exists a positive solution of (27) which contradicts the assumption of the theorem.

(II) When $m = 0$ (in this case n is odd), therefore

$$(-1)^k x^{[k]} > 0 \quad \text{for } k = 0, \dots, n. \quad (48)$$

Since $x^\Delta < 0$ eventually, then $\lim_{t \rightarrow \infty} x(t) = l \geq 0$. Assume that $l > 0$. Then for sufficiently large $T_2 \in [T_1, \infty)_\mathbb{T}$, we have $x(\varphi_j(t)) \geq l$ for $t \geq T_2$ and $j = 0, 1, \dots, N$. It follows that

$$\phi_{\gamma_j}(x(\varphi_j(t))) \geq l^{\gamma_j} \geq L \quad \text{for } t \in [T_2, \infty)_\mathbb{T}, \quad (49)$$

where $L := \inf_{0 \leq j \leq N} \{l^{\gamma_j}\} > 0$. Then from (1), we have

$$\begin{aligned} -x^{[n]}(t) &= \sum_{j=0}^N p_j(t) \phi_{\gamma_j}(x(\varphi_j(t))) \geq L \sum_{j=0}^N p_j(t) \\ &= L \bar{p}(t). \end{aligned} \quad (50)$$

Integrating from t to $v \in [t, \infty)_\mathbb{T}$, we get

$$-x^{[n-1]}(v) + x^{[n-1]}(t) = L \int_t^v \bar{p}(s) \Delta s. \quad (51)$$

And by (48) we see that $x^{[n-1]}(v) > 0$. Hence by taking limits as $v \rightarrow \infty$ we have

$$x^{[n-1]}(t) \geq L \int_t^\infty \bar{p}(s) \Delta s, \quad (52)$$

which implies

$$\begin{aligned} (x^{[n-2]}(t))^\Delta &\geq L^{1/\alpha_{n-1}} \left[\frac{1}{r_{n-1}(t)} \int_t^\infty \bar{p}(s) \Delta s \right]^{1/\alpha_{n-1}} \\ &= L^{1/\alpha_{n-1}} \bar{p}_1(t). \end{aligned} \quad (53)$$

Again integrating above inequality from t to $v \in [t, \infty)_\mathbb{T}$ and then taking $v \rightarrow \infty$, we get

$$-x^{[n-2]}(t) \geq L^{1/\alpha_{n-1}} \int_t^\infty \bar{p}_1(s) \Delta s, \quad (54)$$

which implies

$$\begin{aligned} &-(x^{[n-3]}(t))^\Delta \\ &\geq L^{1/\alpha[n-2,n-1]} \left[\frac{1}{r_{n-2}(t)} \int_t^\infty \bar{p}_1(s) \Delta s \right]^{1/\alpha_{n-2}} \\ &= L^{1/\alpha[n-2,n-1]} \bar{p}_2(t). \end{aligned} \quad (55)$$

Continuing this process $(n-3)$ times, we find

$$-(x(t))^\Delta \geq L^{1/\alpha[1,n-1]} \bar{p}_{n-1}(t) \quad \text{for } t \in [T_2, \infty)_\mathbb{T}. \quad (56)$$

Integrating above inequality T_2 to $t \in [T_2, \infty)_\mathbb{T}$, we get

$$-x(t) + x(T_2) \geq L^{1/\alpha[1,n-1]} \int_{T_2}^t \bar{p}_{n-1}(s) \Delta s. \quad (57)$$

Hence by (29), we have $\lim_{t \rightarrow \infty} x(t) = -\infty$, which contradicts the fact that $x > 0$ eventually. This shows that $\lim_{t \rightarrow \infty} x(t) = 0$. This completes the proof. $\square$

