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Research Article

Dynamics of Stochastic Coral Reefs Model with Multiplicative Nonlinear Noise

Zaitang Huang[1,2,3,4]

[1] *Yangtze Center of Mathematics and Department of Mathematics, Sichuan University, Chengdu, Sichuan 610064, China*
[2] *School of Mathematics and Statistics, Guangxi Teachers Education University, Nanning, Guangxi 530023, China*
[3] *Key Laboratory of Environment Change and Resources Use in Beibu Gulf, Guangxi Teachers Education University, Nanning, Guangxi 530023, China*
[4] *Guangxi Key Laboratory of Earth Surface Processes and Intelligent Simulation, Guangxi Teachers Education University, Nanning, Guangxi 530023, China*

Correspondence should be addressed to Zaitang Huang; huangzaitang@126.com

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Little seems to be known about the ergodicity of random dynamical systems with multiplicative nonlinear noise. This paper is devoted to discern asymptotic behavior dynamics through the stochastic coral reefs model with multiplicative nonlinear noise. By support theorem and Hörmander theorem, the Markov semigroup corresponding to the solutions is to prove the Foguel alternative. Based on boundary distributions theory, the required conservative operators related to the solutions are further established to ensure the existence a stationary distribution. Meanwhile, the density of the distribution of the solutions either converges to a stationary density or weakly converges to some probability measure.

1. Introduction

The coral reefs equation is one of the most famous ecosystem models [1]:

$$\begin{aligned} \dot{X}(t) &= X\left(\gamma - \gamma X + (\sigma - \gamma) Y - \frac{g}{1-Y}\right), \\ \dot{Y}(t) &= Y(r - d - (\sigma + r) X - rY), \end{aligned} \tag{1}$$

where $\dot{X}(t)$ represents the cover of macroalgae; $\dot{Y}(t)$ represents the cover of corals.

(i) r is the rate that corals recruit to and overgrow algal turfs;

(ii) d is the natural mortality rate of corals;

(iii) σ is the rate that corals are overgrown by macroalgae;

(iv) γ is the rate that macroalgae spread vegetatively over algal turfs;

(v) g is the grazing rate that parrotfish graze macroalgae without distinction from algal turfs.

By the results in Li et al. [2], they discuss all kinds of dynamical behaviors. Recently, system (1) was studied extensively that it exhibits complex dynamical phenomena, including chaos, bifurcation, stability, and attractiveness [1–10].

However, ecosystem in the real world is very often subject to environmental noise due to uncertainty and unknown factors [11–16]. From a biological point of view and the generality of the models considered [17–21], these systems can appear very formal. This paper studies a stochastic coral reefs model where the intrinsic growth rate of the cover of macroalgae, γ, and the one of the cover of corals, r, are perturbed stochastically $\gamma \to \gamma + \alpha \dot{W}_t$ and $r \to r + \alpha \dot{W}_t$. In the paper, we only consider that stochastic coral reefs model can be described by

$$\begin{aligned} dX &= X\left(\gamma - \gamma X + (\sigma - \gamma) Y - \frac{g}{1-Y}\right) dt \\ &\quad + \alpha X^2 dW(t), \\ dY &= Y(r - d - (\sigma + r) X - rY)\, dt + \beta Y^2 dW(t), \end{aligned} \tag{2}$$

where $W(t)$ is a two-sided canonical Brownian motion. α and β represent the intensity of random noise and the differential $X^2dW(t)$ and $Y^2dW(t)$ are to be understood in the sense of multiplicative nonlinear noise. Since the drift and diffusion coefficients of (2) satisfy locally Lipschitz continuous condition, we can apply standard theorems that provide both existence and uniqueness of the positive solution of (2) (see [20]), for any given initial value. However, since the diffusion term of (2) is not linear but nonlinear, the existing powerful classical results [11–21] fail to work here. Nevertheless, this paper discusses the asymptotic behavior of the stochastic coral reefs model with multiplicative nonlinear noise using Fokker-Planck equations. To overcome the difficulty from the diffusion term, based on boundary distributions theory and conservative operators, we show that the density of the distribution of the solutions either converges to a stationary density or weakly converges to some probability measure.

This paper is organized as follows. In Section 2, we study the global attractiveness of the solution for stochastic coral reefs model with multiplicative nonlinear noise. In Section 3, we discuss the ergodicity of the solution for stochastic coral reefs model with multiplicative nonlinear noise.

2. Global Attractiveness

In the section, we study the global attractiveness of the solution for stochastic coral reefs model with multiplicative nonlinear noise.

Proposition 1. *Suppose that $\sigma < d < \gamma < r < 2\gamma$, $0 < g < \gamma$ hold.*

(I) *If the cover of macroalgae is absent, then the cover of corals dies with probability one.*

(II) *If the cover of corals is absent, the quantity of the preys oscillates between 0 and ∞, and there exists a unique stationary distribution with the density $f_*(x)$*

$$f_*(x) = c\exp\left\{-3x + \frac{2\gamma}{\alpha^2}e^{-x} - \frac{\gamma}{\alpha^2}e^{-2x}\right\}, \tag{3}$$

where c is a constant.

Proof. Denoting $X = e^{\xi_t}$, $Y = e^{\eta_t}$, we replace system (2) by

$$\begin{aligned} d\xi &= \left(\gamma - \frac{\alpha^2}{2}e^{2\xi_t} - \gamma e^{\xi_t} - (\gamma-\sigma)e^{\eta_t} - \frac{g}{1-e^{\eta_t}}\right)dt \\ &\quad + \alpha e^{\xi_t}dW, \\ d\eta &= \left(r - d - \frac{\beta^2}{2}e^{2\eta_t} - (\sigma+r)e^{\xi_t} - re^{\eta_t}\right)dt \\ &\quad + \beta e^{\eta_t}dW. \end{aligned} \tag{4}$$

Let

$$\begin{aligned} f_1(x,y) &= \gamma - \frac{\alpha^2}{2}e^{2x_t} - \gamma e^{x_t} - (\gamma-\sigma)e^{y_t} - \frac{g}{1-e^{y_t}}, \\ f_2 &= r - d - \frac{\beta^2}{2}e^{2y_t} - (\sigma+r)e^{x_t} - re^{y_t}. \end{aligned} \tag{5}$$

Then, system (4) becomes

$$\begin{aligned} d\xi &= f_1(\xi_t,\eta_t)\,dt + \alpha e^{\xi_t}dW, \\ d\eta &= f_2(\xi_t,\eta_t)\,dt + \beta e^{\eta_t}dW, \end{aligned} \tag{6}$$

or Stratonovich stochastic differential equation

$$\begin{aligned} d\xi &= \left(\gamma - \alpha^2e^{2\xi_t} - \gamma e^{\xi_t} - (\gamma-\sigma)e^{\eta_t} - \frac{g}{1-e^{\eta_t}}\right)dt \\ &\quad + \alpha e^{\xi_t}\circ dW_t, \\ d\eta &= \left(r - d - \beta^2e^{2\eta_t} - (\sigma+r)e^{\xi_t} - re^{\eta_t}\right)dt + \beta e^{\eta_t} \\ &\quad \circ dW_t. \end{aligned} \tag{7}$$

Let $\mathscr{L}$ denote the generator of diffusion (4); that is,

$$\begin{aligned} \mathscr{L}v &= \frac{1}{2}\alpha^2e^{2x}\frac{\partial^2 v}{\partial x^2} + \alpha\beta e^{x+y}\frac{\partial^2 v}{\partial x\partial y} + \frac{1}{2}\beta^2e^{2y}\frac{\partial^2 v}{\partial y^2} \\ &\quad + f_1\frac{\partial v}{\partial x} + f_2\frac{\partial v}{\partial y}. \end{aligned} \tag{8}$$

Then Fokker-Planck equation (FPE) of (4) can be described by

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2}\alpha^2\frac{\partial^2\left(e^{2x}u\right)}{\partial x^2} + \alpha\beta\frac{\partial^2\left(e^{x+y}u\right)}{\partial x\partial y} + \frac{1}{2}\beta^2\frac{\partial^2\left(e^{2y}u\right)}{\partial y^2} \\ &\quad - \frac{\partial(f_1u)}{\partial x} - \frac{\partial(f_2u)}{\partial y}. \end{aligned} \tag{9}$$

(I) If the cover of macroalgae is absent, the quantity $Y = e^{\eta_t}$ of the cover of corals satisfies

$$d\eta = \left(r - d - \frac{\beta^2}{2}e^{2\eta_t} - re^{\eta_t}\right)dt + \beta e^{\eta_t}dW. \tag{10}$$

Fix

$$\begin{aligned} &s_1(x) \\ &= \int_0^x \exp\left\{-\int_0^y \frac{2\left(r - d - \left(\beta^2/2\right)e^{2u_t} - re^{u_t}\right)}{\beta^2e^{2u_t}}du\right\}dy \\ &= \int_0^x \exp\left\{\frac{r+d}{\beta^2} - \frac{2r}{\beta^2e^{y_t}} + \frac{r-d}{\beta^2e^{2y_t}} + y\right\}dy. \end{aligned} \tag{11}$$

It is easy to see that $\lim_{x\to+\infty}s_1(x) = +\infty$ and $\lim_{x\to-\infty}s_1(x) > -\infty$. Then, we can obtain

$$\begin{aligned} &\lim_{t\to+\infty}\eta_t = -\infty \\ &\text{or equivalently } \lim_{t\to+\infty}Y_t = 0 \quad \text{a.s.} \end{aligned} \tag{12}$$

It implies that without the cover of macroalgae, the cover of corals dies with probability one.

