Research Article

The Hyper-Wiener Index of Trees of Order n with Diameter d

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The hyper-Wiener index is a kind of extension of the Wiener index, used for predicting physicochemical properties of organic compounds. The hyper-Wiener index $WW(G)$ is defined as $WW(G) = (1/2) \sum_{u,v \in V(G)} (d_G(u, v) + d_G^2(u, v))$ with the summation going over all pairs of vertices in G , and $d_G(u, v)$ denotes the distance of the two vertices u and v in the graph G . In this paper, we obtain the second-minimum hyper-Wiener indices among all the trees with n vertices and diameter d and characterize the corresponding extremal graphs.

1. Introduction

Let G be a simple graph of order n with vertex set $V(G)$ and edge set $E(G)$. The distance between two vertices u, v of G , denoted by $d_G(u, v)$ or $d(u, v)$, is defined as the minimum length of the paths between u and v in G . The diameter of a graph G is the maximum distance between any two vertices of G .

The Wiener index of a graph G , denoted by $W(G)$, is one of the oldest topological indexes, which was first introduced by Wiener [1] in 1947. It is defined as $W(G) = \sum_{u,v \in V(G)} d_G(u, v)$ where the summation goes over all pairs of vertices of G . The hyper-Wiener index of acyclic graphs was introduced by Randić in 1993 [2]. Then Klein et al. [3] extended the definition for all connected graphs, as a generalization of the Wiener index. Similar to the symbol $W(G)$ for the Wiener index, the hyper-Wiener index is traditionally denoted by $WW(G)$. The hyper-Wiener index of a graph G is defined as

$$WW(G) = \frac{1}{2} \left(\sum_{u,v \in V(G)} d_G(u, v) + \sum_{u,v \in V(G)} d_G^2(u, v) \right). \quad (1)$$

Let $S(G) = \sum_{u,v \in V(G)} d_G^2(u, v)$. Then

$$WW(G) = \frac{1}{2} W(G) + \frac{1}{2} S(G). \quad (2)$$

We denote $D_G(u) = \sum_{v \in V(G)} d_G(u, v)$, $DD_G(u) = \sum_{v \in V(G)} d_G^2(u, v)$, and then

$$\begin{aligned} W(G) &= \frac{1}{2} \sum_{u \in V(G)} D_G(u), \\ S(G) &= \frac{1}{2} \sum_{u \in V(G)} DD_G(u). \end{aligned} \quad (3)$$

Recently, the properties and uses of the hyper-Wiener index have received a lot of attention. Feng and Ilić [4] studied hyper-Wiener indices of graphs with given matching number. Feng et al. [5] researched the hyper-Wiener index of unicyclic graphs. Feng et al. [6] discussed the hyper-Wiener index of bicyclic graphs. Gutman [7] obtained the relation between hyper-Wiener and Wiener index. Xu and Trinajstić [8] discussed hyper-Wiener index of graphs with cut edges. M.-H. Liu and B. L. Liu [9] determined trees with

FIGURE 1

the seven smallest and fifteen greatest hyper-Wiener indices. Yu et al. [10] studied the hyper-Wiener index of trees with given parameters.

All graphs considered in this paper are finite and simple. Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G)$. For a vertex $v \in V(G)$, the degree and the neighborhood of v are denoted by $d_G(v)$ and $N_G(v)$ (or written as $d(v)$ and $N(v)$ for short). A vertex v of degree 1 is called pendant vertex. An edge $e = uv$ incident with the pendant vertex v is called a pendant edge. For a subset U of $V(G)$, let $G - U$ be the subgraph of G obtained from G by deleting the vertices of U and the edges incident with them. Similarly, for a subset E' of $E(G)$, we denote by $G - E'$ the subgraph of G obtained from G by deleting the edges of E' . If $U = \{v\}$ and $E' = \{uv\}$, the subgraphs $G - U$ and $G - E'$ will be written as $G - v$ and $G - uv$ for short, respectively. For any two nonadjacent vertices u and v in graph G , we use $G + uv$ to denote the graph obtained from G by adding a new edge uv . Denote by S_n and P_n the star and the path on n vertices, respectively.