Theorem 5. Assume that for sufficiently large $T \in [t_0, \infty)_\mathbb{T}$ the second-order dynamic equation

$$\begin{aligned} &[r_m(t) \phi_{\alpha_m}(z^\Delta(t))]^\Delta + Q_m(t) \phi_{\alpha_m \gamma_0/\alpha[1,n]}(z(\varphi(t))) \\ &= 0, \quad \text{for } \varphi(t) \in (T, \infty)_\mathbb{T}, \end{aligned} \quad (58)$$

is oscillatory, where

$$Q_m(t) := \frac{P_{n-m-1}(t) [R_m(\varphi(t), T_1)]^{\gamma_0/\alpha[m+1,n]}}{[R_1(\varphi(t), T_1)]^{\alpha_m \gamma_0/\alpha[1,n]}}, \quad (59)$$

for every integer number $m \in \{1, \dots, n-1\}$ with $m+n$ being odd. Then

- (1) if n is even, then every solution of (1) is oscillatory,
- (2) if n is odd and (29) holds, then every solution of (1) either is oscillatory or tends to zero eventually.

Proof. Assume that (1) has a nonoscillatory solution x on $[t_0, \infty)_\mathbb{T}$. Then, without loss of generality, $x(t) > 0$ and $x(\varphi_j(t)) > 0$, $j = 0, 1, 2, \dots, N$ on $[t_0, \infty)_\mathbb{T}$. By Lemma 3, there exists an integer m , $0 \leq m < n$, with $n+m$ being odd such that (24) and (25) hold for $t \geq T_1 \in [t_0, \infty)_\mathbb{T}$.

(I) When $m \geq 1$, as seen in the proof of Theorem 4, we obtain, for $t \in [T_1, \infty)_\mathbb{T}$,

$$-[x^{[m]}(t)]^\Delta \geq \phi_{\alpha[m+1,n]}^{-1} [\phi_{\gamma_0}(x(\varphi(t)))] P_{n-m-1}(t), \quad (60)$$

$$x^{[m-1]}(t) \geq \phi_{\alpha_m}^{-1}(x^{[m]}(t)) R_1(t, T_1). \quad (61)$$

Hence, we have

$$\left[\frac{x^{[m-1]}(t)}{R_1(t, T_1)} \right]^\Delta \leq 0, \quad \text{for } t \in (T_1, \infty)_\mathbb{T}. \quad (62)$$

Since for $t \in (T_1, \infty)_{\mathbb{T}}$,

$$x^{[m-1]}(t) = \frac{x^{[m-1]}(t)}{R_1(t, T_1)} R_1(t, T_1), \quad (63)$$

we have

$$\begin{aligned} & (x^{[m-2]}(t))^{\Delta} \\ &= \phi_{\alpha_{m-1}}^{-1} \left(\frac{x^{[m-1]}(t)}{R_1(t, T_1)} \right) \left[\frac{R_1(t, T_1)}{r_{m-1}(t)} \right]^{1/\alpha_{m-1}}. \end{aligned} \quad (64)$$

It follows from (62) that we have, for $t \in (T_1, \infty)_{\mathbb{T}}$,

$$\begin{aligned} x^{[m-2]}(t) &\geq x^{[m-2]}(t) - x^{[m-2]}(T_1) \\ &= \int_{T_1}^t \phi_{\alpha_{m-1}}^{-1} \left(\frac{x^{[m-1]}(s)}{R_1(s, T_1)} \right) \left[\frac{R_1(s, T_1)}{r_{m-1}(s)} \right]^{1/\alpha_{m-1}} \Delta s \\ &\geq \phi_{\alpha_{m-1}}^{-1} \left(\frac{x^{[m-1]}(t)}{R_1(t, T_1)} \right) \int_{T_1}^t \left[\frac{R_1(s, T_1)}{r_{m-1}(s)} \right]^{1/\alpha_{m-1}} \Delta s \\ &= \phi_{\alpha_{m-1}}^{-1} \left(\frac{x^{[m-1]}(t)}{R_1(t, T_1)} \right) R_2(t, T_1). \end{aligned} \quad (65)$$

Continuing this process, we have

$$x(t) \geq \phi_{\alpha_{[1, m-1]}}^{-1} \left(\frac{x^{[m-1]}(t)}{R_1(t, T_1)} \right) R_m(t, T_1). \quad (66)$$