(II) If the cover of corals is absent, the quantity $X = e^{\xi_t}$ of the cover of macroalgae satisfies

$$d\xi = \left(\gamma - \frac{\alpha^2}{2}e^{2\xi_t} - \gamma e^{\xi_t}\right)dt + \alpha e^{\xi_t}dW. \tag{13}$$

Let

$$\begin{aligned} s_2(x) &= \int_0^x \exp\left\{-\int_0^y \frac{2\left(\gamma - \left(\alpha^2/2\right)e^{2u_t} - \gamma e^{u_t}\right)}{\alpha^2 e^{2u_t}}du\right\}dy \\ &= \int_0^x \exp\left\{\frac{\gamma}{\alpha^2} + \frac{\gamma}{\alpha^2 e^{2y_t}} - \frac{2\gamma}{\alpha^2 e^{y_t}} + y\right\}dy. \end{aligned} \tag{14}$$

It easily shows that $\lim_{x\to+\infty} s_2(x) = +\infty$ and $\lim_{x\to-\infty} s_2(x) > -\infty$. We have

$$\begin{aligned} &\limsup_{t\to+\infty} \xi_t = \infty; \\ &\liminf_{t\to+\infty} \xi_t = -\infty \\ \text{or equivalently } &\limsup_{t\to+\infty} X_t = \infty; \\ &\liminf_{t\to+\infty} X_t = 0 \quad \text{a.s.} \end{aligned} \tag{15}$$

It implies that, without the predators, the quantity of the preys oscillates between 0 and ∞. Furthermore, there exists a stationary distribution of system (13) with the density $f_*(x)$ satisfying the FPE

$$\begin{aligned} &\frac{1}{2}\frac{d^2}{dx^2}\left[\alpha^2 e^{2x} f_*(x)\right] \\ &\quad - \frac{d}{dx}\left[\left(\gamma - \frac{\alpha^2}{2}e^{2x_t} - \gamma e^{x_t}\right) f_*(x)\right] = 0. \end{aligned} \tag{16}$$

Solving (16), then we get

$$\begin{aligned} y(x) &= \exp\left\{-3x + \frac{2\gamma}{\alpha^2}e^{-x} - \frac{\gamma}{\alpha^2}e^{-2x}\right\} \\ &\quad \cdot \left[c + k\int \exp\left\{x - \frac{2\gamma}{\alpha^2}e^{-x} + \frac{\gamma}{\alpha^2}e^{-2x}\right\}dx\right], \end{aligned} \tag{17}$$

where c and k are real numbers. With the conditions $y(x) \geq 0$ and

$$\int_{-\infty}^{+\infty} y(x)\,dx = 1, \tag{18}$$

it means that

$$\begin{aligned} &K = 0, \\ &\frac{1}{c} = \int_{-\infty}^{+\infty} \exp\left\{-3x + \frac{2\gamma}{\alpha^2}e^{-x} - \frac{\gamma}{\alpha^2}e^{-2x}\right\}dx \\ &\quad = \int_0^{+\infty} y^2 \exp\left\{\frac{2\gamma}{\alpha^2}y - \frac{\gamma}{\alpha^2}y^2\right\}dy. \end{aligned} \tag{19}$$

It easily shows that

$$\begin{aligned} \frac{1}{c} &= \exp\left\{\frac{\gamma}{\alpha}\right\}\frac{\alpha}{2\gamma\sqrt{2\gamma}} \\ &\quad \cdot \int_{-\sqrt{2\gamma}/\alpha}^{\infty} \left(\alpha u + \sqrt{2\gamma}\right)^2 \exp\left\{-\frac{u^2}{2}\right\}du = \frac{\alpha^2}{2\gamma} \\ &\quad + \frac{\sqrt{\pi}\alpha\left(2\gamma + \alpha^2\right)}{2\gamma\sqrt{\gamma}}\left[1 - \Phi\left(-\frac{\sqrt{2\gamma}}{\alpha}\right)\right]\exp\left\{\frac{\gamma}{\alpha^2}\right\}, \end{aligned} \tag{20}$$

where $\Phi(x)$ is standard normal distribution function. Thus, we get

$$f_*(x) = c\exp\left\{-3x + \frac{2\gamma}{\alpha^2}e^{-x} - \frac{\gamma}{\alpha^2}e^{-2x}\right\}. \tag{21}$$

By ergodic theorem, if ξ_t is a solution of system (13), then we have

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t h(\xi_s)\,ds = \int_{\infty}^{\infty} h(x) f_*(x)\,dx \quad \text{a.s.} \tag{22}$$

Furthermore, ξ_t converges in probability to $f_*(x)$ when $t \to +\infty$. Define

$$\begin{aligned} m &= \int_{\infty}^{\infty} e^x f_*(x)\,dx \\ &= x\int_0^{+\infty} y\exp\left\{\frac{2\gamma}{\alpha^2}y - \frac{\gamma}{\alpha^2}y^2\right\}dy. \end{aligned} \tag{23}$$

It is easy to see that

$$\begin{aligned} m &= \frac{c\alpha^2}{2\gamma} + \frac{\sqrt{\pi}c\alpha}{\sqrt{\gamma}}\left[1 - \Phi\left(-\frac{\sqrt{2\gamma}}{\alpha}\right)\right]\exp\left\{\frac{\gamma}{\alpha^2}\right\} \\ &= \frac{\gamma\sqrt{\gamma}\alpha^2 + 2\sqrt{\pi}\gamma^2\alpha\left[1 - \Phi\left(-\sqrt{2\gamma}/\alpha\right)\right]\exp\left\{\gamma/\alpha^2\right\}}{\gamma\sqrt{\gamma}\alpha^2 + \sqrt{\pi}\alpha\left(2\gamma^2 + \gamma\alpha^2\right)\left[1 - \Phi\left(-\sqrt{2\gamma}/\alpha\right)\right]\exp\left\{\gamma/\alpha^2\right\}}. \end{aligned} \tag{24}$$

□

Theorem 2. *Suppose that $\alpha > 0$ and $\beta > 0$ hold. Then there exists a constant λ such that*

$$\begin{aligned} &\limsup_{t\to\infty}\frac{1}{t}\left[\frac{4}{\lambda^2}\ln(x+y) + \int_0^t (x+y)^2\,dt\right] \\ &\quad \leq \frac{4\max\{\gamma, r-d\}}{\lambda^2} \end{aligned} \tag{25}$$

holds with probability 1, where x and y are the solution of (2) with the initial condition $(x_0, y_0) \in \mathfrak{R}^2$.

Proof. Define the Lyapunov function on $\mathfrak{R}^2$

$$V(x, y) = \ln(x + y). \tag{26}$$

Applying Itô's formula to function (26), we get

$$\begin{aligned} dV(x,y) = {} & \left[\frac{x}{x+y}\left(\gamma-\gamma x+(\sigma-\gamma)y-\frac{g}{1-y}\right)\right. \\ & \left.-\frac{1}{2}\alpha^2\frac{x^4}{(x+y)^2}-\alpha\beta\frac{x^2y^2}{(x+y)^2}\right]dt \\ & +\left[\frac{y}{x+y}(r-d-(\sigma+r)x-ry)\right. \\ & \left.-\frac{1}{2}\beta^2\frac{y^4}{(x+y)^2}\right]dt+\frac{\alpha x^2+\beta y^2}{x+y}dW. \end{aligned} \tag{27}$$

That is,

$$\begin{aligned} V(x,y) = {} & V(0) \\ & +\int_0^t\left[\frac{x}{x+y}\left(\gamma-\gamma x+(\sigma-\gamma)y-\frac{g}{1-y}\right)\right. \\ & \left.-\frac{1}{2}\alpha^2\frac{x^4}{(x+y)^2}-\frac{\alpha\beta x^2y^2}{(x+y)^2}\right]ds \\ & +\int_0^t\left[\frac{y}{x+y}(r-d-(\sigma+r)x-ry)\right. \\ & \left.-\frac{1}{2}\beta^2\frac{y^4}{(x+y)^2}\right]ds+M(t), \end{aligned} \tag{28}$$

where

$$M(t)=\int_0^t\frac{\alpha x^2+\beta y^2}{x+y}dW \tag{29}$$

is a local martingale with quadratic form:

$$\langle M(t)\rangle=\int_0^t\left(\frac{\alpha x^2+\beta y^2}{x+y}\right)^2dt. \tag{30}$$

Fix $0<\varepsilon<1$. For any $\kappa\geq 1$, by martingale inequality, we have

$$P\left\{\sup_{0\leq t\leq\kappa}\left[M(t)-\frac{\varepsilon}{4}\langle M(t)\rangle\right]>\frac{4\ln\kappa}{\varepsilon}\right\}\leq\frac{1}{\kappa^2}. \tag{31}$$

By using Borel-Cantelli theorem, we can choose a set $\Omega'\subset\Omega$ with $P(\Omega')=1$ and for any $\omega\in\Omega$ there is a $\kappa_0(\omega)$ such that $\forall\kappa\geq\kappa_0(\omega)$

$$\sup_{0\leq t\leq\kappa}\left[M(t)-\frac{\varepsilon}{4}\langle M(t)\rangle\right]\leq\frac{4\ln\kappa}{\varepsilon}. \tag{32}$$

It implies that

$$M(t)\leq\frac{\varepsilon}{4}\langle M(t)\rangle+\frac{4\ln\kappa}{\varepsilon}\quad\forall 0\leq t\leq\kappa, \tag{33}$$

for $\omega\in\Omega'$ and $\kappa\geq\kappa_0(\omega)$. Substituting (33) into (28), we get

$$\begin{aligned} & V(x,y)+\frac{1}{4}\int_0^t\frac{(\alpha x+\beta y)^2}{(x+y)^2}dt \\ & \quad\leq V(0) \\ & \qquad+\int_0^t\left[\frac{x}{x+y}\left(\gamma-\gamma x+(\sigma-\gamma)y-\frac{g}{1-y}\right)\right]ds \\ & \qquad+\int_0^t\left[\frac{y}{x+y}(r-d-(\sigma+r)x-ry)\right]ds \\ & \qquad-\int_0^t\frac{(1-\varepsilon)\left(\alpha x^2+\beta y^2\right)^2}{4(x+y)^2}ds+\frac{4\ln\kappa}{\varepsilon}, \end{aligned} \tag{34}$$

for $0\leq t\leq\kappa$ and for almost ω and $\kappa\geq\kappa_0(\omega)$. Moreover, there exists a constant λ satisfying

$$\alpha x^2+\beta y^2\geq\lambda(x+y)^2. \tag{35}$$

By inequality (35), we get

$$\frac{\left(\alpha x^2+\beta y^2\right)^2}{4(x+y)^2}dt\geq\lambda^2\frac{(x+y)^2}{4}dt, \tag{36}$$

it means that

$$\begin{aligned} & V(x,y)+\frac{1}{4}\int_0^t\frac{(\alpha x+\beta y)^2}{(x+y)^2}dt \\ & \quad\geq V(x,y)+\frac{\lambda^2}{4}\int_0^t(x+y)^2dt. \end{aligned} \tag{37}$$

Moreover, there exists a positive constant $\mu=\max\{\gamma,r-d\}$ satisfying

$$\begin{aligned} \gamma x+(r-d)y & \leq\mu(x+y), \\ -\gamma x^2-(r+\gamma)xy-ry^2 & \leq 0. \end{aligned} \tag{38}$$

Therefore, we get

$$\begin{aligned} & \frac{1}{x+y}\left(\gamma x-\gamma x^2+(\sigma-\gamma)xy-\frac{gx}{1-y}+(r-d)y\right. \\ & \quad\left.-(\sigma+r)xy-ry^2\right)\leq\mu. \end{aligned} \tag{39}$$

Then, we have

$$\begin{aligned} & V(x,y)+\frac{1}{4}\int_0^t\frac{(\alpha x+\beta y)^2}{(x+y)^2}dt \\ & \quad\leq V(0)+\int_0^t\mu ds+\frac{4\ln\kappa}{\varepsilon}\leq V(0)+\mu t+\frac{4\ln\kappa}{\varepsilon}, \end{aligned} \tag{40}$$

for any $\omega\in\Omega'$, $\kappa\geq\kappa_0(\omega)$, and $0\leq t\leq\kappa$.