A tree is a connected acyclic graph. Let T be a tree of order n with diameter d . If $d = n - 1$, then $T \cong P_n$; and if $d = 2$, then $T \cong S_n$. Therefore, in the following, we assume that $3 \leq d \leq n - 2$. Let $\mathcal{T}_{n,d} = \{T : T \text{ is a tree with order } n \text{ and diameter } d, 3 \leq d \leq n - 2\}$. Liu and Pan [11] characterized the minimum and second-minimum Wiener indices of trees in the set $\mathcal{T}_{n,d}$ ($3 \leq d \leq n - 2$), and Yu et al. [10] characterized the minimum hyper-Wiener index of all trees on n vertices with diameter d . Motivated by these articles, we will characterize the second-minimum hyper-Wiener indices of trees in the set $\mathcal{T}_{n,d}$ ($3 \leq d \leq n - 2$) in this paper.

2. Preliminaries

Lemma 1 (see [8]). *Let H , X , and Y be three connected graphs disjoint in pair. Suppose that u, v are two vertices of H , v_1 is a vertex of X , and u_1 is a vertex of Y . Let G be the graph obtained from H , X , and Y by identifying v with v_1 and u with u_1 , respectively. Let G_1 be the graph obtained from H , X , and Y by identifying three vertices v, v_1 , and u_1 , and let G_2 be the*

graph obtained from H , X , and Y by identifying three vertices u, v_1 , and u_1 . Then one has

$$WW(G_1) < WW(G) \quad (4)$$

$$\text{or } WW(G_2) < WW(G).$$

By Lemma 1, we have the following result.

Corollary 2. *Let G be a graph and $u, v \in V(G)$. Suppose that $G_{p,q}^*$ is the graph obtained from G by attaching p, q pendant vertices to u, v , respectively. Then*

$$WW(G_{p-i,q+i}^*) < WW(G_{p,q}^*), \quad 1 \leq i \leq p, \quad (5)$$

or

$$WW(G_{p+i,q-i}^*) < WW(G_{p,q}^*), \quad 1 \leq i \leq q. \quad (6)$$

Let G_1, G_2 be two connected graphs with $V(G_1) \cap V(G_2) = \{v\}$. Let $G_1 \vee G_2$ be a graph with $V(G_1) \cup V(G_2)$ as its vertex set and $E(G_1) \cup E(G_2)$ as its edge set. We have the following result.

Lemma 3 (see [8]). *Let H be a connected graph, T_m be a tree of order m , and $V(H) \cap V(T_m) = \{v\}$. Then*

$$WW(H \vee T_m) \geq WW(H \vee S_m), \quad (7)$$

and equality holds if and only if $H \vee T_m \cong H \vee S_m$, where v is the center of star S_m .

Lemma 4 (see [6]). *Let G be a connected graph of order n , v be a pendant vertex of G , and $vw \in E(G)$. Then*

- (1) $W(G) = W(G - v) + D_{G-v}(w) + n - 1$;
- (2) $S(G) = S(G - v) + DD_{G-v}(w) + 2D_{G-v}(w) + n - 1$.

By Lemma 4 and the definition of hyper-Wiener index, we have the following result.

Corollary 5. *Let G be a connected graph of order n , v be a pendant vertex of G , and $vw \in E(G)$. Then*

$$WW(G) = WW(G - v) + \frac{1}{2}DD_{G-v}(w) + \frac{3}{2}D_{G-v}(w) + n - 1. \quad (8)$$

Lemma 6 (see [9]). Let G be a connected graph on $n \geq 2$ vertices and v be a vertex of G . Let $G_{k,l}$ be the graph obtained from G by attaching two new paths $P : vv_1v_2 \cdots v_k$ and $Q : vv_1u_2 \cdots u_l$ of length k and l at v , respectively, where $u_1, \dots, u_l$ and $v_1, \dots, v_k$ are distinct new vertices. Let $G_{k+1,l-1} = G_{k,l} - u_{l-1}u_l + v_ku_l$. If $k \geq l \geq 1$, then

$$WW(G_{k,l}) < WW(G_{k+1,l-1}). \quad (9)$$

Let v_1 be a vertex of graph G , and $v_2, \dots, v_{t+s}, u_0$ are distinct new vertices (not in G). Let G' be the graph $G \vee P$, where P is a new path $v_1v_2 \cdots v_{t+s}$. Let $M_{t,t+s} = G' + v_tu_0$ and $M_{t+i,t+s} = G' + v_{t+i}u_0$, where $1 \leq i \leq s$.