Then for $\varphi(t) \in (T_1, \infty)_{\mathbb{T}}$

$$x(\varphi(t)) \geq \phi_{\alpha_{[1, m-1]}}^{-1} \left(\frac{x^{[m-1]}(\varphi(t))}{R_1(\varphi(t), T_1)} \right) R_m(\varphi(t), T_1). \quad (67)$$

From (60) and (67), we get

$$\begin{aligned} -[x^{[m]}(t)]^{\Delta} &\geq \phi_{\alpha_m \gamma_0 / \alpha_{[1, n]}} \left(\frac{x^{[m-1]}(\varphi(t))}{R_1(\varphi(t), T_1)} \right) \\ &\cdot P_{n-m-1}(t) [R_m(\varphi(t), T_1)]^{\gamma_0 / \alpha_{[m+1, n]}} = Q_m(t) \\ &\cdot \phi_{\alpha_m \gamma_0 / \alpha_{[1, n]}} (x^{[m-1]}(\varphi(t))). \end{aligned} \quad (68)$$

Set $z(t) := x^{[m-1]}(t) > 0$; we have, for $t \geq T_1$,

$$\begin{aligned} & -[r_m(t) \phi_{\alpha_m} (z^{\Delta}(t))]^{\Delta} \\ & \geq Q_m(t) \phi_{\alpha_m \gamma_0 / \alpha_{[1, n]}} (z(\varphi(t))), \end{aligned} \quad (69)$$

or

$$\begin{aligned} & [r_m(t) \phi_{\alpha_m} (z^{\Delta}(t))]^{\Delta} + Q_m(t) \phi_{\alpha_m \gamma_0 / \alpha_{[1, n]}} (z(\varphi(t))) \\ & \leq 0. \end{aligned} \quad (70)$$

In view of Lemma 1, there exists a positive solution of (58) which contradicts the assumption of the theorem.

(II) When $m = 0$, as shown in the proof of Theorem 4, we show that if (29) holds, then $\lim_{t \rightarrow \infty} x(t) = 0$. This completes the proof. $\square$

Remark 6. The conclusion of Theorems 4 and 5 remains intact if assumption (29) is replaced by one of the following conditions:

$$\begin{aligned} & \int_T^{\infty} \bar{P}_0(t) \Delta t = \infty, \\ & \int_T^{\infty} \bar{P}_1(t) \Delta t = \infty, \\ & \vdots \\ & \text{or } \int_T^{\infty} \bar{P}_{n-2}(t) \Delta t = \infty. \end{aligned} \quad (71)$$

Theorem 7. Assume that

$$\int_{t_0}^{\infty} \bar{P}_2(t) \Delta t = \infty. \quad (72)$$

And for sufficiently large $T \in [t_0, \infty)_{\mathbb{T}}$, the first-order dynamic equation

$$\begin{aligned} & z^{\Delta}(t) + K_{n-1}(t) \phi_{\gamma_0 / \alpha_{[1, n]}} (z(\varphi(t))) = 0, \\ & \text{for } \varphi(t) \in [T, \infty)_{\mathbb{T}}, \end{aligned} \quad (73)$$

is oscillatory, where

$$K_{n-1}(t) := p(t) [R_{n-1}(\varphi(t), T)]^{\gamma_0}. \quad (74)$$

Then

- (1) if n is even, every solution of (1) is oscillatory,
- (2) if n is odd, then every solution of (1) either is oscillatory or tends to zero eventually.

Theorem 8. Assume that (72) holds and, for sufficiently large $T \in [t_0, \infty)_{\mathbb{T}}$, the second-order dynamic equation

$$\begin{aligned} & [r_{n-1}(t) \phi_{\alpha_{n-1}} (z^{\Delta}(t))]^{\Delta} \\ & + Q_{n-1}(t) \phi_{\gamma_0 / \alpha_{[1, n-2]}} (z(\varphi(t))) = 0, \\ & \text{for } \varphi(t) \in (T, \infty)_{\mathbb{T}}, \end{aligned} \quad (75)$$

is oscillatory, where

$$Q_{n-1}(t) := \frac{p(t) [R_{n-1}(\varphi(t), T_1)]^{\gamma_0}}{[R_1(\varphi(t), T_1)]^{\gamma_0 / \alpha_{[1, n-2]}}}. \quad (76)$$

Then

- (i) if n is even, every solution of (1) is oscillatory,
- (ii) if n is odd, every solution of (1) either is oscillatory or tends to zero eventually.