If $\kappa - 1 \le t \le \kappa$ with $\kappa \ge \kappa_0(\omega)$, then we get

$$\begin{aligned} &\frac{1}{t}\left[V(x,y) + \frac{\lambda^2}{4}\int_0^t (x+y)^2\,dt\right] \\ &\quad \le \mu + \frac{1}{t}\left[V(0) + \frac{4\ln\kappa}{\varepsilon}\right]. \end{aligned} \tag{41}$$

It means that

$$\limsup_{t\to\infty}\frac{1}{t}\left[V(x,y) + \frac{\lambda^2}{4}\int_0^t (x+y)^2\,dt\right] \le \mu, \tag{42}$$

or

$$\begin{aligned} &\limsup_{t\to\infty}\frac{1}{t}\left[\frac{4}{\lambda^2}V(x,y) + \int_0^t (x+y)^2\,dt\right] \\ &\quad \le \frac{4\max\{\gamma, r-d\}}{\lambda^2}. \end{aligned} \tag{43}$$

The proof is completed. □

Theorem 3. *Suppose that $\alpha > 0$ and $\beta > 0$ hold. Then, with probability 1, we have*

$$\limsup_{t\to+\infty}\frac{\ln(x+y)}{\ln t} \le 1. \tag{44}$$

Proof. Define C^2-function:

$$V(x,y) = e^t \ln(x+y). \tag{45}$$

Applying Itô's formula to function (44), we have

$$\begin{aligned} dV(x,y) = &\left[e^t \ln(x+y)\right. \\ &\left. + \frac{xe^t}{x+y}\left(\gamma - \gamma x + (\sigma-\gamma)y - \frac{g}{1-y}\right)\right]dt \\ &- \frac{e^t\left(\alpha x^2 + \beta y^2\right)^2}{2(x+y)^2}dt \\ &+ \left[\frac{ye^t}{x+y}(r - d - (\sigma+r)x - ry)\right]dt + e^t \\ &\cdot \frac{\alpha x^2 + \beta y^2}{x+y}dW. \end{aligned} \tag{46}$$

That is,

$$\begin{aligned} V(x,y) = V(0) &+ \int_0^t \left[e^s \ln(x+y)\right. \\ &\left. + \frac{xe^s}{x+y}\left(\gamma - \gamma x + (\sigma-\gamma)y - \frac{g}{1-y}\right)\right]ds \\ &+ \int_0^t \left[\frac{ye^s}{x+y}(r - d - (\sigma+r)x - ry)\right]ds \\ &- \int_0^t \frac{e^t\left(\alpha x^2 + \beta y^2\right)^2}{2(x+y)^2}ds + M(t), \end{aligned} \tag{47}$$

where

$$M(t) = \int_0^t e^s \frac{\alpha x^2 + \beta y^2}{x+y}dW \tag{48}$$

is a local martingale with quadratic form:

$$\langle M(t)\rangle = \int_0^t e^{2s}\left(\frac{\alpha x^2 + \beta y^2}{x+y}\right)^2 ds. \tag{49}$$

By using Borel-Cantelli theorem and martingale inequality, for $\varepsilon < 1$, $\theta > 1$, and $\rho > 0$, for almost $\omega \in \Omega$, there is a $\kappa_0(\omega)$ such that $\forall\kappa \ge \kappa_0(\omega)$, and we have

$$M(t) \le \frac{\varepsilon e^{-\kappa\rho}}{2}\langle M(t)\rangle + \frac{\theta e^{\kappa\rho}\ln\kappa}{\varepsilon}, \quad 0 \le t \le \kappa\rho. \tag{50}$$

By (47) and (50), we get

$$\begin{aligned} V(x,y) \le V(0) &+ \int_0^t \left[e^s \ln(x+y)\right. \\ &\left. + \frac{xe^s}{x+y}\left(\gamma - \gamma x + (\sigma-\gamma)y - \frac{g}{1-y}\right)\right]ds \\ &+ \int_0^t \left[\frac{ye^s}{x+y}(r - d - (\sigma+r)x - ry)\right. \\ &\left. - \frac{e^s\left(\alpha x^2 + \beta y^2\right)^2}{2(x+y)^2}ds\right]ds \\ &+ \int_0^t \frac{\varepsilon e^{-\kappa\rho}}{2}e^{2s}\left(\frac{\alpha x^2 + \beta y^2}{x+y}\right)^2 ds + \frac{4\theta e^{\kappa\rho}\ln\kappa}{\varepsilon}. \end{aligned} \tag{51}$$

Moreover, there exists a constant λ satisfying

$$\alpha x^2 + \beta y^2 \ge \lambda(x+y)^2. \tag{52}$$

By inequality (52), it easily shows that

$$\frac{\left(\alpha x^2 + \beta y^2\right)^2}{(x+y)^2}dt \ge \lambda^2(x+y)^2\,dt. \tag{53}$$

Moreover, there exists a positive constant μ satisfying

$$\begin{aligned} \gamma x + (r-d)y &\le \mu(x+y), \\ -\gamma x^2 - (r+\gamma)xy - ry^2 - \frac{gx}{1-y} &\le 0. \end{aligned} \tag{54}$$

Combining (51), (52), and (53), we get

$$\begin{aligned} V(x,y) \le V(0) &+ \int_0^t \left[e^s \ln(x+y) + e^s\mu(x+y)\right]ds \\ &- \int_0^t e^s\lambda^2(x+y)^2\frac{1-\varepsilon e^{-\kappa\rho+s}}{2}ds \\ &+ \frac{\theta e^{\kappa\rho}\ln\kappa}{\varepsilon} \end{aligned} \tag{55}$$

for any $0 \le s \le \kappa\rho$.

It is easy to see that there exists a real number K independent of κ satisfying

$$\ln(x+y)+\mu(x+y)-\lambda^2(x+y)^2\frac{1-\varepsilon e^{-\kappa\rho+s}}{2}\le K. \quad (56)$$

From (55) and (56), we get

$$\begin{aligned} V(x,y) &\le V(0)+\int_0^t e^s K ds+\frac{\theta e^{\kappa\rho}\ln\kappa}{\varepsilon} \\ &= V(0)+K\left(e^t-1\right)+\frac{\theta e^{\kappa\rho}\ln\kappa}{\varepsilon} \end{aligned} \quad (57)$$

for any $0\le s\le\kappa\rho$. Then, we have

$$\ln(x+y)\le e^{-t}V(0)+K\left(1-e^{-t}\right)+e^{-t}\frac{\theta e^{\kappa\rho}\ln\kappa}{\varepsilon}. \quad (58)$$

If $(\kappa-1)\rho\le t\le\kappa\rho$ and $\kappa\ge\kappa_0(\omega)$, we obtain

$$\begin{aligned} \frac{\ln(x+y)}{\ln t} &\le \frac{e^{-(\kappa-1)\rho}}{\ln(\kappa-1)\rho}(V(0)-K)+\frac{K}{\ln(\kappa-1)\rho} \\ &\quad+\frac{\theta e^{\rho}\ln\kappa}{\varepsilon\ln(\kappa-1)\rho}. \end{aligned} \quad (59)$$

Letting $k\to\infty$, we get

$$\limsup_{t\to\infty}\frac{\ln(x+y)}{\ln t}\le\frac{\theta e^{\rho}}{\varepsilon}. \quad (60)$$

By (60), for every $\rho>0$, $\varepsilon<1$, and $\theta>1$, then by letting $\rho\to0$, $\theta\to1$, and $\varepsilon\to1$, we obtain

$$\limsup_{t\to\infty}\frac{\ln(x+y)}{\ln t}\le 1. \quad (61)$$

The proof is completed. □

Theorem 4. *Suppose that $\sigma<d<\gamma<r<2\gamma$, $0<g<\gamma$ hold. Let $\overline{\xi}_t$ denote the solution of the following the equation:*

$$d\overline{\xi}_t=\left(\gamma-\frac{\alpha^2}{2}e^{2\overline{\xi}_t}-\gamma e^{\overline{\xi}_t}\right)dt+\alpha e^{\overline{\xi}_t}dW \quad (62)$$

with the initial value $\overline{\xi}_0=\xi_0$. Let $\overline{\eta}_t$ denote the solution of the following the equation:

$$d\overline{\eta}_t=\left(r-d-\frac{\beta^2}{2}e^{2\overline{\eta}_t}-re^{\overline{\eta}_t}\right)dt+\beta e^{\overline{\eta}_t}dW. \quad (63)$$

with the initial value $\overline{\eta}_0=\eta_0$.

Then, with probability 1, there exist $\overline{\xi}_t\ge\xi_t$ and $\overline{\eta}_t\ge\eta_t$.