Lemma 7 (see [9]). Suppose G is a connected graph on $n \geq 2$ vertices or has only one vertex. If $t \geq s \geq 1$, then $WW(M_{t,t+s}) \leq WW(M_{t+i,t+s})$, where $1 \leq i \leq s$.

3. Main Results

In this section, we will give the second-minimum hyper-Wiener index in the set $\mathcal{T}_{n,d}$ ($3 \leq d \leq n-2$). In order to formulate our results, we need to define some trees as follows.

Let $T_{n,d}(p_1, \dots, p_{d-1})$ be a tree of order n obtained from a path $P_{d+1} = v_0v_1 \cdots v_{d-1}v_d$ by attaching p_i pendant vertices to v_i , $1 \leq i \leq d-1$, respectively, where $n = d+1 + \sum_{i=1}^{d-1} p_i$, $p_i \geq 0$, $i = 1, 2, \dots, d-1$. Denote $T_{n,d,i} = T_{n,d}(\underbrace{0, \dots, 0}_{i-1}, 0, n-d-i, 0, \dots, 0)$ (see Figure 1(b)), and we note that $T_{n,d,i} \cong T_{n,d,d-i}$.

Denote $Z_{n,d,i,j} = T_{n,d}(\underbrace{0, \dots, 0}_{i-1}, n-d-2, \underbrace{0, \dots, 0}_{j-i-1}, 1, 0, \dots, 0)$ (see Figure 1(a)).

For $2 \leq i \leq d-2$, let $X_{n,d,i}$ be a graph obtained from $T_{n-1,d,i}$ by attaching a pendant vertex to one pendant vertex of $T_{n-1,d,i}$ except for v_0, v_d (see Figure 1(d)). Then $X_{n,d,i} \cong X_{n,d,d-i}$. Let $Y_{n,d,i}$ be a graph obtained from $T_{d+2,d,i}$ by attaching $n-d-2$ pendant vertices to one pendant vertex of $T_{d+2,d,i}$ except for v_0, v_d (see Figure 1(c)). Then $Y_{n,d,i} \cong Y_{n,d,d-i}$.

Yu et al. [10] characterized that $T_{n,d,\lfloor d/2 \rfloor}$ had the minimum hyper-Wiener index of all trees on n vertices with diameter d . In the following, we firstly give possible trees with the second-minimum hyper-Wiener indices in the set $\mathcal{T}_{n,d}$ ($3 \leq d \leq n-2$).

Denote $\mathcal{T}_1 = \{T_{n,d,i} : 1 \leq i \leq d-1\}$, $\mathcal{T}_2 = \{X_{n,d,i} : 2 \leq i \leq d-2\}$, $\mathcal{T}_3 = \{Y_{n,d,i} : 2 \leq i \leq d-2\}$, and $\mathcal{T}_4 = \{Z_{n,d,i,j} : 1 \leq i < j \leq d-1\}$.

Lemma 8. Let $T \in \mathcal{T}_{n,d}$. Then $WW(T) \geq WW(T_{n,d,i})$, or $WW(T) \geq WW(X_{n,d,i})$, or $WW(T) \geq WW(Y_{n,d,i})$, or $WW(T) \geq WW(Z_{n,d,i,j})$.

Proof. If $T \in \mathcal{T}_1$, the conclusion is obvious. If $T \notin \mathcal{T}_1$, let $P_{d+1} = v_0v_1 \cdots v_{d-1}v_d$ be a path of length d in T with $d(v_0) = d(v_d) = 1$. Let $V_d = \{v_i : d(v_i) \geq 3, 1 \leq i \leq d-1\}$. Since $n \geq d+2$, $V_d \neq \emptyset$, we consider two cases.

Case 1 ($|V_d| \geq 2$). By Lemma 3,

$$WW(T) \geq WW(T_{n,d}(p_1, \dots, p_{d-1})). \quad (10)$$

By Corollary 2, $WW(T_{n,d}(p_1, \dots, p_{d-1})) \geq WW(T_{n,d}(\underbrace{0, \dots, 0}_{i-1}, \underbrace{0, \dots, 0}_{j-i-1}, q_j, 0, \dots, 0)) \geq WW(Z_{n,d,i,j})$,

where $p_1 + \dots + p_{d-1} = q_i + q_j$.