Proof of Theorems 7 and 8. Assume that (1) has a nonoscillatory solution x on $[t_0, \infty)_{\mathbb{T}}$. Then, without loss of generality, it is sufficiently large, such that $x(t) > 0$ and $x(\varphi_j(t)) > 0$, for $t \in [t_0, \infty)_{\mathbb{T}}$ and $j = 0, 1, \dots, N$. By Lemma 3, there exists

an integer m , $0 \leq m < n$, with $n + m$ being odd such that (24) and (25) hold for $t \geq T_1 \in [t_0, \infty)_{\mathbb{T}}$.

(I) When $m \geq 1$, we claim that (72) implies that $m = n - 1$. In fact, if $1 \leq m \leq n - 3$, then for $t \geq T_1$

$$\begin{aligned} x^{[n]}(t) &< 0, \\ x^{[n-1]}(t) &> 0, \\ x^{[n-2]}(t) &< 0, \\ x^{[n-3]}(t) &> 0. \end{aligned} \quad (77)$$

Since $x^\Delta(t) > 0$ on $[T_1, \infty)_{\mathbb{T}}$, then $x(t) > x(T_1) := c_1 > 0$ for $t \geq T_1$. Then there exists $T_2 \in [T_1, \infty)_{\mathbb{T}}$ such that $x(\varphi_j(t)) \geq l$ for $t \geq T_2$ and $j = 0, 1, \dots, N$. It follows that

$$\phi_{\gamma_j}(x(\varphi_j(t))) \geq l^{\gamma_j} \geq L, \quad \text{for } t \in [T_2, \infty)_{\mathbb{T}}, \quad (78)$$

where $L := \inf_{0 \leq j \leq N} \{l^{\gamma_j}\} > 0$. Then from (1), we have

$$\begin{aligned} -x^{[n]}(t) &= \sum_{j=0}^N p_j(t) \phi_{\gamma_j}(x(\varphi_j(t))) \geq L \sum_{j=0}^N p_j(t) \\ &= L \bar{p}(t). \end{aligned} \quad (79)$$

Integrating from t to $v \in [t, \infty)_{\mathbb{T}}$, we get

$$-x^{[n-1]}(v) + x^{[n-1]}(t) = L \int_t^v \bar{p}(s) \Delta s. \quad (80)$$

And by (48) we see that $x^{[n-1]}(v) > 0$. Hence by taking limits as $v \rightarrow \infty$ we have

$$x^{[n-1]}(t) \geq L \int_t^\infty \bar{p}(s) \Delta s, \quad (81)$$

which implies

$$\begin{aligned} (x^{[n-2]}(t))^\Delta &\geq L^{1/\alpha_{n-1}} \left[\frac{1}{r_{n-1}(t)} \int_t^\infty \bar{p}(s) \Delta s \right]^{1/\alpha_{n-1}} \\ &= L^{1/\alpha_{n-1}} \bar{p}_1(t). \end{aligned} \quad (82)$$

Integrating above inequality from t to $v \in [t, \infty)_{\mathbb{T}}$ and then taking $v \rightarrow \infty$, we get

$$-x^{[n-2]}(t) \geq L^{1/\alpha_{n-1}} \int_t^\infty \bar{p}_1(s) \Delta s, \quad (83)$$

which implies

$$\begin{aligned} -(x^{[n-3]}(t))^\Delta &\geq L^{1/\alpha[n-2, n-1]} \left[\frac{1}{r_{n-2}(t)} \int_t^\infty \bar{p}_1(s) \Delta s \right]^{1/\alpha_{n-2}} \\ &= L^{1/\alpha[n-2, n-1]} \bar{p}_2(t). \end{aligned} \quad (84)$$

