Research Article

Chaos Control on a Duopoly Game with Homogeneous Strategy

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We study the dynamics of a nonlinear discrete-time duopoly game, where the players have homogenous knowledge on the market demand and decide their outputs based on adaptive expectation. The Nash equilibrium and its local stability are investigated. The numerical simulation results show that the model may exhibit chaotic phenomena. Quasiperiodicity is also found by setting the parameters at specific values. The system can be stabilized to a stable state by using delayed feedback control method. The discussion of control strategy shows that the effect of both firms taking control method is better than that of single firm taking control method.

1. Introduction

An oligopoly is a market form in which a market or industry is dominated by a small number of sellers. Thus, they have power to decide the market price and the yields. The behavior of one player inevitable influences the other players' behavior. To maximize their profits, each oligopolist is likely to be aware of the actions of the others.

Cournot, in 1838, introduced the first formal theory of oligopoly [1]. In Cournot model, each oligopolist assumes that other firms hold their outputs constant, and all oligopolists select a quantity based on others' outputs to maximize profits. However, oligopoly game becomes complex when the players adopt dynamic strategy depending on their previous outputs and their rivals' outputs.

Expectations play an important role in modeling economic phenomena. A firm can choose its expectation rules to adjust its production. There exist three different firms' expectations: naive, bounded rational, and adaptive [2]. The case of where both players adopt bounded rational expectation and that of one takes bounded rational expectation and the other takes adaptive expectation have been studied by many researchers [3–10]. However, there are few literatures that focus on the case of where both players adopt adaptive expectation.

Recently, the dynamics of duopoly game has been studied by many researchers. Chaotic phenomena in duopoly game were first found by Puu [11]. Later, Kopel [12] studied the stability of the duopoly game with different demand functions

and different cost functions. Chaos was also found in this paper. Agiza and Elsadany [2] studied the dynamics in the Cournot duopoly game with heterogeneous players; in which case, one player accepts naive expectation, and the other adopts bounded rational expectation. Ma and Ji [4] studied duopoly game with homogenous players in electric power industry, where both players adjust their outputs according to bounded rational rule. Sarafopoulos [5] assumed that one player accepts bounded rational expectation and the other uses adaptive expectation.

Since chaotic phenomena were found in duopoly game, there are also many researchers focusing on the chaos control in duopoly game [6, 13–17]. However, to our best knowledge, there are few literatures that focus on the efficiency of chaos control. In this paper, we explore the dynamics of a homogeneous duopoly game where both players take adaptive expectation. According to the analysis of numerical simulations, the parameters' effects on the stability of the system are obtained. To lead the system to stable state, we try to use DFC method. The efficiency of the case where only one company takes control measure and that of the case where both companies take control measure is compared.

The paper is organized as follows. In Section 2, the model and the parameters are introduced. In Section 3, we study the equilibriums and the stability of the equilibriums of the model. In Section 4, we give the numerical results about the reactions of the two players. In Section 5, we control the system to a stable situation. Section 6 gives the conclusions.

2. The Model

We assume that the duopoly players produce homogenous goods which are perfect substitutes and offer goods at discrete-time periods. Therefore, the duopoly players face the same market demands. They both choose adaptive expectation rule to decide the amounts of the goods in the next period as their response strategy. Assume that the total demand is reciprocal to price p ; therefore, the reverse demand function is

$$p = \frac{1}{q_1 + q_2}, \quad (1)$$

where q_i ($i = 1, 2$) represents the quantity that firm i produced. The cost of firm i contains two parts: one is constant cost d_i and the other is variable cost. The per units cost corresponding to the variable cost is constant, and the value is c_i . Then, the total cost of firm i is

$$C_i(q_i) = c_i q_i + d_i. \quad (2)$$

Now, we present a simple case which will be used to modify players' expectation latter. In this case, both firms have the conception that the other firm will produce at time $t + 1$ as it did at time t . Then, they will decide their outputs based on the amounts their rival player did at time t . Then, the expectation net profits of firm 1 and firm 2 at time $t + 1$ can be expressed as

$$\begin{aligned} \pi_1 &= p * q_1 - C_1(q_1) \\ &= \frac{q_1(t+1)}{q_1(t+1) + q_2(t)} - c_1 q_1(t+1) - d_1, \\ \pi_2 &= p * q_2 - C_2(q_2) \\ &= \frac{q_2(t+2)}{q_1(t) + q_2(t+1)} - c_2 q_2(t+1) - d_2. \end{aligned} \quad (3)$$

The marginal expectation profits of firm 1 and firm 2 at (q_1, q_2) are as follows:

$$\begin{aligned} \frac{\partial \pi_1}{\partial q_1} &= \frac{q_2(t)}{(q_1(t+1) + q_2(t))^2} - c_1, \\ \frac{\partial \pi_2}{\partial q_2} &= \frac{q_1(t)}{(q_1(t) + q_2(t+1))^2} - c_2. \end{aligned} \quad (4)$$