So

$$WW(T) \geq WW(Z_{n,d,i,j}). \quad (11)$$

Case 2 ($|V_d| = 1$). Let $v_i \in V_d$ and $N(v_i) \setminus \{v_{i-1}, v_{i+1}\} = \{w_1, w_2, \dots, w_s\}$ with $d(w_j) \geq 2$, $1 \leq j \leq s$, and $d(w_{r+1}) = \dots = d(w_s) = 1$. Then $r \geq 1$ as $T \notin \mathcal{T}_1$. Let $T_i(w_j)$ be subtrees of $T - v_i$ which contain w_j , and $|V(T_i(w_j))| = s_j + 1$, $1 \leq j \leq r$. By Lemma 3, we can obtain a tree T_1 obtained from $T_{d+s+1,d,i}$ by attaching s_j pendant vertices to w_j , $1 \leq j \leq s$, respectively, such that $WW(T) > WW(T_1)$. By Corollary 2, we can obtain a tree $T_2 \in \mathcal{T}_2 \cup \mathcal{T}_3$ such that $WW(T_1) \geq WW(T_2)$. So $WW(T) \geq WW(X_{n,d,i})$, or $WW(T) \geq WW(Y_{n,d,i})$. $\square$

Lemma 9. Let $P = v_0v_1 \cdots v_d$ be a path of order $d+1$. Then

$$\begin{aligned} D_P(v_j) &= \frac{2j^2 - 2dj + d^2 + d}{2}, \\ DD_P(v_j) &= \frac{6j^2 + 6dj^2 - 6d^2j - 6dj + 2d^3 + 3d^2 + d}{6}, \end{aligned} \quad (12)$$

for $1 \leq j \leq d-1$. Moreover, if $1 \leq i < j \leq d/2$, $D_P(v_i) > D_P(v_j)$, and $DD_P(v_i) > DD_P(v_j)$; if $d/2 \leq i < j \leq (d-1)$, $D_P(v_i) < D_P(v_j)$, and $DD_P(v_i) < DD_P(v_j)$.

Proof. By the definition of $D_G(v)$ and $DD_G(v)$, we have

$$\begin{aligned} D_P(v_j) &= (1 + 2 + \dots + j) + (1 + 2 + \dots + (d-j)) \\ &= \frac{2j^2 - 2dj + d^2 + d}{2}, \\ DD_P(v_j) &= (1^2 + 2^2 + \dots + j^2) + (1^2 + 2^2 + \dots + (d-j)^2) \\ &= \frac{6j^2 + 6dj^2 - 6d^2j - 6dj + 2d^3 + 3d^2 + d}{6}. \end{aligned} \quad (13)$$

Then $D_P(v_i) - D_P(v_j) = (i-j)(i+j-d)$, $DD_P(v_i) - DD_P(v_j) = (d+1)(i-j)(i+j-d)$, and thus the results hold. $\square$

Lemma 10. (1) For any $T \in \mathcal{T}_1$, $WW(T) \geq WW(T_{n,d,\lfloor d/2 \rfloor})$, with equality if and only if $T \cong T_{n,d,\lfloor d/2 \rfloor}$.

(2) For any $T \in \mathcal{T}_2$, $WW(T) \geq WW(X_{n,d,\lfloor d/2 \rfloor})$, with equality if and only if $T \cong X_{n,d,\lfloor d/2 \rfloor}$.

(3) For any $T \in \mathcal{T}_3$, $WW(T) \geq WW(Y_{n,d,\lfloor d/2 \rfloor})$, with equality if and only if $T \cong Y_{n,d,\lfloor d/2 \rfloor}$.

(4) For any $T \in \mathcal{T}_4$, $WW(T) \geq WW(Z_{n,d,\lfloor d/2 \rfloor, \lfloor d/2 \rfloor + 1})$, with equality if and only if $T \cong Z_{n,d,\lfloor d/2 \rfloor, \lfloor d/2 \rfloor + 1}$.