Again, integrating above inequality from T_2 to $t \in [T_2, \infty)_{\mathbb{T}}$ and noting that $x^{[n-3]} > 0$ eventually, we get

$$x^{[n-3]}(T_2) - x^{[n-3]}(t) \geq L^{1/\alpha[n-2, n-1]} \int_{T_2}^t \bar{p}_2(s) \Delta s. \quad (85)$$

Then by (72), we have $\lim_{t \rightarrow \infty} x^{[n-3]}(t) = -\infty$, which contradicts the fact that $x^{[n-3]} > 0$ on $[T_2, \infty)_{\mathbb{T}}$. This shows that if (72) holds, then $m = n - 1$. The rest of proof of (I) is similar to proof (I) of Theorems 4 and 5, respectively, with $m = n - 1$ and hence can be omitted.

(II) When $m = 0$ (in this case n is odd), therefore

$$(-1)^k x^{[k]} > 0 \quad \text{for } k = 0, \dots, n. \quad (86)$$

Since $x^\Delta < 0$ eventually, then $\lim_{t \rightarrow \infty} x(t) = l \geq 0$. Assume that $l > 0$. Then for sufficiently large $T_2 \in [T_1, \infty)_{\mathbb{T}}$, we have $x(\varphi_j(t)) \geq l$ for $t \geq T_2$ and $j = 1, \dots, N$. It follows that

$$\phi_{\gamma_j}(x(\varphi_j(t))) \geq l^{\gamma_j} \geq L \quad \text{for } t \in [T_2, \infty)_{\mathbb{T}}, \quad (87)$$

where $L := \inf_{0 \leq j \leq N} \{l^{\gamma_j}\} > 0$. Then from (1), we have

$$\begin{aligned} -x^{[n]}(t) &= \sum_{j=0}^N p_j(t) \phi_{\gamma_j}(x(\varphi_j(t))) \geq L \sum_{j=0}^N p_j(t) \\ &= L \bar{p}(t). \end{aligned} \quad (88)$$

Integrating from t to $v \in [t, \infty)_{\mathbb{T}}$, we get

$$-x^{[n-1]}(v) + x^{[n-1]}(t) = L \int_t^v \bar{p}(s) \Delta s. \quad (89)$$

And using (48) we see that $x^{[n-1]}(v) > 0$. Hence by taking limits as $v \rightarrow \infty$ we have

$$x^{[n-1]}(t) \geq L \int_t^\infty \bar{p}(s) \Delta s, \quad (90)$$

which implies

$$\begin{aligned} (x^{[n-2]}(t))^\Delta &\geq L^{1/\alpha_{n-1}} \left[\frac{1}{r_{n-1}(t)} \int_t^\infty \bar{p}(s) \Delta s \right]^{1/\alpha_{n-1}} \\ &= L^{1/\alpha_{n-1}} \bar{p}_1(t). \end{aligned} \quad (91)$$

Integrating above inequality from t to $v \in [t, \infty)_{\mathbb{T}}$ and then taking $v \rightarrow \infty$, we get

$$-x^{[n-2]}(t) \geq L^{1/\alpha_{n-1}} \int_t^\infty \bar{p}_1(s) \Delta s, \quad (92)$$

which implies

$$\begin{aligned} -(x^{[n-3]}(t))^\Delta &\geq L^{1/\alpha[n-2, n-1]} \left[\frac{1}{r_{n-2}(t)} \int_t^\infty \bar{p}_1(s) \Delta s \right]^{1/\alpha_{n-2}} \\ &= L^{1/\alpha[n-2, n-1]} \bar{p}_2(t). \end{aligned} \quad (93)$$

Again, integrating above inequality from T_2 to $t \in [T_2, \infty)_{\mathbb{T}}$ and noting that $x^{[n-3]} > 0$ eventually, we get

$$x^{[n-3]}(T_2) - x^{[n-3]}(t) \geq L^{1/\alpha[n-2, n-1]} \int_{T_2}^t \bar{P}_2(s) \Delta s. \quad (94)$$