The firms can make maximize profits when the marginal profits are zero. Then, the reaction outputs of the two firms with respect to their competitor's last outputs are

$$\begin{aligned} q_1(t+1) &= q_1(q_2) = \sqrt{\frac{q_2(t)}{c_1}} - q_2(t), \\ q_2(t+1) &= q_2(q_1) = \sqrt{\frac{q_1(t)}{c_2}} - q_1(t). \end{aligned} \quad (5)$$

In order to make sure that q_i ($i = 1, 2$) is nonnegative, the reaction functions are modified as

$$q_1(t+1) = \begin{cases} \sqrt{\frac{q_2}{c_1}} - q_2, & 0 < q_2 \leq \frac{1}{c_1}, \\ 0, & q_2 > \frac{1}{c_1}, \end{cases} \quad (6)$$

$$q_2(t+1) = \begin{cases} \sqrt{\frac{q_1}{c_2}} - q_1, & 0 < q_1 \leq \frac{1}{c_2}, \\ 0, & q_1 > \frac{1}{c_2}. \end{cases} \quad (7)$$

On the condition that the two firms have adaptive expectation, the firms compute their outputs with weights between their own last outputs and their reaction outputs $q_i(q_j)$ ($i \neq j$). Then, the outputs they produce in the next period are as follows:

$$\begin{aligned} q_1(t+1) &= (1 - v_1) * q_1(t) + v_1 * q_1(q_2), \\ q_2(t+1) &= (1 - v_2) * q_2(t) + v_2 * q_2(q_1), \end{aligned} \quad (8)$$

where $q_1(q_2)$ and $q_2(q_1)$ are as (6) and (7), respectively, and v_1 and v_2 are weights on firms' reaction outputs, respectively. We focus on the dynamics of system (8) in the next section.

3. Equilibrium and Stability

By setting $q_i(t+1) = q_i(t)$, ($i = 1, 2$) in map (8), the equilibrium output points of the dynamic duopoly game can be obtained as the nonnegative solutions of the algebraic system:

$$\begin{aligned} -q_1(t) + \sqrt{\frac{q_2(t)}{c_1}} - q_2(t) &= 0, \\ -q_2(t) + \sqrt{\frac{q_1(t)}{c_2}} - q_1(t) &= 0. \end{aligned} \quad (9)$$

The system has two equilibriums: one is $E_1^*(0, 0)$ and the other is $E_2^*(q_1^*, q_2^*) = (c_2/(c_1 + c_2)^2, c_1/(c_1 + c_2)^2)$. E_1^* has no practical significance because both the outputs of two firms are zero. Hence, we just investigate Nash equilibrium E_2^* .

The Jacobian matrix of the two-dimensional map (8) at equilibrium E_2^* is

$$J(E) = \begin{bmatrix} 1 - v_1 & \frac{1}{2} v_1 \sqrt{\frac{1}{c_1 q_2^*}} - 1 \\ \frac{1}{2} v_2 \sqrt{\frac{1}{c_2 q_1^*}} - 1 & 1 - v_2 \end{bmatrix}. \quad (10)$$

According to Agiza and Elsadany [3], equilibrium E_2^* is locally stable if the following conditions are held:

$$\begin{aligned} 1 - T + D &> 0, \\ 1 + T + D &> 0, \\ 1 - D &> 0, \end{aligned} \quad (11)$$

where

$$\begin{aligned}
 T &= 2 - v_1 - v_2, \\
 D &= v_1 v_2 - v_1 - v_2 - \frac{1}{4} v_1 v_2 \sqrt{\frac{1}{c_1 q_2^* c_2 q_1^*}} + \frac{1}{2} v_1 \sqrt{\frac{1}{c_1 q_1^*}} \\
 &\quad + \frac{1}{2} v_2 \sqrt{\frac{1}{c_2 q_1^*}} \\
 &= v_1 v_2 - v_1 - v_2 - \frac{1}{4} \frac{v_1 v_2 (c_1 + c_2)^2}{c_1 c_2} + \frac{1}{2} \frac{v_1 (c_1 + c_2)}{c_1} \\
 &\quad + \frac{1}{2} \frac{v_2 (c_1 + c_2)}{c_2}.
 \end{aligned} \tag{12}$$

By substituting T and D into (11), the stable conditions of equilibrium E_2^* become

$$\begin{aligned}
 2(c_1 + c_2)(v_1 c_2 + v_2 c_1) - v_1 v_2 (c_1 - c_2)^2 - 4c_1 c_2 &> 0, \\
 2(c_1 + c_2)(v_1 c_2 + v_2 c_1) - v_1 v_2 (c_1 - c_2)^2 \\
 - 8c_1 c_2 (v_1 + v_2) + 12c_1 c_2 &> 0, \\
 -2(c_1 + c_2)(v_1 c_2 + v_2 c_1) + v_1 v_2 (c_1 - c_2)^2 \\
 + 4c_1 c_2 (v_1 + v_2 + 1) &> 0.
 \end{aligned} \tag{13}$$