Proof. Let $Z=e^{-\xi_t}$, $\overline{Z}=e^{-\overline{\xi}_t}$. By Itô's formula, we have

$$\begin{aligned} dZ_t &= \left(\gamma-\gamma Z_t+\alpha^2Z_t^{-1}+(\gamma-\sigma)Z_te^{\eta_t}+\frac{gZ_t}{1-e^{\eta_t}}\right)dt \\ &\quad-\alpha dW. \end{aligned} \quad (64)$$

$$d\overline{Z}_t=\left(\gamma-\gamma\overline{Z}_t+\alpha^2\overline{Z}_t^{-1}\right)dt-\alpha dW.$$

By using comparison theorem, we have $Z_t\ge\overline{Z}_t$ for all $t\ge0$ a.s. It implies that $\mathbb{P}\{\overline{\xi}_t\ge\xi_t\}=1$ for all $t\ge0$. It is easy to see that we show the second assertion $\overline{\eta}_t\ge\eta_t$ by a similar way for all $t\ge0$. The proof is completed. □

Theorem 5. *Suppose that $\sigma<d<\gamma<r<2\gamma$, $0<g<\gamma$ hold. Then the following assertions are true:*

$$\begin{aligned} &\text{(I)}\ \liminf_{t\to+\infty}\frac{1}{t}\int_0^t\left(\frac{\beta^2}{2}e^{2\eta_t}+(\sigma+r)e^{\xi_t}+re^{\eta_t}\right)\ge r-d \quad a.s. \\ &\text{(II)}\ \liminf_{t\to+\infty}\frac{1}{t}\int_0^t\left(\frac{\alpha^2}{2}e^{2\xi_t}+\gamma e^{\xi_t}+(\gamma-\sigma)e^{\eta_t}+\frac{g}{1-e^{\eta_t}}\right) \\ &\quad\ge\gamma \quad a.s. \end{aligned} \quad (65)$$

Proof. From Theorem 3, we get

$$\limsup_{t\to\infty}\frac{\eta_t}{\ln t}\le 1 \quad \text{a.s.} \quad (66)$$

Therefore,

$$\begin{aligned} &r-d-\limsup_{t\to\infty} \\ &\quad\cdot\frac{1}{t}\left[\int_0^t\left(\frac{\beta^2}{2}e^{2\eta_s}+(\sigma+r)e^{\xi_t}+re^{\eta_s}\right)ds\right. \\ &\quad\left.-\int_0^t\beta e^{\eta_s}dW_s\right], \\ &y=\limsup_{t\to\infty}\frac{\eta_t}{t}\le 0. \end{aligned} \quad (67)$$

For any $\varepsilon>0$, by martingale inequality, we have

$$\begin{aligned} &\mathbb{P}\left\{\sup_{0\le t\le k}\left\{\int_0^t-\beta e^{\eta_s}dW_s-\frac{\varepsilon}{2}\int_0^t\beta^2e^{2\eta_s}ds\right\}\ge\frac{2\ln k}{\varepsilon}\right\} \\ &\quad\le\frac{1}{k^2}. \end{aligned} \quad (68)$$

By using Borel-Cantelli theorem, for almost all ω, there is a real constant $k=k(\omega)$ satisfying, for all $n>k$ and $0\le t\le n$,

$$\int_0^t-\beta e^{\eta_s}dW_s-\frac{\varepsilon}{2}\int_0^t\beta^2e^{2\eta_s}ds\le\frac{2\ln k}{\varepsilon}. \quad (69)$$

It means that for $k-1\le t\le k$

$$\frac{1}{t}\left(\int_0^t-\beta e^{\eta_s}dW_s-\frac{\varepsilon}{2}\int_0^t\beta^2e^{2\eta_s}ds\right)\le\frac{2\ln k}{(k-1)\varepsilon}. \quad (70)$$

Then, we get

$$\liminf_{t\to\infty}\frac{1}{t}\left(\int_0^t\beta e^{\eta_s}dW_s+\frac{\varepsilon}{2}\int_0^t\beta^2e^{2\eta_s}ds\right)\ge 0. \quad (71)$$

Combining (67) and (71), it yields

$$\begin{aligned} &\limsup_{t\to\infty}\frac{1}{t}\int_0^t\left(-\frac{\beta^2(1+\varepsilon)}{2}e^{2\eta_s}-(\sigma+r)e^{\xi_t}-re^{\eta_s}\right)ds \\ &\quad\le d-r. \end{aligned} \quad (72)$$

Moreover, by Theorem 4, we get

$$0 < \limsup_{t\to\infty} \frac{1}{t}\int_0^t (\sigma + r) e^{\xi_t} ds \le (\sigma + r) m < \infty \quad \text{a.s.} \tag{73}$$

Therefore,

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t}\int_0^t \left(-\frac{\beta^2}{2} e^{2\eta_s} - (\sigma + r) e^{\xi_t} - r e^{\eta_s}\right) ds \\
&\quad \le \frac{1}{1+\varepsilon} \limsup_{t\to\infty} \frac{1}{t} \\
&\quad \cdot \int_0^t \left(-\frac{\beta^2 (1+\varepsilon)}{2} e^{2\eta_s} - (\sigma + r) e^{\xi_t} - r e^{\eta_s}\right) ds \\
&\quad + \frac{\varepsilon}{1+\varepsilon} \limsup_{t\to\infty} \frac{1}{t}\int_0^t \left(-(\sigma + r) e^{\xi_t} - r e^{\eta_s}\right) ds \\
&\quad \le \frac{1}{1+\varepsilon} (d - r).
\end{aligned} \tag{74}$$

Letting $\varepsilon \to 0$, we have

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t}\int_0^t \left(-\frac{\beta^2}{2} e^{2\eta_s} - (\sigma + r) e^{\xi_t} - r e^{\eta_s}\right) ds \\
&\quad \le (d - r).
\end{aligned} \tag{75}$$

The first assertion is proven. Using similar way, the second assertion is also proven. The proof is completed. □

3. Ergodicity

In the section, we discuss the ergodicity of the solution for stochastic coral reefs model with multiplicative nonlinear noise.

Theorem 6. *Suppose that $\sigma < d < \gamma < r < 2\gamma$, $0 < g < \gamma$ hold.*

(I) *Then, the transition probability function $\mathscr{P}(t, x_0, y_0, \cdot)$ of system (4), that is, $\mathscr{P}(t, x_0, y_0, A) = \mathbb{P}\{(\xi_t, \eta_t) \in A\}$ for an $A \in \mathscr{B}(\mathfrak{R}^2)$, results in a density $k(t, x, y, x_0, y_0)$ for all $t > 0$.*

(II) *Then, system (4) results in integral Markov semigroup.*

Proof. Let $a(x)$ and $b(x)$ denote vector fields on $\mathfrak{R}^d$; then the Lie bracket $[a, b]$ denotes a vector field:

$$[a, b]_j (x) = \sum_{k=1}^{d} \left(a_k \frac{\partial b_j}{\partial x_k}(x) - b_k \frac{\partial a_j}{\partial x_k}(x)\right). \tag{76}$$

Denote

$$\begin{aligned}
&a_0(x, y) \\
&\quad = \begin{pmatrix} \gamma - \alpha^2 e^{2x} - \gamma e^{x_t} + (\sigma - \gamma) e^{y_t} - \dfrac{g}{1 - e^{y_t}} \\ (r - d) - \beta^2 e^{2y} - (\sigma + r) e^{x_t} - r e^{y_t} \end{pmatrix}, \\
&a_1(x, y) = \begin{pmatrix} \alpha e^x \\ \beta e^y \end{pmatrix}.
\end{aligned} \tag{77}$$

Then it is easy to see that

$$\begin{aligned}
a_2 &= [a_0, a_1] = \begin{pmatrix} \alpha e^x \left(\gamma + \alpha^2 e^{2x} - (\gamma - \sigma) e^{y_t} + \dfrac{g}{1 - e^{y_t}}\right) + \beta (\gamma - \sigma) e^{2y} + \dfrac{\beta g e^{2y}}{(1 - e^y)^2} \\ \beta e^y \left((r - d) + \beta^2 e^{2y} - (\sigma + r) e^{x_t}\right) + \beta (\sigma + r) e^{2x} \end{pmatrix}, \\
a_3 &= [a_1, [a_0, a_1]] = \begin{pmatrix} 2\alpha^4 e^{4y} + 2\beta (r - \sigma) e^{3y} + \alpha\beta e^{x+y} \left[(\sigma - r) + \dfrac{g}{(1 - e^y)^2}\right] + \dfrac{2\beta g e^{2y} (2 - e^y)}{(1 - e^y)^3} \\ 2\beta^4 e^{4y} + 2\alpha\beta (\sigma + r) e^{3x} - \beta (\alpha + \beta) (\sigma + r) e^{2x+y} \end{pmatrix}, \\
a_4 &= [a_1, [a_1, [a_0, a_1]]] = \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \end{pmatrix},
\end{aligned} \tag{78}$$

where

$$\begin{aligned}
\Lambda_1 &= -\alpha e^x \left(2\alpha^4 e^{4y} + 2\beta (r - \sigma) e^{3y} + \frac{2\beta g e^{2y} (2 - e^y)}{(1 - e^t)^3}\right) + \beta e^y \left(8\alpha^4 e^{4y} + 6\beta (r - \sigma) e^{3y} + \alpha\beta e^{x+y} \left[(\sigma - r) + \frac{g (3 - e^y)}{(1 - e^y)^3}\right] + \frac{2\beta g e^{2y} \left[3 (1 - e^y)^2 - 1\right]}{(1 - e^y)^4}\right), \\
\Lambda_2 &= \alpha e^x \left[6\alpha\beta (\sigma + r) e^{3x} - 2\beta (\alpha + \beta) (\sigma + r) e^{2x+y}\right] + \beta e^y \left[6\beta^4 e^{4y} - 2\alpha\beta (\sigma + r) e^{3x}\right].
\end{aligned} \tag{79}$$

Assume that there is a point (x, y) and so the vectors $a_1(x, y)$, $a_3(x, y)$, and $a_4(x, y)$ can not span the space $\mathfrak{R}^2$. Then, the vectors $a_1(x, y)$ and $a_3(x, y)$ are parallel; the vectors $a_1(x, y)$ and $a_4(x, y)$ are also parallel. Therefore, we get

$$\begin{aligned}
&\alpha e^x \left(2\beta^4 e^{4y} + 2\alpha\beta(\sigma + r) e^{3x}\right. \\
&\quad \left. - \beta(\alpha + \beta)(\sigma + r) e^{2x+y}\right) = \beta e^y \left(2\alpha^4 e^{4y}\right. \\
&\quad + 2\beta(r - \sigma) e^{3y} + \alpha\beta e^{x+y}\left[(\sigma - r) + \frac{g}{(1 - e^y)^2}\right] \\
&\quad \left. + \frac{2\beta g e^{2y}(2 - e^y)}{(1 - e^y)^3}\right) \\
&\alpha e^x \Lambda_2 = \beta e^y \Lambda_1.
\end{aligned} \tag{80}$$

It is easy to check that equality (80) is impossible.

It implies that the vectors a_1, a_3, a_4 span $\mathfrak{R}^2$ at any point (x, y). Therefore, we obtain the Hörmander condition.

($\mathscr{H}$) For every $(x, y) \in \mathfrak{R}^2$, the vectors

$$\begin{gathered}
a_1(x, y), \\
\left[a_i, a_j\right](x, y)_{0 \le i, j \le 1}, \\
\left[a_i, \left[a_j, a_k\right]\right](x, y)_{0 \le i, j, k \le 1}, \ldots
\end{gathered} \tag{81}$$

span the space $\mathfrak{R}^2$.

By Hypothesis ($\mathscr{H}$) and Hörmander theorem [22–29], then the transition probability function $\mathscr{P}(t, x_0, y_0, \cdot)$ results in a density $k(t, x, y, x_0, y_0)$ and $k \in C^\infty((0, \infty) \times \mathfrak{R}^2 \times \mathfrak{R}^2)$. Thus, the first assertion has been proved.