Proof. Let $v = v_i$ and two paths $P = v_0v_1 \cdots v_i$, $Q = v_iv_{i+1} \cdots v_d$ in Lemma 6. We obtain the results of (1), (2), and (3). In the following, we prove (4).

Let $T = Z_{n,d,i,j}$ be a tree which $WW(T)$ is as small as possible in the set $\mathcal{T}_4$. Let $P = v_0 v_1 \cdots v_{d-1} v_d$ be a path of length d in T with $d(v_0) = d(v_d) = 1$, and v_{d+1} be a pendant vertex of T which is adjacent to v_j . By Lemma 7 and the choice of T , we know that $j = i + 1$. Note that $Z_{n,d,i,i+1} - v_0 \cong Z_{n,d,i-1,i} - v_d$, and then by Corollary 5, we have

$$\begin{aligned} & WW(Z_{n,d,i,i+1}) - WW(Z_{n,d,i-1,i}) \\ &= \frac{3}{2}((n-d-2)(2i-d-1) + 2i-d+1) \\ &+ \frac{1}{2}((n-d-2)(2i-d-1)(d+1) \\ &+ (d+1)(2i-d+1)) = \frac{1}{2}(d+4) \\ &\cdot ((n-d-2)(2i-d-1) + 2i-d+1). \end{aligned} \quad (14)$$

Thus, if $i > \lfloor d/2 \rfloor$, $WW(Z_{n,d,i,i+1}) - WW(Z_{n,d,i-1,i}) > 0$; if $i \leq \lfloor d/2 \rfloor$, $WW(Z_{n,d,i,i+1}) - WW(Z_{n,d,i-1,i}) \leq 0$, with equality if and only if $d = 2i = n - 3$ or $d = 2i + 1 = n - 2$. So, for any $T \in \mathcal{T}_4$, $WW(T) \geq WW(Z_{n,d,\lfloor d/2 \rfloor, \lfloor d/2 \rfloor + 1})$, with equality if and only if $T \cong Z_{n,d,\lfloor d/2 \rfloor, \lfloor d/2 \rfloor + 1}$. $\square$

Lemma 11. Let $3 \leq d \leq n - 3$, and then

- (1) $WW(Z_{n,d,\lfloor d/2 \rfloor, \lfloor d/2 \rfloor + 1}) < WW(X_{n,d,\lfloor d/2 \rfloor}) \leq WW(Y_{n,d,\lfloor d/2 \rfloor})$;
- (2) $WW(Z_{n,d,\lfloor d/2 \rfloor, \lfloor d/2 \rfloor + 1}) < WW(T_{n,d,\lfloor d/2 \rfloor - 1})$.

Proof. (1) We denote $Z = Z_{n,d,\lfloor d/2 \rfloor, \lfloor d/2 \rfloor + 1}$, $X = X_{n,d,\lfloor d/2 \rfloor}$, $Y = Y_{n,d,\lfloor d/2 \rfloor}$.

Note that $Z_{n,d,\lfloor d/2 \rfloor, \lfloor d/2 \rfloor + 1} - v_{d+1} \cong X_{n,d,\lfloor d/2 \rfloor} - v_{d+2}$, and then by Corollary 5, we have

$$\begin{aligned} & WW(Z_{n,d,\lfloor d/2 \rfloor, \lfloor d/2 \rfloor + 1}) - WW(X_{n,d,\lfloor d/2 \rfloor}) \\ &= \frac{3}{2}(D_{Z-v_{d+1}}(v_{\lfloor d/2 \rfloor + 1}) - D_{X-v_{d+2}}(v_{d+1})) \\ &+ \frac{1}{2}(DD_{Z-v_{d+1}}(v_{\lfloor d/2 \rfloor + 1}) - DD_{X-v_{d+2}}(v_{d+1})) \\ &= \frac{3}{2}\left(2 - 2d + 2\left\lfloor \frac{d}{2} \right\rfloor\right) \\ &+ \frac{1}{2}\left(4 - 2\left(d - \left\lfloor \frac{d}{2} \right\rfloor\right)^2 - 2\left(d - \left\lfloor \frac{d}{2} \right\rfloor\right)\right) \\ &= 5 - 4d + 4\left\lfloor \frac{d}{2} \right\rfloor - \left(d - \left\lfloor \frac{d}{2} \right\rfloor\right)^2 \\ &\leq 5 - 4d + 4\left\lfloor \frac{d}{2} \right\rfloor \leq 5 - 2d < 0. \end{aligned} \quad (15)$$