In view of (10), the eigenvalues associated with the equilibrium $E_2^*(q_1^*, q_2^*)$ are

$$\begin{aligned}
 \lambda_1 &= \frac{T + \sqrt{T^2 - 4D}}{2}, \\
 \lambda_2 &= \frac{T - \sqrt{T^2 - 4D}}{2}.
 \end{aligned} \tag{14}$$

Now, suppose that

$$\begin{aligned}
 T^2 - 4D &< 0, \\
 |\lambda_i| &= 1, \quad i = 1, 2.
 \end{aligned} \tag{15}$$

Then, $D = 1$; namely,

$$\begin{aligned}
 v_1 v_2 - v_1 - v_2 - \frac{1}{4} \frac{v_1 v_2 (c_1 + c_2)^2}{c_1 c_2} + \frac{1}{2} \frac{v_1 (c_1 + c_2)}{c_1} \\
 + \frac{1}{2} \frac{v_2 (c_1 + c_2)}{c_2} &= 1.
 \end{aligned} \tag{16}$$

It follows from (16) that

$$v_1 = \frac{1 + v_2 - v_2 (c_1 + c_2) / 2c_2}{v_2 - 1 - v_2 (c_1 + c_2)^2 / 4c_2 + (c_1 + c_2) / 2c_2} = v_1^*. \tag{17}$$

Then, (q_1^*, q_2^*, v_1^*) is a candidate for Neimark-Sacker bifurcation point [18].

FIGURE 1: Bifurcation diagram with respect to parameter c_1 against variable q_1 with 500 iterations of map (8) for $v_1 = v_2 = 0.5$, $c_2 = 2$.

4. Numerical Simulations

In this section, numerical simulation results corresponding to model (8) are presented. Bifurcation diagrams and phase portraits with respect to different parameters are used to show complex dynamical behaviors of the duopoly model.

Figure 1 presents the bifurcation diagram of system (8) with respect to parameter c_1 (per unit cost of firm 1) against variable q_1 for $v_1 = v_2 = 0.5$, $c_2 = 2$. It is seen that the system is in periodic state for $c_1 > 0.18$. As c_1 decreases, periodic motion and chaotic motion occur alternatively. And the system is driven to chaos through quasiperiodic route. The main region where the system appears chaotic behavior is the range with $c_1 \in [0.082, 0.121]$. There exist many period orbits, such as period 7 orbit with $c_1 \in [0.051, 0.066]$ and period 8 orbit with $c_1 \in [0.014, 0.017]$, in model (8). For $c_1 > 0.18$, the stable output of firm 1 decreases with per unit cost c_1 increasing, which is consistent with the fact that the comparative superiority of firm 1 decreases as per unit cost c_1 increasing.

Now, set $v_2 = 0.7$, $c_1 = 0.14$, and $c_2 = 2$. It follows from (17) that $v_1^* \approx 0.2958$. Figure 2 shows the bifurcation diagram of map (8) with respect to parameter v_1 . At $v_1^* \approx 0.2958$, a Neimark-Sacker bifurcation occurs. Nash equilibrium E_2^* is locally stable for $v_1 < 0.2985$ and loses its stability for $v_1 > 0.2985$. The system evolves from Nash equilibrium E_2^* into chaos with parameter v_1 (weight on firm 1's reaction function) increasing, through the mechanism of quasiperiodic route.

Figure 3 depicts the trajectories of outputs of firm 1 and firm 2 in the phase space (q_1, q_2) for $v_1 = 0.5$, $v_2 = 0.5$, $c_2 = 2$, and the initial point $(0.4, 0.4)$. The red point is Nash equilibrium. A periodic attractor with $c_1 = 0.13$ is shown in Figure 3(b), while a quasiperiodic attractor with $c_1 = 0.12$ is shown in Figure 3(c). For $c_1 = 0.19$, the outputs of firm 1 and firm 2 converge to the Nash equilibrium, which is shown in Figure 3(a).

Figure 4 depicts the trajectories of outputs of firm 1 and firm 2 in the phase space (q_1, q_2) for $v_2 = 0.7$, $c_1 = 0.14$,

FIGURE 2: Bifurcation diagram with respect to parameter v_1 against variable q_1 with 500 iterations of map (8) for $v_2 = 0.7$, $c_1 = 0.14$, $c_2 = 2$.