From the first assertion, it easily shows that for any $t > 0$, (ξ_t, η_t) of system (4) results in the density $u(t, x, y)$ satisfying the FPE (9). Furthermore, we get

$$u(t, x, y) = \iint_{-\infty}^{\infty} k(t, x, y, x_1, y_1)\, \nu(x_1, y_1)\, dx_1 dy_1. \tag{82}$$

Defining the operator $\{P(t)\}_{t \ge 0}$ is

$$\begin{aligned}
P(t)\, \nu(x, y) &= u(t, x, y) \\
&= \iint_{-\infty}^{\infty} k(t, x, y, x_1, y_1)\, \nu(x_1, y_1)\, dx_1 dy_1
\end{aligned} \tag{83}$$

for any $t > 0$, $\nu \in D$. By using continuation theorem of operator and assertion (I), it easily shows that the operator $\{P(t)\}_{t \ge 0}$ is an integral Markov semigroup. The proof is completed. □

Theorem 7. *Suppose that $\sigma < d < \gamma < r < 2\gamma$, $0 < g < \gamma$ hold. Then there is no more than three solution curves such that $x = g(y)$ satisfying rank $D_{x_0, y_0, \phi} = 2$ if $x_\phi(T) \ne g(y_\phi(T))$.*

Proof. Let $(x_0, y_0) \in \mathfrak{R}^2$ and $\phi \in L^2([0, T]; \mathfrak{R})$. We consider the following system:

$$\begin{aligned}
x'_\phi &= \gamma - \alpha^2 e^{2x_t} + (\alpha\phi(t) - \gamma) e^{x_t} - (\gamma - \sigma) e^{y_t} \\
&\quad - \frac{g}{1 - e^{y_t}}, \\
y'_\phi &= r - d - \beta^2 e^{2y_t} - (\sigma + r) e^{x_t} + (\beta\phi(t) - r) e^{y_t}.
\end{aligned} \tag{84}$$

Denote

$$\begin{aligned}
\overline{f}_1 &= \gamma - \alpha^2 e^{2x_t} + (\alpha\phi(t) - \gamma) e^{x_t} - (\gamma - \sigma) e^{y_t} \\
&\quad - \frac{g}{1 - e^{y_t}}, \\
\overline{f}_2 &= r - d - \beta^2 e^{2y_t} - (\sigma + r) e^{x_t} + (\beta\phi(t) - r) e^{y_t}.
\end{aligned} \tag{85}$$

System (84) becomes

$$\begin{aligned}
x'_\phi &= \overline{f}_1(x_\phi, y_\phi), \\
y'_\phi &= \overline{f}_2(x_\phi, y_\phi).
\end{aligned} \tag{86}$$

We show that (x_ϕ, y_ϕ) denote the solution of system (84) with the initial value $x_\phi(0) = x_0$, $y_\phi(0) = y_0$ and $F : C([0, T], \mathfrak{R}) \to \mathfrak{R}^2$ defined as $F(h) = (x_{\phi+H}(T), y_{\phi+H}(T))$. By using the perturbation method, we get the Frechet derivative $D_{x_0, y_0, \phi}$ of F. Let $f = (\overline{f}_1, \overline{f}_2)$ and $\Lambda(t) = f'(\overline{f}_1, \overline{f}_2)$. $Q(t, s)$ denote the fundamental matrix of the following differential equation:

$$\dot{Y} = \Lambda(t) Y. \tag{87}$$

That is, $\partial Q(t, s)/\partial t = \Lambda(t) Q(t, s)$ and $Q(ts, s) = I$ for $s \le t \le T$. Then, we have

$$D_{x_0, y_0; \phi} h = \int_0^T Q(T, s)\, q(x_\phi, y_\phi)\, h(s)\, ds, \tag{88}$$

where $q(x_\phi, y_\phi) = (\alpha e^y, \beta e^\beta)^\top$. Let $\varepsilon \in (0, T)$ and $h(t) = 0$ if $0 \le t \le T - \varepsilon$ and $h(t) = (1/\epsilon)(t - T + \varepsilon)$ if $T - \varepsilon \le t \le T$. By using Taylor formula, we get $Q(T, s) = I - \Lambda(T)(T - s) + o(T - s)$ as $s \to T$. Then, we have

$$\begin{aligned}
D_{x_0, y_0; \phi} h &= \int_{T-\varepsilon}^T \frac{s - T + \varepsilon}{\varepsilon} q(x_\phi, y_\phi)\, ds + \Lambda(T) \\
&\quad \cdot \int_{T-\varepsilon}^T \frac{s - T + \varepsilon}{\varepsilon} (s - T)\, q(x_\phi, y_\phi)\, ds + o(\varepsilon^2).
\end{aligned} \tag{89}$$

Let $\overline{x} = x_\phi(T)$, $\overline{y} = y_\phi(T)$, and $c = \phi(T)$. By using mean value theorem of integration, we get

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{T-\varepsilon}^T \frac{s - T + \varepsilon}{\varepsilon} q(x_\phi(s), y_\phi(s))\, ds = \frac{1}{2} q(\overline{x}, \overline{y}), \\
&\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \Lambda(T) \int_{T-\varepsilon}^T \frac{s - T + \varepsilon}{\varepsilon} (s - T)\, q(x_\phi(s), y_\phi(s))\, ds \\
&\quad = -\frac{1}{6} \Lambda(T)\, q(\overline{x}, \overline{y}).
\end{aligned} \tag{90}$$

It is easy to calculate that we show

$$\Lambda(T) = \begin{pmatrix} -2\alpha^2 e^{2\overline{x}_t} + (\alpha c - \gamma) e^{\overline{x}_t} & -(\gamma - \sigma) e^{\overline{y}_t} - \dfrac{g e^{\overline{y}_t}}{(1 - e^{\overline{y}_t})^2} \\ -(\sigma + r) e^{\overline{x}_t} & -2\beta^2 e^{2\overline{y}_t} + (\beta c - r) e^{\overline{y}_t} \end{pmatrix}. \quad (91)$$

Therefore, we obtain

$$\Lambda(T) q(\overline{x}, \overline{y}) = \begin{pmatrix} -2\alpha^3 e^{3\overline{x}_t} + \alpha(\alpha c - \gamma) e^{2\overline{x}_t} - \beta(\gamma - \sigma) e^{2\overline{y}_t} - \dfrac{\beta g e^{2\overline{y}_t}}{(1 - e^{\overline{y}_t})^2} \\ -\alpha(\sigma + r) e^{2\overline{x}_t} - 2\beta^3 e^{3\overline{y}_t} + \beta(\beta c - r) e^{2\overline{y}_t} \end{pmatrix}. \quad (92)$$

If the two vectors $q(\overline{x}, \overline{y})$ and $\Lambda(T)$ are not linearly dependent, then we get the rank $D_{x_0, y_0, \phi} = 2$. Since the between $q(\overline{x}, \overline{y})$ and $\Lambda(T) q(\overline{x}, \overline{y})$ is linear dependence, thus, it is easy to see that $\overline{x}$ and $\overline{y}$ denote the solution of the following differential equation:

$$\det \begin{pmatrix} \alpha e^{\overline{x}} & -2\alpha^3 e^{3\overline{x}_t} + \alpha(\alpha c - \gamma) e^{2\overline{x}_t} - \beta(\gamma - \sigma) e^{2\overline{y}_t} - \dfrac{\beta g e^{2\overline{y}_t}}{(1 - e^{\overline{y}_t})^2} \\ \beta e^{\overline{y}} & -\alpha(\sigma + r) e^{2\overline{x}_t} - 2\beta^3 e^{3\overline{y}_t} + \beta(\beta c - r) e^{2\overline{y}_t} \end{pmatrix} = 0. \quad (93)$$

That is,

$$\begin{aligned} &\alpha^2 \left(2\alpha\beta e^{\overline{y}} - (\sigma + r)\right) e^{3\overline{x}_t} - \left(\alpha\beta(\alpha c - \gamma) e^{\overline{y}}\right) e^{2\overline{x}_t} \\ &\quad + \left(\alpha\beta(\beta c - r) e^{2\overline{y}_t} - 2\alpha\beta^3 e^{3\overline{y}_t}\right) e^{\overline{x}} \\ &\quad + \left(\beta^2(\gamma - \sigma) + \frac{\beta^2 g}{(1 - e^{\overline{y}_t})^2}\right) e^{3\overline{y}} = 0. \end{aligned} \quad (94)$$

It easily knows that there is no more than three solution curves satisfying (94), and the graph of a function $\overline{x} = g(\overline{y})$ represents each solution curve. The proof is completed. □

Theorem 8. *Suppose that $\sigma < d < \gamma < r < 2\gamma$, $0 < g < \gamma$, $r + g < d + \gamma$ hold. Hypothesis condition $\mathscr{H}_1$: there is a point z^* satisfying $g_2(y, z^*) \le 0$ for all $y \in \mathfrak{R}$ and $(y_\phi, z_\phi) \in E$. The following assertions are true.*

(I) *If Hypothesis condition $\mathscr{H}_1$ does not hold, then system (84) is controllable in E.*

(II) *If Hypothesis condition $\mathscr{H}_1$ holds, then system (84) is controllable in E_i $(i = 1, 2)$, where*

$$\begin{aligned} g_2(y_\phi, z_\phi) &= -\gamma z + \alpha\beta^{-1}(r - d - \gamma) e^{-y} + \frac{\alpha^2 - \alpha\beta^{-1} e^{-y}}{z + \beta^{-1}\alpha e^{-y(t)}} \\ &\quad + \frac{g\left(z + \beta^{-1}\alpha e^{-y(t)}\right)}{1 - e^y} + (z(\gamma - \sigma) - \alpha\beta) e^y \\ &\quad + \frac{\alpha(\gamma - \sigma) + \alpha\beta + \beta r}{\beta}, \\ E &= \{(y, z) \mid z > -\alpha\beta e^{-y}\}, \\ E_1 &= \left\{(x, y) \mid e^{-x} - \beta\alpha^{-1} e^{-y} < a^*\right\} \\ &= \{(y, z) \in E \mid z \le a^*\}, \\ E_2 &= \left\{(x, y) \mid e^{-x} - \beta\alpha^{-1} e^{-y} > (\gamma - \sigma)^{-1} \alpha\beta\right\} \\ &= \left\{(y, z) \mid z > (\gamma - \sigma)^{-1} \alpha\beta\right\}. \end{aligned} \quad (95)$$