Namely,

$$WW(Z_{n,d,\lfloor d/2 \rfloor, \lfloor d/2 \rfloor + 1}) < WW(X_{n,d,\lfloor d/2 \rfloor}). \quad (16)$$

Note that $X_{d+3,d,\lfloor d/2 \rfloor} \cong Y_{d+3,d,\lfloor d/2 \rfloor}$, and then by Lemma 4, we have

$$\begin{aligned} & WW(X_{n,d,\lfloor d/2 \rfloor}) - WW(Y_{n,d,\lfloor d/2 \rfloor}) \\ &= \frac{1}{2}(W(X_{n,d,\lfloor d/2 \rfloor}) - W(Y_{n,d,\lfloor d/2 \rfloor})) \\ &+ \frac{1}{2}(S(X_{n,d,\lfloor d/2 \rfloor}) - S(Y_{n,d,\lfloor d/2 \rfloor})) \\ &= \frac{1}{2}(W(X - v_{d+3}) - W(Y - v_{d+3})) \\ &+ D_{X-v_{d+3}}(v_{\lfloor d/2 \rfloor}) - D_{Y-v_{d+3}}(v_{d+1}) \\ &+ \frac{1}{2}(S(X - v_{d+3}) - S(Y - v_{d+3})) \\ &+ DD_{X-v_{d+3}}(v_{\lfloor d/2 \rfloor}) - DD_{Y-v_{d+3}}(v_{d+1}) \\ &+ 2(D_{X-v_{d+3}}(v_{\lfloor d/2 \rfloor}) - D_{Y-v_{d+3}}(v_{d+1})) \\ &= \frac{1}{2}(W(X_{n-1,d,\lfloor d/2 \rfloor}) - W(Y_{n-1,d,\lfloor d/2 \rfloor}) - d + 1) \\ &+ \frac{1}{2}\left(S(X_{n-1,d,\lfloor d/2 \rfloor}) - S(Y_{n-1,d,\lfloor d/2 \rfloor}) + 5\right. \\ &\quad \left. - \left(\left\lfloor \frac{d}{2} \right\rfloor + 1\right)^2 - \left(d - \left\lfloor \frac{d}{2} \right\rfloor + 1\right)^2\right) + 2(-d + 1) \\ &< \frac{1}{2}(W(X_{n-1,d,\lfloor d/2 \rfloor}) - W(Y_{n-1,d,\lfloor d/2 \rfloor}) - d + 1) \\ &+ \frac{1}{2}(S(X_{n-1,d,\lfloor d/2 \rfloor}) - S(Y_{n-1,d,\lfloor d/2 \rfloor}) - 2d + 7) \\ &< WW(X_{n-1,d,\lfloor d/2 \rfloor}) - WW(Y_{n-1,d,\lfloor d/2 \rfloor}) < \cdots \\ &< WW(X_{d+3,d,\lfloor d/2 \rfloor}) - WW(Y_{d+3,d,\lfloor d/2 \rfloor}) = 0. \end{aligned} \quad (17)$$

Namely,

$$WW(X_{n,d,\lfloor d/2 \rfloor}) \leq WW(Y_{n,d,\lfloor d/2 \rfloor}). \quad (18)$$

(2) By Lemma 9, $D_{P_{d+1}}(v_{\lfloor d/2 \rfloor + 1}) \leq D_{P_{d+1}}(v_{\lfloor d/2 \rfloor - 1})$, and $DD_{P_{d+1}}(v_{\lfloor d/2 \rfloor + 1}) \leq DD_{P_{d+1}}(v_{\lfloor d/2 \rfloor - 1})$. By Corollary 5 and Lemma 9,