FIGURE 3: (a) Phase portrait of map (8) for $v_1 = 0.5$, $v_2 = 0.5$, $c_1 = 0.19$, $c_2 = 2$. (b) Phase portrait of map (8) for $v_1 = 0.5$, $v_2 = 0.5$, $c_1 = 0.13$, $c_2 = 2$. (c) Phase portrait of map (8) for $v_1 = 0.5$, $v_2 = 0.5$, $c_1 = 0.12$, $c_2 = 2$.

FIGURE 4: (a) Phase portrait of map (8) for $v_1 = 0.3$, $v_2 = 0.7$, $c_1 = 0.14$, $c_2 = 2$. (b) Phase portrait of map (8) for $v_1 = 0.37$, $v_2 = 0.7$, $c_1 = 0.14$, $c_2 = 2$. (c) Phase portrait of map (8) for $v_1 = 0.48$, $v_2 = 0.7$, $c_1 = 0.14$, $c_2 = 2$.

$c_2 = 2$, and the initial point $(0.4, 0.4)$. The red point is Nash equilibrium. For $v_1 = 0.3$, the outputs of firm 1 and firm 2 converge to the Nash equilibrium, which is shown in Figure 4(a). Figure 4(b) shows a quasiperiodic attractor with $v_1 = 0.37$ and Figure 4(c) shows a periodic attractor with $v_1 = 0.48$.

5. Chaos Control in the Dynamic Output System

Any firm does not want to adjust its outputs too frequently and largely. So it is important for firms to control their outputs to a stable process. Chaos control has been studied by many researchers since chaos has been found in economy [6, 13–17]. In DFC (delayed feedback control) method [19], the most common control function is

$$u_t = K(q(t) - q(t-1)). \quad (18)$$

We consider the case that only firm 1 adopts control function to make the system stable, while firm 2 does not realize that it will generate chaos in the future on the condition that firm 1 does not take any measure to prevent this case. Then, system (8) can be rewritten as

$$q_1(t+1) = (1 - v_1) * q_1(t) + v_1 * q_1(q_2) + u_{1,t}, \quad (19)$$

$$q_2(t+1) = (1 - v_2) * q_2(t) + v_2 * q_2(q_1),$$

where $u_{1,t} = K_1(q_1(t) - q_1(t-1))$.

To examine the effects of the control function, we compare the output process of firm 1 before and after adding the control function. Set $v_1 = 0.85$, $v_2 = 0.7$, $c_1 = 0.2$, and $c_2 = 2$ in model (8), which may exhibit chaos. The bifurcation diagram with respect to the control parameter K_1 is given in Figure 5. The system is stable for $K_1 < -0.22$. For $K_1 = -0.3$, the outputs processes of firm 1 before and after using control function are shown in Figure 6. It is seen that the control function effectively leads the outputs to a stable state.

FIGURE 5: Bifurcation diagram with respect to parameter K_1 against variable q_1 with 500 iterations of map (19) for $v_1 = 0.85$, $v_2 = 0.7$, $c_1 = 0.2$, $c_2 = 2$.

FIGURE 6: (a) Time series of output of firm 1 before using control function. (b) Time series of output of firm 1 after only firm 1 uses control function. (c) Time series of output of firm 1 after both firms use control function.

Now, consider the case that both firms realize that they will generate chaos in the future if they do not take any measure to control it. Therefore, both firms use control function to stabilize their outputs. Then, system (8) can be rewritten as

$$\begin{aligned} q_1(t+1) &= (1 - v_1) * q_1(t) + v_1 * q_1(q_2) + u_{1,t}, \\ q_2(t+1) &= (1 - v_2) * q_2(t) + v_2 * q_2(q_1) + u_{2,t}, \end{aligned} \quad (20)$$

where $u_{1,t} = K_1(q_1(t) - q_1(t-1))$ and $u_{2,t} = K_2(q_2(t) - q_2(t-1))$. The outputs of firm 1 are shown in Figure 6(c) for $K_1 = -0.3$ and $K_2 = -0.3$. By comparing Figure 6(b) with Figure 6(c), it is convenient to conclude that the effect of both firms taking control method is better than that of single firm taking control method.

6. Conclusion

In this paper, we have investigated a Cournot duopoly model where both the players decide their outputs weighting on their own previous outputs and the optimal outputs on the condition that their rival produces as their previous step. The Nash equilibrium and its local stability were analyzed. The numerical simulations show that the changes of marginal cost c_1 and weight factor v_1 may lead the Nash equilibrium to be unstable and the system into chaotic state. The system can quickly arrive at the Nash equilibrium by taking DFC method with a suitable controlling parameter. The effect of both firms taking control method is better than that of single firm taking control method.

In view of the fact that firms may not produce any goods if they suffer losses, it is needed to take firms' profits into consideration in modeling the outputs of firms. This problem will be investigated in our future research.

Competing Interests

The authors declare that they have no competing interests.

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