Proof. Let

$$z_\phi(t) = e^{-x_\phi(t)} - \beta^{-1}\alpha e^{-y_\phi(t)}. \quad (96)$$

Then, system (84) can be replaced by the following differential equations:

$$\begin{aligned} y'_\phi &= \beta e^{y_\phi}\phi + g_1(y_\phi, z_\phi), \\ z'_\phi &= g_2(y_\phi, z_\phi), \end{aligned} \quad (97)$$

where

$$\begin{aligned} g_1(y_\phi, z_\phi) &= r - d - \frac{\sigma + r}{z + \alpha\beta^{-1} e^{-y}} - r e^y - \beta^2 e^{2y}, \\ g_2(y_\phi, z_\phi) &= -\gamma z + \alpha\beta^{-1}(r - d - \gamma) e^{-y} \\ &\quad + \frac{\alpha^2 - \alpha\beta^{-1} e^{-y}}{z + \beta^{-1}\alpha e^{-y(t)}} \\ &\quad + \frac{g\left(z + \beta^{-1}\alpha e^{-y(t)}\right)}{1 - e^y} \\ &\quad + (z(\gamma - \sigma) - \alpha\beta) e^y \\ &\quad + \frac{\alpha(\gamma - \sigma) + \alpha\beta + \beta r}{\beta}. \end{aligned} \quad (98)$$

Denote

$$E = \{(y, z) \mid z > -\alpha\beta e^{-y}\}. \quad (99)$$

By (96), it is easy to see that $(y_\phi, z_\phi) \in E$ for any $\phi \in C([0, T], \mathfrak{R})$. There is a point z^* satisfying $g_2(y, z^*) \le 0$ for

all $y \in \mathfrak{R}$ and $(y_\phi, z_\phi) \in E$. It is easily know that there exists $z^* \in [0, (\gamma-\sigma)^{-1}\alpha\beta]$; let

$$a^* = \inf\{z^* \mid g_2(y, z^*) \le 0 \ \forall y \in \mathfrak{R},\ (y, z^*) \in E\}, \quad (100)$$

$$\begin{aligned} E_1 &= \left\{(x, y) \mid e^{-x} - \beta\alpha^{-1}e^{-y} < a^*\right\} \\ &= \{(y, z) \in E \mid z \le a^*\}, \\ E_2 &= \left\{(x, y) \mid e^{-x} - \beta\alpha^{-1}e^{-y} > (\gamma-\sigma)^{-1}\alpha\beta\right\} \\ &= \left\{(y, z) \mid z > (\gamma-\sigma)^{-1}\alpha\beta\right\}. \end{aligned} \quad (101)$$

Step 1. Fix $z_0 > z_1$. Due to $\lim_{y\to-\infty} g(y, z)_2 = -\infty$ uniformly in $z \in [z_1, z_0]$, it shows that there is y_0 satisfying $g_2(y_0, z) \le -1$ for any $z \in [z_1, z_0]$. We take

$$\phi = -\frac{g_1(y_0, z_1)}{\beta e^{y_0}}, \quad (102)$$

where $z(t)$ denotes the solution of

$$\begin{aligned} z'_\phi &= g_2(y_0, z_1), \\ z(0) &= z_0. \end{aligned} \quad (103)$$

It implies that system (97) results in the solution $(y_\phi, z_\phi) = (y_0, z_1)$ and $z_\phi(0) = z_0$. Due to $z'_\phi = g_2(y_0, z_1) \le -1$ whenever z_ϕ, we can choose a $T > 0$ satisfying $z_\phi(T) = z_1$.

Step 2. Fix $(\gamma-\sigma)^{-1}\alpha\beta < z_0 < z_1$. Due to $\lim_{y\to\infty} g_2(y, z) = \infty$ uniformly in $z \in [z_1, z_0]$, we can choose y_0 satisfying $g_2(y_0, z) \ge 1$ for any $z \in [z_1, z_0]$. By using similar way to Step 1, it easily shows that there is a function $\phi(t)$ and $T > 0$ such that (97) results in a solution $y_\phi(0) \equiv y_0$, $z_\phi(0) \equiv z_0$, and $z_\phi(T) = z_1$.

Step 3. Let $z_0 < 0$ and

$$\bar{z} = \ln\left[\frac{-\alpha}{\beta z_0}\right]. \quad (104)$$

Since $\lim_{y\to\bar{z}^{-1}} g_2(y, z_0) = \infty$, it easily knows that there is real number c_1 and $y_0 \in \mathfrak{R}$ satisfying $g(y_0, z)_2 \ge 1$ for any $z \in [z_0 - c_1, z_0 + c_1]$ and $(y_0, z) \in E$. In the case, we can choose a control function ϕ satisfying $y_\phi \equiv y_0$, $z_\phi(0) = z_0$, and $z_\phi(T) = z_0 + c_1$ for some $T > 0$.

Step 4. If Hypothesis condition $\mathscr{H}_1$ holds. It is easy to see that $z^* \in [0, (\gamma-\sigma)^{-1}\alpha\beta]$. Based on the definition of a^*, for any $\varepsilon > 0$, there are $\mu_1 > 0$ and $\mu_2 > 0$ satisfying the property: if (z_0, z_1) such that $z_1 - \mu_1 < z_0 < z_1 < a^* - \varepsilon$, one can choose a y_0 such that $g(y_0, z)_2 > \mu_2$ for $z \in [z_0, z_1]$ and $(y_0, z) \in E$. Hence, we can choose a control function ϕ satisfying $y_\phi \equiv y_0$, $z_\phi(0) = z_0$, and $z_\phi(T) = z_1$ for some $T > 0$.

Step 5. If Hypothesis condition $\mathscr{H}_1$ does not hold. Then there are μ_1 and $\mu_2 > 0$ satisfying the property: if (z_0, z_1) such that $z_1 - \mu_1 < z_0 < z_1$ and $z_0, z_1 \in [0, (\gamma-\sigma)^{-1}\alpha\beta]$, one can choose a y_0 such that $g(y_0, z)_2 > \mu_2$ for $z \in [z_0, z_1]$ and $(y_0, z) \in E$. By the same way as before, we can choose a control function ϕ satisfying $y_\phi(0) = y_0$, $z_\phi(0) = z_0$, and $z_\phi(T) = z_1$ for some $T > 0$.

Step 6. Let $y_0 \in \mathfrak{R}$, $K > 0$, $L_1 > L_0$, and $0 < \varepsilon < \min(K/4, (L_1 - L_0)/4)$ and $[y_0, y_0 + K] \times [L_0, L_1] \subset E$. Denote

$$\begin{aligned} m^* = \max\{&|g_1(y, z)| + |g_2(y, z)| \mid (y, z) \\ &\in [y_0, y_0 + K] \times [L_0, L_1]\}, \end{aligned} \quad (105)$$

and $t_0 = \varepsilon/m^*$. For every $z_0 \in [L_0 + \varepsilon, L_1 - \varepsilon] \times [L_0, L_1]$, it easily knows that system (97) with $y_\phi(0) = y_0$, $z_\phi(0) = z_0$ results in the solution such that, for all $t \in [0, t_0]$,

$$\begin{aligned} y_\phi(t_0) &\in \left(y_0 + \frac{K}{2}, y_0 + K\right), \\ z_\phi(t) &\in [z_0 - \varepsilon, z_0 + \varepsilon]. \end{aligned} \quad (106)$$

From (97), we get

$$\beta\phi e^{y_\phi} - m^* \le y'_\phi \le \beta\phi e^{y_\phi} - m^*. \quad (107)$$

It is easy to calculate that the solution of the equation $y' = ay + b$ is

$$y = y_0 + bt + \ln\frac{b}{ae^{y_0} + b - ae^{y_0+bt}}. \quad (108)$$

By using comparison theorem, we have, for any $t \in [0, t_0]$,

$$\begin{aligned} y_\phi &\ge y_0 - m^*t + \ln m^* \\ &\quad - \ln\left[\beta\phi e^{y_0}\left(e^{-m^*t} - 1\right) + m^*\right], \\ y_\phi &\le y_0 + m^*t + \ln m^* \\ &\quad - \ln\left[\beta\phi e^{y_0}\left(1 - e^{m^*t}\right) + m^*\right]. \end{aligned} \quad (109)$$

Thus,

$$\begin{aligned} y_\phi &\ge y_0 - \varepsilon + \ln m^* - \ln\left[\beta\phi e^{y_0}\left(e^{-\varepsilon} - 1\right) + m^*\right], \\ y_\phi &\le y_0 + \varepsilon + \ln m^* - \ln\left[\beta\phi e^{y_0}\left(1 - e^{\varepsilon}\right) + m^*\right]. \end{aligned} \quad (110)$$

Therefore, (I) term of (106) can be proven if we find a constant ϕ satisfying

$$\begin{aligned} &y_0 - \varepsilon + \ln m^* - \ln\left[\beta\phi e^{y_0}\left(e^{-\varepsilon} - 1\right) + m^*\right] \\ &\quad > y_0 + \frac{K}{2}, \end{aligned} \quad (111)$$

$$y_0 + \varepsilon + \ln m^* - \ln\left[\beta\phi e^{y_0}\left(1 - e^{\varepsilon}\right) + m^*\right] < y_0 + K. \quad (112)$$

It is obvious that (111) is equivalent to

$$\phi \in \left(\frac{m^*\left(1 - e^{-\varepsilon-K/2}\right)}{\beta e^{y_0}\left(1 - e^{-\varepsilon}\right)}; \frac{m^*}{\beta e^{y_0}\left(1 - e^{-\varepsilon}\right)}\right), \quad (113)$$

and (112) is equivalent to

$$\phi < \frac{m^* \left(1 - e^{-\varepsilon - K}\right)}{\beta e^{y_0} \left(e^{\varepsilon} - 1\right)}. \tag{114}$$

Obviously, for ε is small enough, we get

$$\frac{m^* \left(1 - e^{-\varepsilon - K/2}\right)}{\beta e^{y_0} \left(1 - e^{-\varepsilon}\right)} < \frac{m^* \left(1 - e^{-\varepsilon - K}\right)}{\beta e^{y_0} \left(e^{\varepsilon} - 1\right)}. \tag{115}$$

In other words, we can find ϕ such that (I) term of (106) is true. For the second assertion, we get, for all $0 \le t \le t_0$,

$$\begin{aligned} \left|z_\phi(t) - z_\phi(0)\right| &= \left|\int_0^t g_2\left(y_\phi(s), z_\phi(s)\right) ds\right| \\ &\le \int_0^{t_0} m^* ds = m^* t_0 = \varepsilon. \end{aligned} \tag{116}$$