Then by (72), we have $\lim_{t \rightarrow \infty} x^{[n-3]}(t) = -\infty$, which contradicts the fact that $x^{[n-3]} > 0$ on $[T_2, \infty)_{\mathbb{T}}$. This shows that $\lim_{t \rightarrow \infty} x(t) = 0$. This completes the proof. $\square$

Remark 9. The conclusion of Theorems 7 and 8 remains intact if assumption (72) is replaced by one of the following conditions:

$$\begin{aligned} \int_T^\infty \bar{P}_0(t) \Delta t &= \infty \\ \text{or} \quad \int_T^\infty \bar{P}_1(t) \Delta t &= \infty. \end{aligned} \quad (95)$$

Theorem 10. Assume that

$$\int_T^\infty \bar{P}_0(t) \Delta t = \infty. \quad (96)$$

Then

- (1) if n is even, every solution of (1) is oscillatory,
- (2) if n is odd, then every solution of (1) either is oscillatory or tends to zero eventually.

Proof. Assume that (1) has a nonoscillatory solution x on $[t_0, \infty)_{\mathbb{T}}$. Then, without loss of generality, it is sufficiently large, such that $x(t) > 0$ and $x(\varphi_j(t)) > 0$, for $t \in [t_0, \infty)_{\mathbb{T}}$ and $j = 0, 1, \dots, N$. By Lemma 3, there exists an integer m , $0 \leq m < n$, with $n+m$ being odd such that (24) and (25) hold for $t \geq T_1 \in [t_0, \infty)_{\mathbb{T}}$.

(I) When $m \geq 1$, this implies that $x(t)$ is strictly increasing on $[t_1, \infty)_{\mathbb{T}}$. Then for sufficiently large $T_2 \in [T_1, \infty)_{\mathbb{T}}$, we have $x(\varphi_j(t)) \geq l$ for $t \geq T_2$. It follows that

$$\phi_{\gamma_j}(x(\varphi_j(t))) \geq l^{\gamma_j} \geq L \quad \text{for } t \in [T_2, \infty)_{\mathbb{T}}, \quad (97)$$

where $L := \inf_{0 \leq j \leq N} \{l^{\gamma_j}\} > 0$. Equation (1) becomes

$$\begin{aligned} -(x^{[n-1]}(t))^\Delta &= \sum_{j=0}^N p_j(t) \phi_{\gamma_j}(x(\varphi_j(t))) \geq L \sum_{j=0}^N p_j(t) \\ &= L \bar{p}(t) \quad \text{for } t \in [t_2, \infty)_{\mathbb{T}}. \end{aligned} \quad (98)$$

Replacing t by s in (98), integrating from t_2 to $t \in [t_2, \infty)_{\mathbb{T}}$, we obtain

$$-x^{[n-1]}(t) + x^{[n-1]}(t_2) \geq L \int_{t_2}^t \bar{p}(s) \Delta s. \quad (99)$$

Hence by (96), we have $\lim_{t \rightarrow \infty} x^{[n-1]}(t) = -\infty$, which contradicts the fact that $x^{[n-1]}(t) > 0$ eventually.

(II) When $m = 0$ (in this case n is odd), therefore

$$(-1)^k x^{[k]} > 0 \quad \text{for } k = 0, \dots, n. \quad (100)$$

This implies that $x(t)$ is strictly decreasing on $[t_1, \infty)_{\mathbb{T}}$. Then $\lim_{t \rightarrow \infty} x(t) = l \geq 0$. Assume that $l > 0$. Then for sufficiently large $t_2 \in [t_1, \infty)_{\mathbb{T}}$, we have $x(\varphi_j(t)) \geq l$ for $t \geq t_2$. It follows that

$$\phi_{\gamma_j}(x(\varphi_j(t))) \geq l^{\gamma_j} \geq L \quad \text{for } t \in [t_2, \infty)_{\mathbb{T}}, \quad (101)$$

where $L := \inf_{0 \leq j \leq N} \{l^{\gamma_j}\} > 0$. As is case (I), we get a contradiction with the fact that $x^{[n-1]}(t) > 0$ eventually. This shows that $\lim_{t \rightarrow \infty} x(t) = 0$. This completes the proof. $\square$

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

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