$$\begin{aligned} & WW(Z_{n,d,\lfloor d/2 \rfloor, \lfloor d/2 \rfloor + 1}) - WW(T_{n,d,\lfloor d/2 \rfloor - 1}) \\ &= WW(T_{n-1,d,\lfloor d/2 \rfloor}) - WW(T_{n-1,d,\lfloor d/2 \rfloor - 1}) + \frac{3}{2} \\ &\cdot D_{P_{d+1}}(v_{\lfloor d/2 \rfloor + 1}) - \frac{3}{2}D_{P_{d+1}}(v_{\lfloor d/2 \rfloor - 1}) + \frac{1}{2} \\ &\cdot DD_{P_{d+1}}(v_{\lfloor d/2 \rfloor + 1}) - \frac{1}{2}DD_{P_{d+1}}(v_{\lfloor d/2 \rfloor - 1}) + 3(n-d \\ &- 2) \leq WW(T_{n-1,d,\lfloor d/2 \rfloor}) - WW(T_{n-1,d,\lfloor d/2 \rfloor - 1}) \\ &+ 3(n-d-2) = \frac{1}{2}(DD_{T_{n-2,d-1,\lfloor d/2 \rfloor - 1}}(v_1) \end{aligned}$$

$$\begin{aligned}
& -DD_{T_{n-2,d-1,\lfloor d/2 \rfloor-1}}(v_{d-1}) + 3D_{T_{n-2,d-1,\lfloor d/2 \rfloor-1}}(v_1) \\
& - 3D_{T_{n-2,d-1,\lfloor d/2 \rfloor-1}}(v_{d-1}) + 3(n-d-2) \\
& = \frac{1}{2} \left(\left(\left\lfloor \frac{d}{2} \right\rfloor \right)^2 - \left(d - \left\lfloor \frac{d}{2} \right\rfloor + 1 \right)^2 \right. \\
& \quad \left. + 3 \left(\left\lfloor \frac{d}{2} \right\rfloor - \left(d - \left\lfloor \frac{d}{2} \right\rfloor + 1 \right) \right) \right) (n-d-2) + 3(n \\
& \quad - d - 2) = \frac{1}{2} \left(-d^2 - 5d + 2 + 8 \left\lfloor \frac{d}{2} \right\rfloor + 2d \left\lfloor \frac{d}{2} \right\rfloor \right) \\
& \quad \cdot (n-d-2) \leq \frac{1}{2} (-d^2 - 5d + 2 + 4d + d^2) (n-d \\
& \quad - 2) = \frac{1}{2} (-d+2) (n-d-2) < 0.
\end{aligned} \tag{19}$$

Namely,

$$WW(Z_{n,d,\lfloor d/2 \rfloor, \lfloor d/2 \rfloor+1}) < WW(T_{n,d,\lfloor d/2 \rfloor-1}). \tag{20}$$

□

Theorem 12. Let $T \in \mathcal{T}_{n,d} \setminus \{T_{n,d,\lfloor d/2 \rfloor}\}$. Then

$$WW(T) \geq WW(Z_{n,d,\lfloor d/2 \rfloor, \lfloor d/2 \rfloor+1}), \tag{21}$$

with equality if and only if $T \cong Z_{n,d,\lfloor d/2 \rfloor, \lfloor d/2 \rfloor+1}$.

Proof. If $d = n - 2$, note that $\mathcal{T}_{n,n-2}$ contains no other trees than $T_{n,d,\lfloor d/2 \rfloor}, T_{n,d,\lfloor d/2 \rfloor-1}, \dots, T_{n,d,2}, T_{n,d,1}$, and by Lemma 6, take $v = v_j$ and two paths $P = v_0 v_1 \dots v_j$, $Q = v_j v_{j+1} \dots v_d$, and we have $WW(T_{n,d,\lfloor d/2 \rfloor}) < WW(T_{n,d,\lfloor d/2 \rfloor-1}) < \dots < WW(T_{n,d,2}) < WW(T_{n,d,1})$.

So,

$$WW(T) \geq WW(T_{n,d,\lfloor d/2 \rfloor-1}), \tag{22}$$

with equality if and only if $T \cong T_{n,d,\lfloor d/2 \rfloor-1} \cong Z_{n,d,\lfloor d/2 \rfloor, \lfloor d/2 \rfloor+1}$.

If $3 \leq d \leq n - 3$, by Lemmas 8, 10, and 11, we have

$$WW(T) \geq WW(Z_{n,d,\lfloor d/2 \rfloor, \lfloor d/2 \rfloor+1}), \tag{23}$$

with equality if and only if $T \cong Z_{n,d,\lfloor d/2 \rfloor, \lfloor d/2 \rfloor+1}$. □

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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