From (106), it easily shows that for $(y_1, z_1) \in (y_0, y_0 + K/2] \times [L_0 + 2\varepsilon, L_1 - 2\varepsilon]$ there is a $z_0 \in [z_1 - \varepsilon, z_1 + \varepsilon]$ and a $T \in (0, t_0)$ satisfying $y_\phi(T) = y_1$ and $z_\phi(T) = z_1$. By using similar proofs, $(y_0 - K/2, y_0]$. Therefore, for any $(y_1, z_1) \in (y_0 - K/2, y_0 + K/2] \times [L_0 + 2\varepsilon, L_1 - 2\varepsilon]$, we obtain that there is a $z_0 \in [z_1 - \varepsilon, z_1 + \varepsilon]$ and a $T \in (0, t_0)$ satisfying $y_\phi(T) = y_1$ and $z_\phi(T) = z_1$. The proof is completed. □

Theorem 9. *Suppose that $\sigma < d < \gamma < r < 2\gamma$, $0 < g < \gamma$, $r + g < d + \gamma$ hold. The following assertions are true.*

(I) *If Hypothesis condition $\mathscr{H}_1$ does not hold, then there is a constant $T > 0$ satisfying $k(T, x, y, x_0, y_0) > 0$ for each $(x_0, y_0) \in \mathfrak{R}^2$ and for almost every $(x, y) \in \mathfrak{R}^2$.*

(II) *If Hypothesis condition $\mathscr{H}_1$ holds, then there is a constant $T > 0$ such that $k(T, x, y, x_0, y_0) > 0$ for any point $(x_0, y_0) \in \mathfrak{R}^2$ and almost every $(x, y) \in \mathfrak{R}^2$ such that (x_0, y_0) and (x, y) in E_i $(i = 1, 2)$, where $z = e^{-x} - \beta^{-1}\alpha e^{-y}$, $k(T, x, y, x_0, y_0) > 0$ is a measurable function.*

Proof. By using continuity theory, we can find a continuity function $\phi \in C(0, T; \mathfrak{R})$. If there exists $\phi \in C(0, T; \mathfrak{R})$ such that the derivative $D_{x_0, y_0, \phi}$ is the rank 2, then we have $k(T, x, y, x_0, y_0) > 0$. Therefore, the proof is completed. □

Theorem 10. *Suppose that $\sigma < d < \gamma < r < 2\gamma$, $0 < g < \gamma$, $r + g < d + \gamma$ hold. If Hypothesis condition $\mathscr{H}_1$ holds. Then, $\limsup_{t\to\infty} \zeta_t \le a^*$, where a^* is defined in (100).*

Proof. System (7) becomes

$$\begin{aligned} d\eta_t &= g_1\left(\eta_t, \zeta_t\right) dt + \beta e^{y_t} \circ dWt, \\ d\zeta_t &= g_2\left(\eta_t, \zeta_t\right) dt. \end{aligned} \tag{117}$$

To prove $\inf_{t\ge 0}\zeta_t < a^*$ s.a, we assume that $\inf_{t\ge 0}\zeta_t \ge a^*$ for all $\omega \in \Omega$ with $\mathbb{P}(\Omega) > 0$. Due to $\zeta_t \ge a^*$ for all $t \ge 0$, we get $g_1(y_t, \zeta_t) \le g_1(y_t, a^*)$ for $\omega \in \Omega$ and $y \in \mathfrak{R}$. Generally, with $\mathbb{P}(\Omega) < 1$, the comparison theorem fails to work here since the diffusion term is nonlinear. By $\lambda(t) = e^{-\eta_t}$, we have

$$d\lambda(t) = -\lambda(t)\, g_1\left(-\ln \lambda(t), \zeta_t\right) dt - \beta dW_t, \tag{118}$$

where $\overline{\lambda}$ denote the solution of

$$d\overline{\lambda}(t) = -\overline{\lambda}(t)\, g_1\left(-\ln \overline{\lambda}(t), \zeta_t\right) dt - \beta dW_t, \tag{119}$$

with $\overline{\lambda}(0) = \lambda(0)$. Since $-y g_1(\ln y, \zeta_t) \ge -y g_1(\ln y, a^*)$ on Ω for all $t > 0$, it is easy to see that $\lambda(t, \omega) \ge \overline{\lambda}(t, \omega)$. Hence, we get $\eta_t \le \eta_t^*(\omega)$ for $\omega \in \Omega$, where $\eta_t^* = -\ln \overline{\lambda}$ is the solution of

$$\begin{aligned} d\eta_t^* &= g_1\left(\eta_t^*, a^*\right) dt + \beta e^{\eta_t^*} \circ dW_t, \\ \eta_0^* &= \eta_0. \end{aligned} \tag{120}$$

Denote

$$\begin{aligned} s_3(x) = \int_0^x \exp\left\{-\int_0^y \frac{2\left(r - d - \left(\beta^2/2\right) e^{2\eta_t} - (\sigma + r)/\left(a^* + \alpha\beta^{-1}e^{-u}\right) - re^{\eta_t}\right)}{\beta^2 e^{2u}} du\right\} dy = \int_0^x \exp\left\{\frac{r + d}{\beta^2} + \frac{r - d}{\beta^2 e^{2y}}\right. \\ \left. - \frac{2r}{\beta^2 e^y} + y + \int_0^y \frac{2(\sigma + r)}{\left(a^* + \alpha\beta^{-1}e^{-u}\right)\beta^2 e^{2u}} du\right\} dy. \end{aligned} \tag{121}$$

It is obvious that $\lim_{t\to\infty} s_3(x) = \infty$, $\lim_{t\to-\infty} s_3(x) > -\infty$. Thus, we get $\lim_{t\to\infty}\eta_t^* = -\infty$ a.s. Hence, we have $\lim_{t\to\infty}\eta_t = -\infty$ on Ω. Moreover, by the definition of g_2, there exists a $M_1 > 0$ satisfying $g_2(y, z) \le -1$ for all $y \le -M_1$, $z \ge 0$. Therefore, for any $\omega \in \Omega$, there is a $M_2 > 0$ such that $\eta_t \le -M_1$ for $t \ge M_2$, it means that $g_2(\eta_t(\omega)\zeta_t(\omega)) \le -1$. By using the second equation of (117), we get $\lim_{t\to\infty}\zeta_t(\omega) = -\infty$, which contradicts our assumption that $\inf_{t\ge 0}\zeta_t \ge a^*$ on Ω.

Next, we prove that $\limsup_{t\to\infty}\zeta_t \le a^*$ a.s. It easily knows that

$$g_2(y, z) = \frac{G(y, z)}{e^{-y}\left(z + \alpha\beta^{-1}e^{-y}\right)}, \tag{122}$$

where

$$\begin{aligned} G(y, z) = z\left(z(\gamma - \sigma) - \alpha\beta\right) + \alpha^2\beta^{-2}(r - d - \gamma) \\ \cdot e^{-3y} + \left[\alpha\beta^{-1}(r - d - \gamma) z - \gamma\alpha\beta^{-1} z - \alpha\beta^{-1}\right. \end{aligned}$$

$$+ \alpha\beta^{-2}\left(\alpha\left(\gamma-\sigma\right)+\alpha\beta+\beta r\right)\Big] e^{-2y} + \Big[\alpha^2 - \gamma z^2 + \alpha\beta^{-1}\left(z\left(\gamma-\sigma\right)-\alpha\beta\right) + \left(\alpha\left(\gamma-\sigma\right)+\alpha\beta+\beta r\right)\beta^{-1}z\Big] e^{-y} \tag{123}$$

is a polynomial of order 3 of the variable e^{-y}. Based on the definition of a^*, there is no more than one point $c_0 \in \mathfrak{R}$ satisfying $g_2(c_0, a^*) = 0$. Then, we have $g_2(y, a^*) \le 0$ for all $y \in \mathfrak{R}$ that for every $\tau > 0$, $g_2(y, a^*) < 0$ for all $y > c_0 + \tau$ or $y < c_0 - \tau$. By continuity theorem, we can choose an $\varepsilon > 0$ and a "rectangle" such that

$$B = \left(-\infty, c_0 - \tau\right] \times \left[a^*, a^* + k\right] \cup \left[c_0 + \tau, \infty\right) \times \left[a^* + k\right] \tag{124}$$

with $k > 0$ satisfying $g_2(y, z) < -\varepsilon$ for all $(y, z) \in A$. By using the Markov property, we get $\limsup_{t\to\infty}\zeta_t \le a^*$ a.s. The proof is complete. □

Theorem 11. *Suppose that $\sigma < d < \gamma < r < 2\gamma$, $0 < g < \gamma$, $r + g < d + \gamma$ hold. If Hypothesis condition $\mathscr{H}_1$ holds, then E_1 is an invariant set; namely, if $(\eta_0, \zeta_0) \in E_1$, then $(\eta_t, \zeta_t) \in E_1$ for all $t > 0$.*

Proof. System (117) results in a solution (η_t, ζ_t) satisfying $(\eta_0, \zeta_0) \in E_1$ and $(\eta_{t_1}(\omega), \zeta_{t_1}(\omega)) \in E_1$ with some $t_1 > 0$, $\omega \in \Omega$. Based on the continuity of the path $\zeta_t(\omega)$, it is easy to show that there is $0 \le t_0 < \bar{t}_0 < t_1$ satisfying

$$\zeta_{t_0}(\omega) = a^*, \quad \zeta_s(\omega) < \zeta_t(\omega) \quad \forall t_0 < s < t < \bar{t}_0. \tag{125}$$

The Property (P). For any $\tau > 0$ there are two constants $k > 0$ and $\varepsilon > 0$ satisfying $g_2(y, z) < -\varepsilon$ for all $(y, z) \in B$, where

$$B = \left(-\infty, c_0 - \tau\right] \times \left[a^*, a^* + k\right] \cup \left[c_0 + \tau, \infty\right) \times \left[a^* + k\right]. \tag{126}$$

The property (P) has been proven in Theorem 10. Since the equation $d\zeta_t = g_2(\eta_t(\omega), \zeta_t(\omega))dt$, (125) and the property (P) that $\eta_t(\omega) = c_0$ for any $t \in [t_0, \bar{t}_0]$, we get $d\zeta_t/dt = g_2(c_0, \zeta_t(\omega))$. From (125), it easily shows that there is a decreasing sequence $\{s_n\}_{n=1}^{\infty}$ satisfying $\lim_{n\to\infty} s_n = t_0$ and $g_2(c_0, \zeta_{s_n}(\omega)) > 0$ for any $n = 1, 2, \ldots$. It is obvious that the result contradicts the property (P). Therefore, the proof is complete. □

Theorem 12. *Suppose that $\sigma < d < \gamma < r < 2\gamma$, $0 < g < \gamma$, $r + g < d + \gamma$, $(\sigma + r)m > r - d$ hold, where m is defined by (23). Then $\lim_{t\to+\infty}\eta_t = -\infty$; the distribution of the Markov process ξ_t weakly converges to the probability measure with the density f_* when $t \to \infty$, where the Markov process (ξ_t, η_t) denotes a solution of (4) with $(\xi_0, \eta_0) \in \mathfrak{R}^2$.*

Proof. By Theorem 4, it is easy to see that $\xi_s \le \bar{\xi}_s$; then we get

$$\eta_t = \eta_0 + \int_0^t \left(r - d - \frac{\beta^2}{2}e^{2\eta_s} - (\sigma + r)e^{\xi_t} - re^{\eta_s}\right) ds + \int_0^t \beta e^{\eta_s} dW_s. \tag{127}$$

Therefore,

$$\frac{\eta_t}{t} \le \frac{\eta_0}{t} + \frac{\beta\int_0^t e^{\eta_s}dW_s - \left(\beta^2/2\right)\int_0^t e^{2\eta_s}ds}{t} - \frac{(\sigma + r)}{t}\int_0^t e^{\xi_t}ds + r - d. \tag{128}$$

Based on the proof of Theorem 5, we have

$$\limsup_{t\to\infty}\frac{\beta\int_0^t e^{\eta_s}dW_s - \left(\beta^2/2\right)\int_0^t e^{2\eta_s}ds}{t} \le 0 \quad \text{a.a.} \tag{129}$$

Moreover, by ergodic theorem, we get

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t e^{\xi_t}ds = \int_{-\infty}^{+\infty} e^x f_*(x)\,dx = m. \tag{130}$$

From (23) and (128)–(130), we have

$$\limsup_{t\to\infty}\frac{\eta_t}{t} \le r - d - (\sigma + r)m < 0. \tag{131}$$

Then

$$\lim_{t\to+\infty}\eta_t = -\infty. \tag{132}$$

Thus, for sufficiently small $\varepsilon > 0$, it is easy to see that there exist t_0 and a set Ω_ε satisfying $\mathrm{Prob}(\Omega_\varepsilon) > 1-\varepsilon$ and $(\gamma-\sigma)e^{y_t} + g/(1 - e^{y_t}) \le \varepsilon$ for $t \ge t_0$ and $\omega \in \Omega_\varepsilon$. From the inequalities

$$\alpha dW + \left(\gamma - \frac{\alpha^2}{2} - \gamma e^{\xi_t} - \varepsilon\right)dt \le d\xi_t \le \alpha dW + \left(\gamma - \frac{\alpha^2}{2} - \gamma e^{\xi_t}\right)dt, \tag{133}$$

it easily shows that the distribution of the Markov process ξ_t weakly converges to the probability measure which possesses the density f_*. The proof is complete. □

Theorem 13. *Suppose that $\sigma < d < \gamma < r < 2\gamma$, $0 < g < \gamma$, $r + g < d + \gamma$, $(\sigma + r)m < r - d$ hold; then there is a stationary distribution in system (4).*

Proof. By using the former random variables X_t and Y_t. It is easy to see that (X_t, Y_t) is a Markov process on $\mathfrak{R}_+^2$ and $\gamma = m\gamma + m'(\alpha^2/2)$, where

$$m' = \int_{\mathfrak{R}} e^{2x} f_*(x)\,dx. \tag{134}$$

Moreover,

$$\limsup_{t\to\infty}\int_0^t X_s^2 ds = \limsup_{t\to\infty}\int_0^t e^{2\xi_s} ds \le \lim_{t\to\infty} e^{2\bar{\xi}_s} ds = m'. \tag{135}$$

Therefore, by using the second assertion of Theorem 5, we get

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t\left(Y_s + \frac{g(\gamma-\sigma)^{-1}}{1-e^{Y_s}} + \frac{\gamma}{\gamma-\sigma}X_s\right)ds \ge \frac{\gamma}{\gamma-\sigma}m. \tag{136}$$

Furthermore, by Theorem 5, we have

$$\frac{\gamma}{(\gamma-\sigma)(\sigma+r)}\liminf_{t\to\infty}\frac{1}{t}\cdot\int_0^t\left(-(\sigma+r)X_s - rY_s - \frac{\beta^2}{2}Y_s^2\right)ds \ge -\frac{\gamma(r-d)}{(\gamma-\sigma)(\sigma+r)}. \tag{137}$$

From (136) and (137), we get

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t\left(-\frac{\gamma(r+\sigma-\gamma)}{(\gamma-\sigma)(\sigma+r)}Y_t - \frac{\beta^2\gamma}{2(\gamma-\sigma)(\sigma+r)}Y_2^2\right)ds \ge \frac{\gamma((\sigma+r)m-(r-d))}{(\gamma-\sigma)(\sigma+r)}. \tag{138}$$

By Theorem 5 again, we have

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t\left(rY_s + \frac{\beta^2}{2}Y_t^2\right)ds \ge (r-d) - m(\sigma+r) > 0. \tag{139}$$

Moreover, we get that the inequality

$$\int_0^t Y_s ds \le \sqrt{t\int_0^t Y_s^2 ds}. \tag{140}$$

Then there are two constants $M_1 > 0$ and $M_2 > 0$ such that

$$2M_1 \le \liminf_{t\to\infty}\frac{1}{t}\int_0^t Y_s^2 ds \le \limsup_{t\to\infty}\frac{1}{t}\int_0^t Y_s^2 ds \le M_2. \tag{141}$$

From the inequality

$$2M_1 \le \liminf_{t\to\infty}\frac{1}{t}\int_0^t Y_s^2 ds, \tag{142}$$

it is easy to see that

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t 1_{\{Y_s^2>M_1\}}Y_s^2 ds > M_1 > 0. \tag{143}$$

By Holder's inequality, we get

$$\frac{1}{t}\int_0^t \mathbb{E}\left[1_{\{Y_s^2>M_1\}}Y_s^2\right]ds \le \left(\frac{1}{t}\int_0^t \mathbb{E}\left[1_{\{Y_s^2>M_1\}}\right]^{(p+1)/p}ds\right)^{p/(p+1)} \cdot\left(\frac{1}{t}\int_0^t \mathbb{E}\left[Y_s\right]^{2+2p}ds\right)^{1/(p+1)}. \tag{144}$$

For $0 < p < 1/2$, there is a constant M_3 satisfying

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}\left[Y_s\right]^{2+2p}ds < M_3. \tag{145}$$

Thus, we get

$$M_1 \le \liminf_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}\left[1_{\{Y_s^2>M_1\}}Y_s^2\right]ds \le \left(\liminf_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{P}\left\{Y_s^2 > M_1\right\}ds\right)^{p/(p+1)} \cdot\left(\limsup_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}\left[Y_s\right]^{2+2p}ds\right)^{1/(p+1)}. \tag{146}$$

That is,

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{P}\left\{Y_s^2 > M_1\right\}ds \ge \varepsilon_0 = \frac{M_1^{(p+1)/p}}{M_3^{1/p}}. \tag{147}$$

Furthermore, there is a constant $H > 0$ such that

$$\liminf_{t\to\infty}\mathbb{P}\left\{X_t + Y_t \le H\right\} \ge 1 - \frac{\varepsilon_0}{2}. \tag{148}$$

It means that

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t Q_t(A)ds \ge \frac{\varepsilon_0}{2} > 0, \tag{149}$$

where Q_t is the semigroup related to the random variable (X_t, Y_t) and

$$A = \left\{y \ge \sqrt{M_1}, x + y \le H\right\}. \tag{150}$$

From (149) and Theorem 1 in [29, pp 36], there is a stationary distribution λ in $\mathfrak{R}_+^2$ for the random variable (X_t, Y_t) satisfying $\lambda(A) > 0$. Because the boundary $A_1 = \{x = 0\} \times \mathfrak{R}_+$ is invariant under Q_t and $\lim_{t\to\infty} Y_t = 0$ if $X_0 = 0$, then we get that $\lambda(A_1) > 0$. Thus, there is a stationary distribution λ on int $\mathfrak{R}^2$; that is, there is a stationary distribution related to the random variable (ξ_t, η_t). The proof is complete. □

Theorem 14. *Suppose that $\sigma < d < \gamma < r < 2\gamma$, $0 < g < \gamma$, $r + g < d + \gamma$ hold. Then the distribution of the random variable (ξ_t, η_t) exists a density $u(t, x, y)$ satisfying (8). Furthermore, the following assertions are true*

(I) *Assume* $(\sigma + r)m < r - d$; *if Hypothesis condition* $\mathscr{H}_1$ *does not hold, then the Markov semigroup* $\{P(t)\}_{t\geq 0}$ *is asymptotically stable on* $\mathfrak{R}^2$; *that is, there is a stationary density* $u_*(t, x, y)$ *of (8) satisfying*

$$\lim_{t\to\infty} \iint_{\mathfrak{R}^2} \left|u(t, x, y) - u_*(x, y)\right| dx\, dy = 0. \tag{151}$$

(II) *Assume* $(\sigma + r)m < r - d$; *if Hypothesis condition* $\mathscr{H}_1$ *holds, then* E_2 *is a transient set and* E_1 *is an invariant set. Furthermore, the integral Markov semigroup* $\{P(t)\}_{t\geq 0}$ *is asymptotically stable on* E_1. *It implies that support* $u_* \subset E_1$ *and*

$$\lim_{t\to\infty} \iint_{E_1} \left|u(t, x, y) - u_*(x, y)\right| dx\, dy = 0. \tag{152}$$

(III) *Assume* $(\sigma+r)m > r-d$; *then* $\lim_{t\to\infty}\eta_t = -\infty$ *a.s. and the distribution of random variable* ξ_t *weakly converges to the probability measure with the density* $f_*(x)$ *when* $t \to \infty$.

Proof. From Theorem 6, then the distribution of the random variable (ξ_t, η_t) results in a density $u(t, x, y)$ satisfying (8). By Theorem 9, for every $f \in D$, we get

$$\int_0^\infty P(t)\, f dt > 0 \quad \text{a.e. on } \mathfrak{R}^2 \text{ or on } E_1. \tag{153}$$

By Corollary 1 in [23, pp 248], it is easy to see that the integral Markov semigroup $\{P(t)\}_{t\geq 0}$ is asymptotically stable or sweeping. Based on Theorems 10, 11, and 13, it is easy to see that (I) assertion and (II) assertion are directly proven. By Theorem 12, (III) assertion is directly proven. The proof is complete. □

Competing Interests

The author declares that he has no competing interests.

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